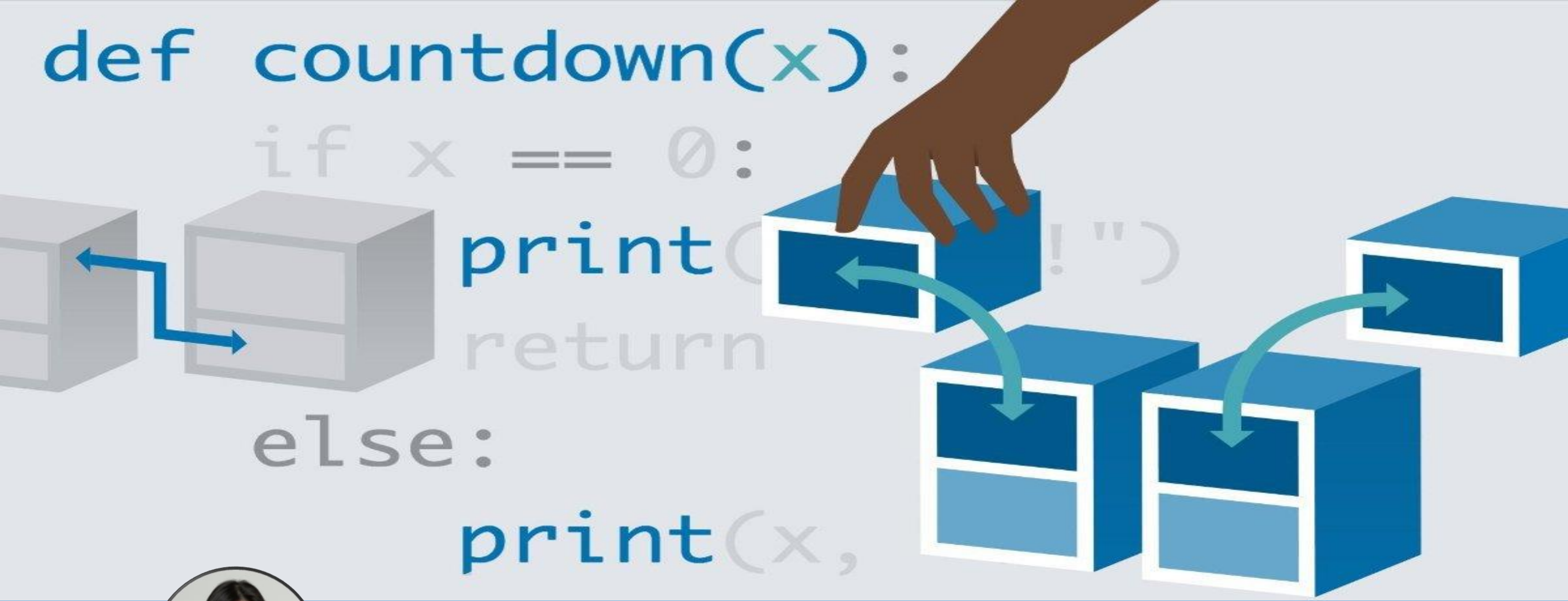

Algorithm Design and Analysis
Introduction

http://ada.miulab.tw slido: #ADA2022

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(Slides modified from hil)

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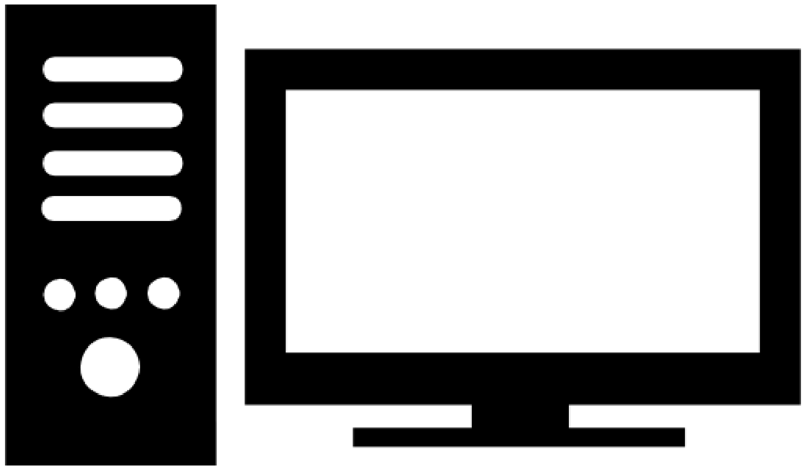
Outline

- Terminology
  - Problem (問題)
  - Problem instance (個例)
  - Computation model (計算模型)
  - Algorithm (演算法)
  - The hardness of a problem (難度)
- Algorithm Design & Analysis Process
- Review: Asymptotic Analysis
- Algorithm Complexity
- Problem Complexity

Efficiency Measurement = Speed

- Why we care?
  - Computers may be fast, but they are not infinitely fast
  - Memory may be inexpensive, but it is not free



Terminology

Textbook Ch. 1 – The Role of Algorithms in Computing

Problem (問題)

The champion problem

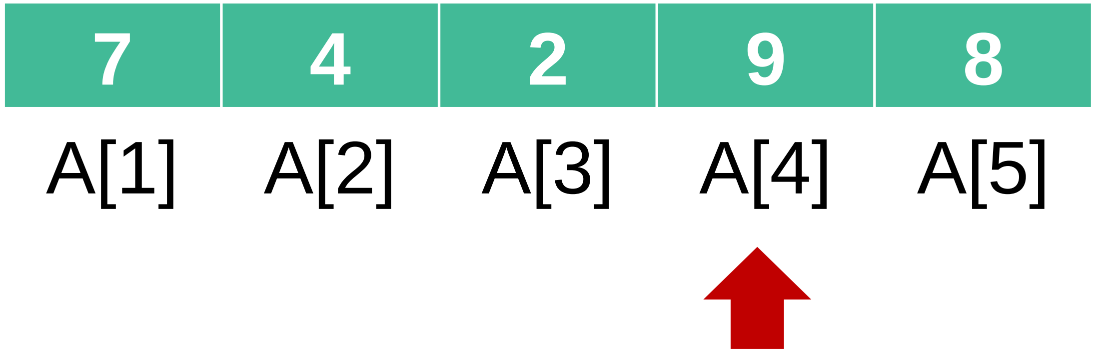
- Input: n distinct integers $A[1], A[2], \ldots, A[n]$.
- Output: the index i with $1 \leq i \leq n$ such that

$$A[i] = \max_{1 \leq j \leq n} A[j].$$

Problem Instance (個例)

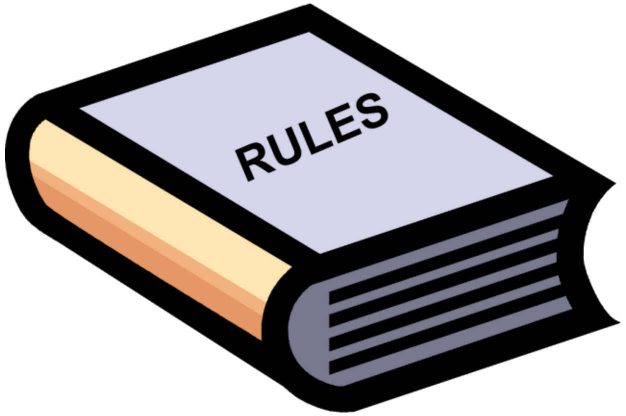
- An instance of the champion problem

$$5 \text{ distinct integers } 7, 4, 2, 9, 8.$$



Computation Model (計算模型)

- Each problem must have its rule (遊戲規則)
- Computation model (計算模型) = rule (遊戲規則)
- The problems with different rules have different hardness levels



Hardness (難易程度)

- How difficult to solve a problem
  - Example: how hard is the champion problem?
  - Following the comparison-based rule

What does “solve (解)” mean?

What does “difficult (難)” mean?

Problem Solving (解題)

- Definition of “solving” a problem
  - Giving an algorithm (演算法) that produces a correct output for any instance of the problem.

Algorithm (演算法)

- Algorithm: a detailed step-by-step instruction
  - Must follow the game rules
  - Like a step-by-step recipe
  - Programming language doesn't matter

  → problem-solving recipe (technology)
- If an algorithm produces a correct output for <u>any instance</u> of the problem

  → this algorithm "solves" the problem

Hardness (難度)

- Hardness of the problem
  - How much effort the best algorithm needs to solve any problem instance
- 防禦力
  - 看看最厲害的賽亞人要花多少攻擊力才能打贏對手

攻擊力 50000

防禦力 100000

Algorithm Design & Analysis Process

Algorithm Design & Analysis Process

1. Formulate a **problem**
2. Develop an **algorithm**

Design Step

3. Prove the **correctness**
4. Analyze **running time/space** requirement

Analysis Step

1. Problem Formulation

The champion problem

- Input: n distinct integers $A[1], A[2], \ldots, A[n]$.
- Output: the index i with $1 \leq i \leq n$ such that

$$A[i] = \max_{1 \leq j \leq n} A[j].$$

2. Algorithm Design

- Create a detailed recipe for solving the problem
  - Follow the comparison-based rule
    - 不准偷看信封的內容
    - 請別人幫忙「比大小」

- Algorithm: 擂台法

```
int i, j;
j = 1;
for (i = 2;  i <= n;  i++)
    if (A[i] > A[j])
        j = i;
return j;
```

Q1: Is this a comparison-based algorithm?

Q2: Does it solve the champion

3. Correctness of the Algorithm

- Prove by contradiction (反證法)

The algorithm solves the champion problem.

Proof Let j^* be the correct answer. That is, $A[j^*] = \max\{A[1], \ldots, A[n]\}$.

- If $j^* = 1$, then Step 5 is never reached. Therefore, 1 is correctly returned.
- If $j^* > 1$, then in the iteration of the for-loop with $i = j^*$, j becomes j^*. By definition of j^*, $A[j^*] > A[i]$ holds for each $i = j^* + 1, \ldots, n$. Therefore, in the remaining iterations of the for-loop, the value of j does not change. Hence, at the end of the algorithm, j^* is correctly returned.

```
1.  int i, j;
2.  j = 1;
3.  for (i = 2; i <= n; i++)
4.      if (A[i] > A[j])
5.          j = i;
6.  return j;
```

Hardness of The Champion Problem

- How much effort the best algorithm needs to solve any problem instance
  - Follow the comparison-based rule
    - 不准偷看信封的內容
    - 請別人幫忙「比大小」
- Effort: we first use the <u>times of comparison</u> for measurement

```
int i, j;
j = 1;
for (i = 2;  i <= n; i++)
    if (A[i] > A[j])
        j = i;
return j;
```

(n - 1) comparisons

Hardness of The Champion Problem

- The hardness of the champion problem is (n - 1) comparisons
  a) There is an algorithm that can solve the problem using at most (n – 1) comparisons
     - This can be proved by 擂臺法, which uses (n – 1) comparisons for any problem instance
  b) For any algorithm, there exists a problem instance that requires (n - 1) comparisons
     - Why?

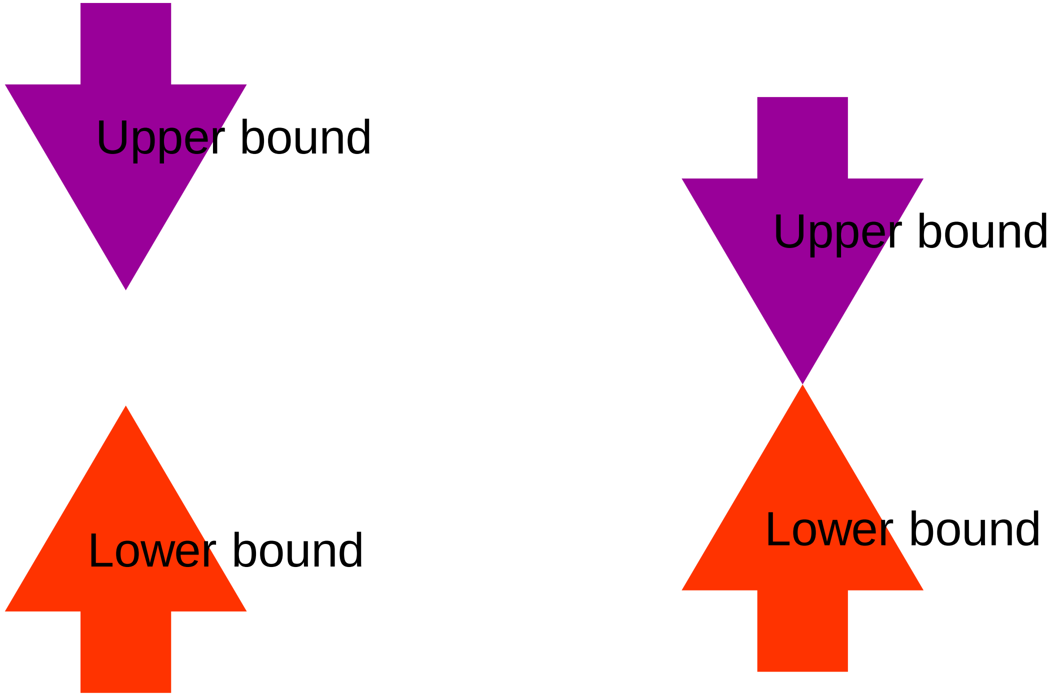
Hardness of The Champion Problem

- Q: Is there an algorithm that only needs $n - 2$ comparisons?
- A: Impossible!
- Reason
  - A single comparison only decides a loser
  - If there are only $n - 2$ comparisons, the most number of losers is $n - 2$
  - There exists a least 2 integers that did not lose

→ any algorithm cannot tell who the champion is

Finding Hardness

- Use the **upper bound** and the **lower bound**
- When they meet each other, we know the <u>hardness of the problem</u>



Hardness of The Champion Problem

- Upper bound
  - how many comparisons are sufficient to solve the champion problem
  - Each algorithm provides an upper bound
  - The smarter algorithm provides tighter, lower, and better upper bound

多此一舉擂臺法

```
int i, j;
j = 1;
for (i = 2;  i <= n;  i++)
    if ((A[i] > A[j]) && (A[j] < A[i]))
        j = i;
return j;
```

→ (2*n* - 2) comparisons

- Lower bound
  - how many comparisons in the worst case are necessary to solve the champion problem
  - Some arguments provide different lower bounds
  - Higher lower bound is better

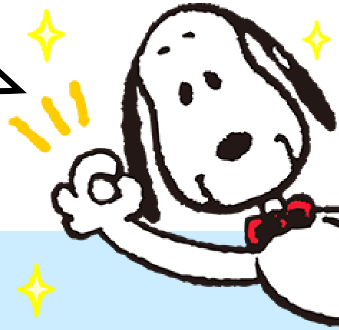
Every integer needs to be in the comparison once
→ (*n*/2) comparisons

When upper bound = lower bound, the problem is solved.
→ We figure out the hardness of the problem

4. Algorithm Analysis

- The majority of researchers in algorithms studies the time and space required for solving problems in two directions
  - Upper bounds: designing and analyzing algorithms
  - Lower bounds: providing arguments
- When the upper and lower bounds match, we have an optimal algorithm and the problem is completely resolved

$O \quad \Omega \quad \Theta \quad o \quad \omega$



Asymptotic Analysis

Edmund Landau
(1877-1938)

Donald E. Knuth
(1938-)

Motivation

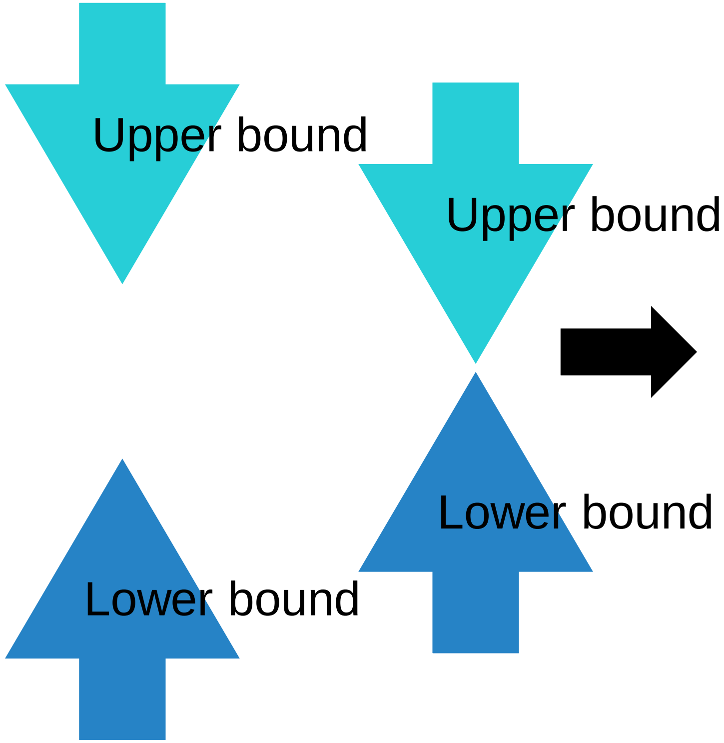
- The hardness of the champion problem is exactly $n-1$ comparisons
- Different problems may have different 「難度量尺」
  - cannot be interchangeable
- Focus on the standard growth of the function to ignore the unit and coefficient effects

Goal: Finding Hardness

- For a problem *P*, we want to figure out
  - The hardness (complexity) of this problem *P* is $\Theta\big(f(n)\big)$
    - n is the instance size of this problem *P*
    - $f(n)$ is a function
    - $\Theta\big(f(n)\big)$ means that “it exactly equals to the growth of the function”
- Then we can argue that under the comparison-based computation model
  - The hardness of the champion problem is $\Theta(n)$
  - The hardness of the sorting problem is $\Theta(n \log n)$

Goal: Finding Hardness

- Use the upper bound and the lower bound
- When they match, we know the hardness of the problem



use $O(f(n))$ and $o(f(n))$

upper bound is $O(h(n))$ & lower bound is $\Omega(h(n))$
→ the problem complexity is exactly $\Theta(h(n))$

use $\Omega(g(n))$ and $\omega(g(n))$

Goal: Finding Hardness

- First learn how to analyze / measure the effort an algorithm needs
  - Time complexity
  - Space complexity
- Focus on **worst-case** complexity
  - “average-case” analysis requires the assumption about the probability distribution of problem instances

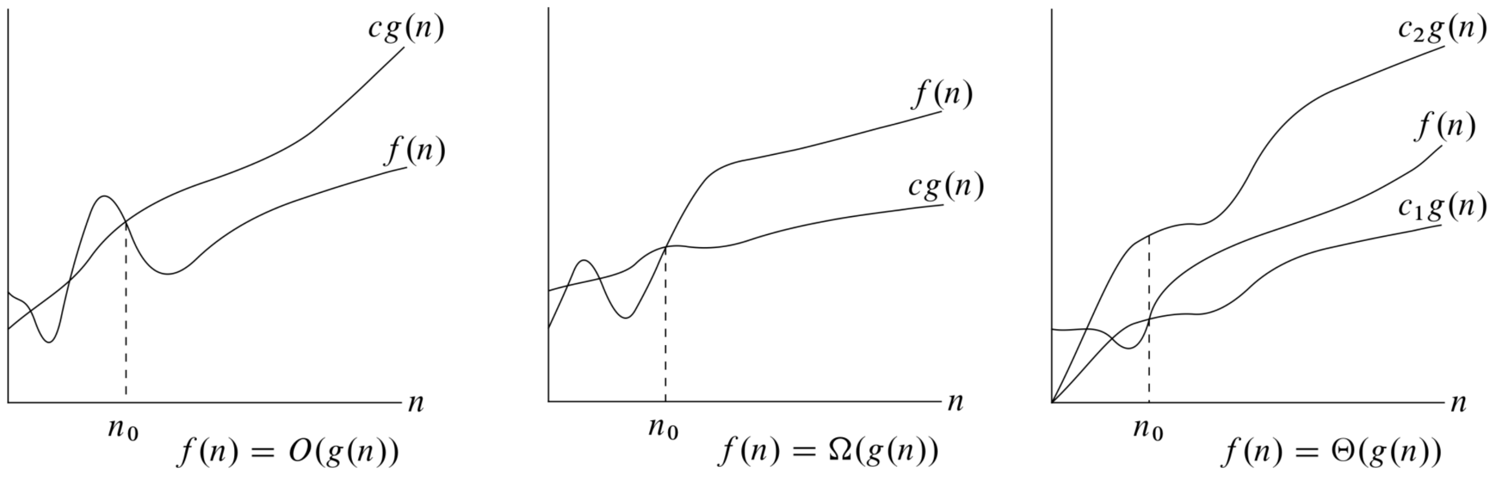
Types	Description
Worst Case	Maximum running time for **any instance** of size n
Average Case	**Expected** running time for a **random instance** of size n
Amortized	Worse-case running time for **a series of** operations

Review of Asymptotic Notation
(Textbook Ch. 3.1)

- $f(n)$ = time or space of an algorithm for an input of size n
- Asymptotic analysis: focus on the ***growth*** of $f(n)$ as $n \to \infty$

Review of Asymptotic Notation
(Textbook Ch. 3.1)

- $f(n)$ = time or space of an algorithm for an input of size n
- Asymptotic analysis: focus on the ***growth*** of $f(n)$ as $n \to \infty$
- O, or Big-Oh: **upper** bounding function
- Ω, or Big-Omega: **lower** bounding function
- Θ, or Big-Theta: **tightly** bounding function



Formal Definition of Big-Oh
(Textbook Ch. 3.1)

- For any two functions $f(n)$ and $g(n)$,

$$f(n) = O(g(n))$$

if there exist positive constants n_0 and c s.t.

$$0 \leq f(n) \leq c \cdot g(n)$$

for all $n \geq n_0$.

$g(n)$的某個常數倍$c \cdot g(n)$可以在n夠大時壓得住$f(n)$

$f(n) = O(g(n))$

- Intuitive interpretation
  - $f(n)$ does not grow faster than $g(n)$
- Comments
  1) $f(n) = O(g(n))$ roughly means $f(n) \leq g(n)$ in terms of rate of growth
  2) "=" is not "equality", it is like "**ϵ (belong to)**"

     The equality is $\{f(n)\} \subseteq O(g(n))$
  3) We do not write $O(g(n)) = f(n)$
- Note
  - $f(n)$ and $g(n)$ can be negative for some integers n
  - In order to compare using asymptotic notation O, both have to be non-negative for sufficiently large n
  - This requirement holds for other notations, i.e. $\Omega, \Theta, o, \omega$

Review of Asymptotic Notation
(Textbook Ch. 3.1)

- Benefit
  - Ignore the low-order terms, units, and coefficients
  - Simplify the analysis
- Example: $f(n) = 5n^3 + 7n^2 - 8$
  - Upper bound: $f(n) = O(n^3)$, $f(n) = O(n^4)$, $f(n) = O(n^3 \log_2 n)$
  - Lower bound: $f(n) = \Omega(n^3)$, $f(n) = \Omega(n^2)$, $f(n) = \Omega(n \log_2 n)$
  - Tight bound: $f(n) = \Theta(n^3)$

"=" doesn't mean "equal to"

- Q: $f(n) = O(n^3)$ and $f(n) = O(n^4)$, so $O(n^3) = O(n^4)$?
  - $O(n^3)$ represents **a set of functions** that are upper bounded by cn^3 for some constant *c* when *n* is large enough
  - In asymptotic analysis, "**=**" means "**ϵ (belong to)**"

Exercise: $100n^2 = O(n^3 - n^2)$?

- Draft.

$$100n^2 \leq 100(n^3 - n^2)$$

$$\leftarrow 200n^2 \leq 100n^3$$

$$\leftarrow 2 \leq n$$

- Let $n_0 = 2$ and $c = 100$

$$100n^2 \leq 100(n^3 - n^2)$$

holds for $n \geq 2$

$$100n^2 = O(n^3 - n^2)$$

Exercise: $n^2 = O(n)$?

- Disproof.
  - Assume for a contradiction that there exist positive constants c and n_0 s.t.

$$n^2 \leq cn$$

holds for any integer n with $n \geq n_0$.

  - Assume $n = 1 + \lceil \max(n_0, c) \rceil$

and because $n > n_0, n > c$, it follows that

$$n^2 > cn$$

  - Due to contradiction, we know that

$$n^2 \neq O(n)$$

Rules
(Textbook Ch. 3.1)

The following statements hold for any real-valued functions $f(n)$ and $g(n)$, where there is a constant n_0 such that $f(n)$ and $g(n)$ are nonnegative for any integer $n \geq n_0$.

- Rule 1: $f(n) = O(f(n))$.
- Rule 2: If c is a positive constant, then $c \cdot O(f(n)) = O(f(n))$.
- Rule 3: If $f(n) = O(g(n))$, then $O(f(n)) = O(g(n))$.
- Rule 4: $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$.
- Rule 5: $O(f(n) \cdot g(n)) = f(n) \cdot O(g(n))$.

Other Notations
(Textbook Ch. 3.1)

$$f(n) = O(g(n)) \rightarrow f(n) \leq g(n) \text{ in rate of growth}$$
$$f(n) = \Omega(g(n)) \rightarrow f(n) \geq g(n) \text{ in rate of growth}$$
$$f(n) = \Theta(g(n)) \rightarrow f(n) = g(n) \text{ in rate of growth}$$
$$f(n) = o(g(n)) \rightarrow f(n) < g(n) \text{ in rate of growth}$$
$$f(n) = \omega(g(n)) \rightarrow f(n) > g(n) \text{ in rate of growth}$$

Goal: Finding Hardness

- First learn how to analyze / measure the effort an algorithm needs
  - Time complexity
  - Space complexity
- Focus on **worst-case** complexity
  - “average-case” analysis requires the assumption about the probability distribution of problem instances

Using O to give upper bounds on the worst-case time complexity of algorithms

Algorithm Analysis

- 擂台法

```
int i, j;                          O(1) time
j = 1;                             O(1) time
for (i = 2;  i <= n;  i++)         O(n) iterations
    if (A[i] > A[j])               O(1) time
        j = i;                     O(1) time
return j;                          O(1) time
```

Adding everything together → an upper bound on the worst-case time complexity

- The worst-case time complexity is

$$O(1) + O(1) + O(n) \cdot (O(1) + O(1)) + O(1)$$

$$
\begin{aligned}
& 3 \cdot O(1) + O(n) \cdot (2O(1)) \\
= & O(1) + O(n) \cdot O(1) && \text{Rule 2} \\
= & O(1) + O(n) && \text{Rule 4} \\
= & O(n) + O(n) && 1 = O(n) \text{ \& Rule 3} \\
= & 2 \cdot O(n) && \text{Rule 2} \\
= & O(n)
\end{aligned}
$$

Sorting Problem

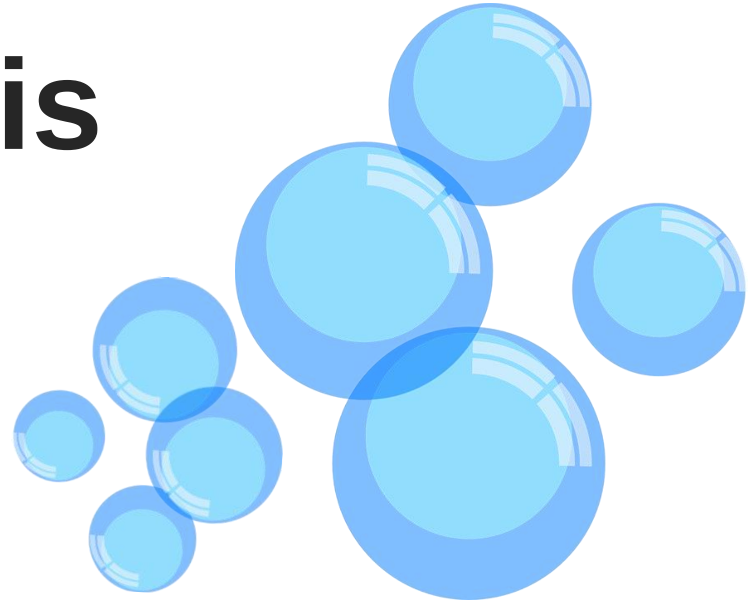
- Input:

  An array A of n distinct integers.

- Output:

  Reorder A such that $A[1] < A[2] < \cdots < A[n]$.

Algorithm Analysis



- Bubble-Sort Algorithm

int *i*, *done*;	$O(1)$ time
do {	$f(n)$ iterations
done = 1;	$O(1)$ time
for (*i* = 1; *i* < *n*; *i* ++) {	$O(n)$ iterations
if (*A*[*i*] > *A*[*i* + 1]) {	$O(1)$ time
exchange *A*[*i*] and *A*[*i* + 1];	$O(1)$ time
done = 0;	$O(1)$ time
}	
}	
} while (*done* == 0)	

$$
\begin{aligned}
& O(1) + f(n) \cdot (O(1) + O(n) \cdot O(1)) \\
= & O(1) + f(n) \cdot O(n) \\
= & f(n) \cdot O(n) \\
= & O(n^2)
\end{aligned}
$$

$f(n) = O(n)$
prove by induction

Example Illustration

7	3	1	4	6	2	5
3	1	4	6	2	5	7
1	3	4	2	5	6	7
1	3	2	4	5	6	7
1	2	3	4	5	6	7

Goal: Finding Hardness

- First learn how to analyze / measure the effort an algorithm needs
  - Time complexity
  - Space complexity
- Focus on **worst-case** complexity
  - “average-case” analysis requires the assumption about the probability distribution of problem instances

Using O to give **upper bounds** on the worst-case time complexity of algorithms

Using Ω to give **lower bounds** on the worst-case time complexity of algorithms

Algorithm Analysis

- 擂台法

int *i*;	$\Omega(1)$ time
int *m* = *A*[1];	$\Omega(1)$ time
for (*i* = 2; *i* <= *n*; *i* ++) {	$\Omega(n)$ iterations
if (*A*[*i*] > *m*)	$\Omega(1)$ time
m = *A*[*i*];	$\Omega(1)$ time
}	
return *m*;	$\Omega(1)$ time

$$\begin{aligned} & 3 \cdot \Omega(1) + \Omega(n) \cdot (2 \cdot \Omega(1)) \\ =& \Omega(1) + \Omega(n) \cdot \Omega(1) \\ =& \Omega(1) + \Omega(n) \\ =& \Omega(n) \end{aligned}$$

Adding everything together
→ a lower bound on the worst-case time complexity?

Algorithm Analysis

- 百般無聊擂台法

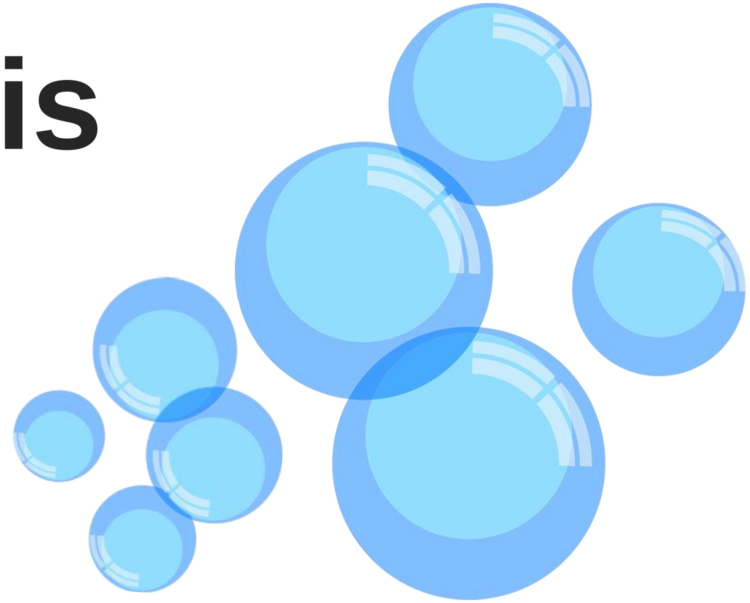
Code	Cost
int *i*;	$\Omega(1)$ time
int *m* = *A*[1];	$\Omega(1)$ time
for (*i* = 2; *i* <= *n*; *i* ++) {	$\Omega(n)$ iterations
if (*A*[*i*] > *m*)	$\Omega(1)$ time
m = *A*[*i*];	$\Omega(1)$ time
if (*i* == *n*)	$\Omega(1)$ time
do *i*++ *n* times	$\Omega(n)$ time
}	
return *m*;	$\Omega(1)$ time

$$
\begin{aligned}
& 3 \cdot \Omega(1) + \Omega(n) \cdot (3 \cdot \Omega(1) + \Omega(n)) \\
=& \Omega(1) + \Omega(n) \cdot \Omega(n) \\
=& \Omega(1) + \Omega(n^2) \\
=& \Omega(n^2)
\end{aligned}
$$

Adding together may result in errors.
The safe way is to analyze using **problem instances**.

e.g. try *A*[*i*] = *i* or *A*[*i*]=2(*n* – *i*) to check the time complexity → $\Omega(1)$

Algorithm Analysis



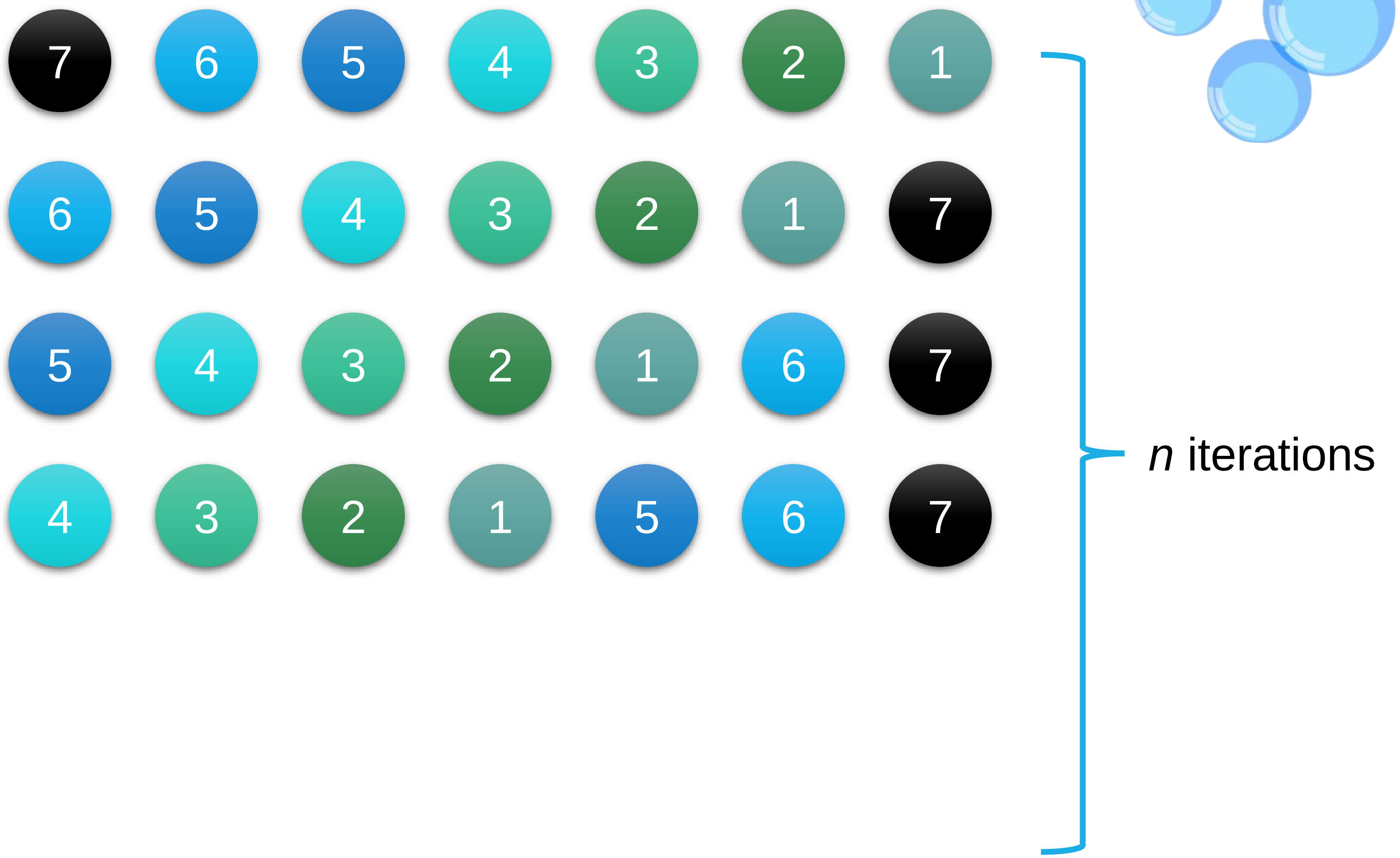
- Bubble-Sort Algorithm

```
int i, done;
do {                                          f(n) iterations
    done = 1;
    for (i = 1; i < n; i ++) {  Ω(n) time
        if (A[i] > A[i + 1]) {
            exchange A[i] and A[i + 1];
             done = 0;
           }
    }
} while (done == 0)
```

When A is decreasing, $f(n) = \Omega(n)$. Therefore, the worst-case time complexity of Bubble-Sort is

$$f(n) \cdot \Omega(n) = \Omega(n^2)$$

Example Illustration



<u>Algorithm</u> **Complexity**

In the worst case, what is the growth of function an algorithm takes

Time Complexity of an Algorithm

- We say that the (worst-case) time complexity of Algorithm *A* is $\Theta(f(n))$ if

1. Algorithm *A* runs in time $O(f(n))$ &
2. Algorithm *A* runs in time $\Omega(f(n))$ (in the worst case)
    - An input instance $I(n)$ s.t. Algorithm A runs in $\Omega(f(n))$ for each n

Tightness of the Complexity

- If we say that the time complexity analysis about $O(f(n))$ is tight
- = the algorithm runs in time $\Omega(f(n))$ in the worst case
- = (worst-case) time complexity of the algorithm is $\Theta(f(n))$
  - Not over-estimate the worst-case time complexity of the algorithm

- If we say that the time complexity analysis of Bubble-Sort algorithm about $O(n^2)$ is tight
- = Time complexity of Bubble-Sort algorithm is $\Omega(n^2)$
- = Time complexity of Bubble-Sort algorithm is $\Theta(n^2)$

Algorithm Analysis

- 百般無聊擂台法

1.	int *i*;	$O(1)$ time
2.	int *m* = *A*[1];	$O(1)$ time
3.	for (*i* = 2; *i* <= *n*; *i* ++) {	$O(n)$ iterations
4.	if (*A*[*i*] > *m*)	$O(1)$ time
5.	*m* = *A*[*i*];	$O(1)$ time
6.	if (*i* == *n*)	$O(1)$ time
7.	do *i*++ *n* times	$O(n)$ time
8.	}	
9.	return *m*;	$O(1)$ time

The worst-case time complexity of 「百般無聊擂臺法」is $\Theta(n)$.

non-tight analysis

$$3 \cdot O(1) + O(n) \cdot (3 \cdot O(1) + O(n))$$
$$= O(1) + O(n) \cdot O(n)$$
$$= O(1) + O(n^2)$$
$$= O(n^2)$$

tight analysis

Step 3 takes $O(n)$ iterations for the for-loop, where only last iteration takes $O(n)$ time and the rest take $O(1)$ time.
The steps 3-8 take time

$$O(n) \cdot O(1) + 1 \cdot O(n) = O(n)$$

The same analysis holds for $\Omega(n)$

Algorithm Comparison

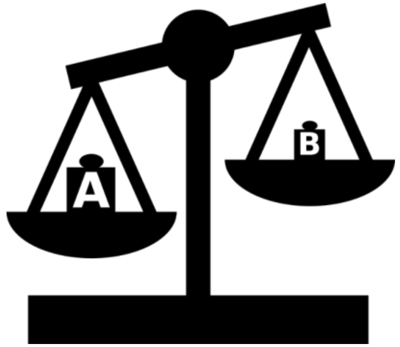
- Q: can we say that Algorithm 1 is a better algorithm than Algorithm 2 if
  - Algorithm 1 runs in $O(n)$ time
  - Algorithm 2 runs in $O(n^2)$ time

- A: No! The algorithm with a lower upper bound on its worst-case time does not necessarily have a lower time complexity.

Comparing *A* and *B*

- Algorithm *A* is no worse than Algorithm *B* in terms of worst-case time complexity if there exists a positive function $f(n)$ s.t.
  - Algorithm *A* runs in time $O\big(f(n)\big)$ &
  - Algorithm *B* runs in time $\Omega\big(f(n)\big)$ in the worst case
- Algorithm *A* is (strictly) better than Algorithm *B* in terms of worst-case time complexity if there exists a positive function $f(n)$ s.t.
  - Algorithm *A* runs in time $O\big(f(n)\big)$ &
  - Algorithm *B* runs in time $\omega\big(f(n)\big)$ in the worst case

  or
  - Algorithm *A* runs in time $o\big(f(n)\big)$ &
  - Algorithm *B* runs in time $\Omega\big(f(n)\big)$ in the worst case



Problem **Complexity**

In the worst case, what is the growth of the function the optimal algorithm of the problem takes

Time Complexity of a Problem

- We say that the (worst-case) time complexity of Problem *P* is $\Theta(f(n))$ if
1. The time complexity of Problem *P* is $O(f(n))$ &
   - There exists an $O(f(n))$-time algorithm that solves Problem *P*
2. The time complexity of Problem *P* is $\Omega(f(n))$
   - Any algorithm that solves Problem *P* requires $\Omega(f(n))$ time
- The time complexity of the champion problem is $\Theta(n)$ because
1. The time complexity of the champion problem is $O(n)$ &
   - 「擂臺法」is the $O(n)$-time algorithm
2. The time complexity of the champion problem is $\Omega(n)$
   - Any algorithm requires $\Omega(n)$ time to make each integer in comparison at least once

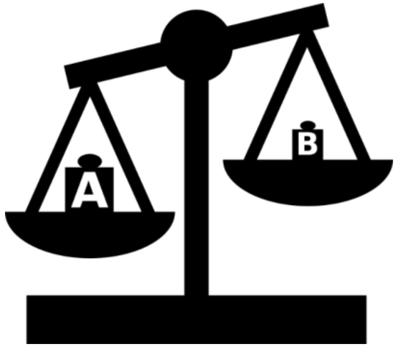
Optimal Algorithm

- If Algorithm *A* is an optimal algorithm for Problem *P* in terms of worst-case time complexity
  - Algorithm *A* runs in time $O\left(f(n)\right)$ &
  - The time complexity of Problem *P* is $\Omega\left(f(n)\right)$ in the worst case
- Examples (the champion problem)
  - 擂台法 → optimal algorithm
    - It runs in $O(n)$ time &
    - Any algorithm solving the problem requires time $\Omega(n)$ in the worst case
  - 百般無聊擂台法 → optimal algorithm
    - It runs in $O(n)$ time &
    - Any algorithm solving the problem requires time $\Omega(n)$ in the worst case

Comparing *P* and *Q*

- Problem *P* is no harder than Problem *Q* in terms of (worst-case) time complexity if there exists a function $f(n)$ s.t.
  - The (worst-case) time complexity of Problem *P* is $O\big(f(n)\big)$ &
  - The (worst-case) time complexity of Problem *Q* is $\Omega\big(f(n)\big)$
- Problem *P* is (strictly) easier than Problem *Q* in terms of (worst-case) time complexity if there exists a function $f(n)$ s.t.
  - The (worst-case) time complexity of Problem *P* is $O\big(f(n)\big)$ &
  - The (worst-case) time complexity of Problem *Q* is $\omega\big(f(n)\big)$

  or
  - The (worst-case) time complexity of Problem *P* is $o\big(f(n)\big)$ &
  - The (worst-case) time complexity of Problem *Q* is $\Omega\big(f(n)\big)$



Concluding Remarks

- Algorithm Design and Analysis Process
  1) Formulate a **problem**
  2) Develop an **algorithm**
  3) Prove the **correctness**
  4) Analyze **running time/space** requirement

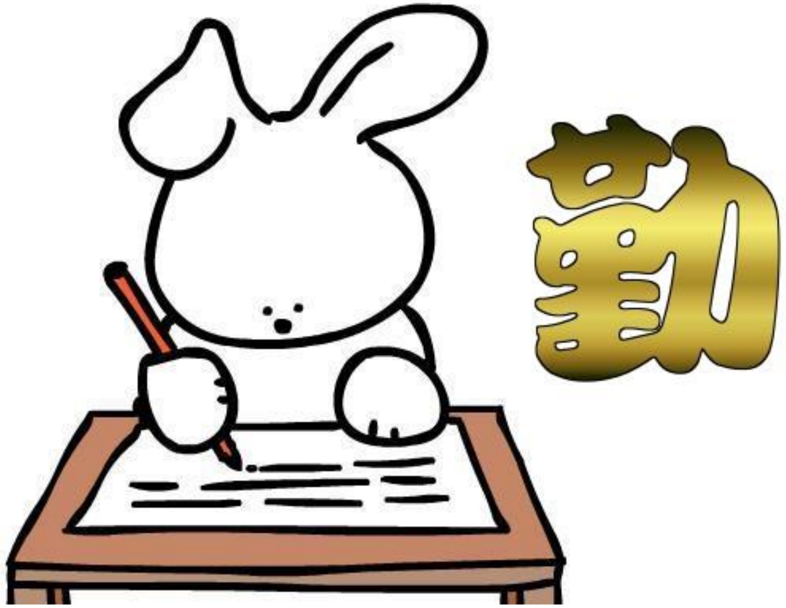
Design Step

Analysis Step

- Usually brute force (暴力法) is not very efficient
- Analysis Skills
  - Prove by contradiction
  - Induction
  - Asymptotic analysis
  - Problem instance
- Algorithm Complexity
  - In the worst case, what is the growth of function an algorithm takes
- Problem Complexity
  - In the worst case, what is the growth of the function the optimal algorithm of the problem takes

Reading Assignment

- Textbook Ch. 3 – Growth of Function



Question?

Important announcement will be sent to @ntu.edu.tw mailbox & post to the course website

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