Slides credited from Hsueh-I Lu, Hsu-Chun Hsiao, & Michael Tsai

Algorithm Design and Analysis
Graph Algorithms (2)

http://ada.miulab.tw

Yun-Nung (Vivian) Chen

國立臺灣大學
National Taiwan University

Midterm Feedback

- Mini-HW
- NTU COOL
- Helpful TAs
- Course recordings (access the channel here)
- Instant feedback

- Grade release
- Course recordings (two classes & last year)
- Homework hints
- TA hour changes

Outline

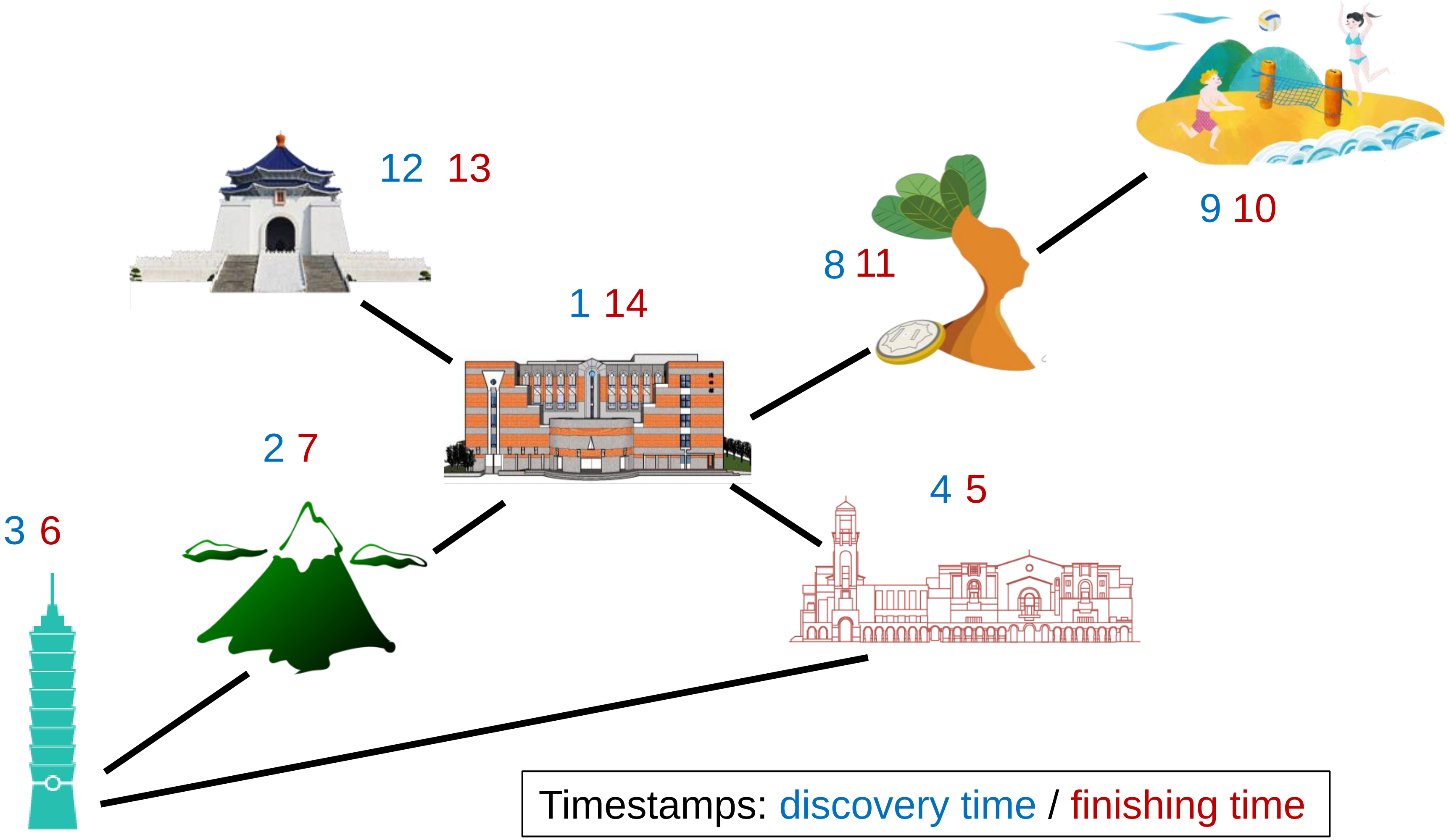
- DFS Applications
  - Connected Components
  - Strongly Connected Components
  - Topological Sorting
- Minimal Spanning Trees (MST)
  - Boruvka’s Algorithm
  - Kruskal’s Algorithm
  - Prim’s Algorithm

Depth-First Search

Textbook Chapter 22.3 – Depth-first search

Depth-First Search (DFS)

- Search as deep as possible and then backtrack until finding a new path

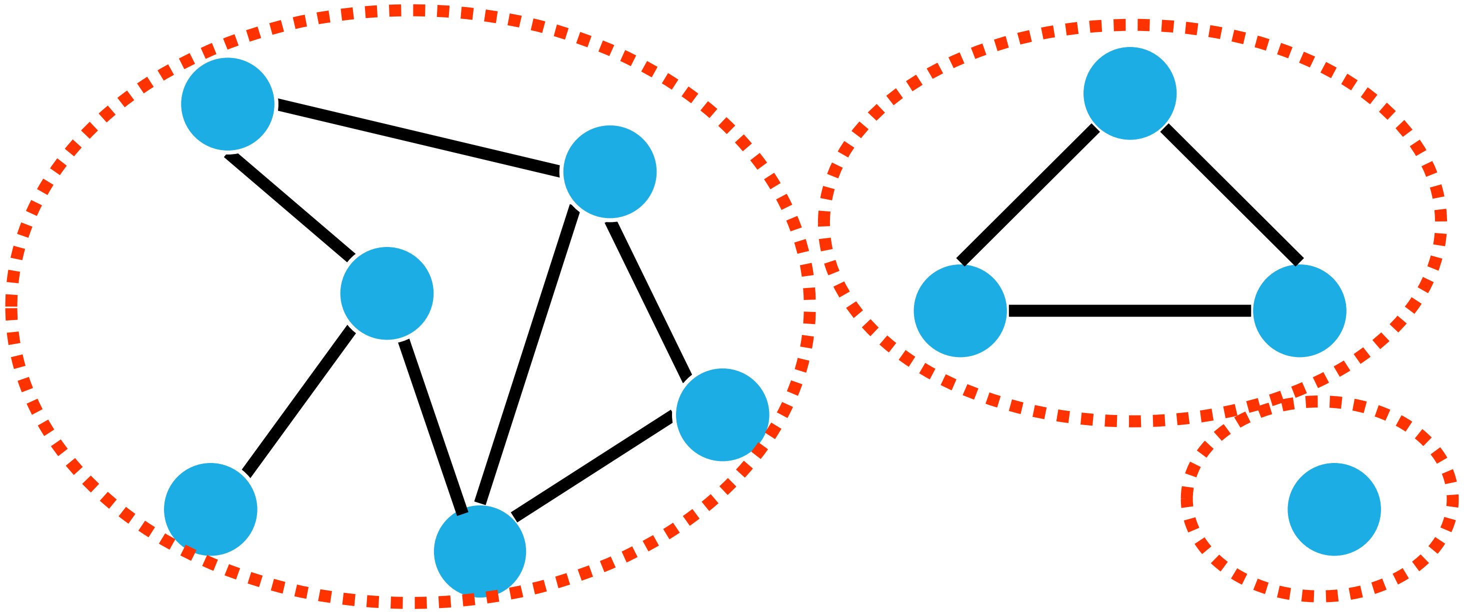


Timestamps: discovery time / finishing time

Connected Components

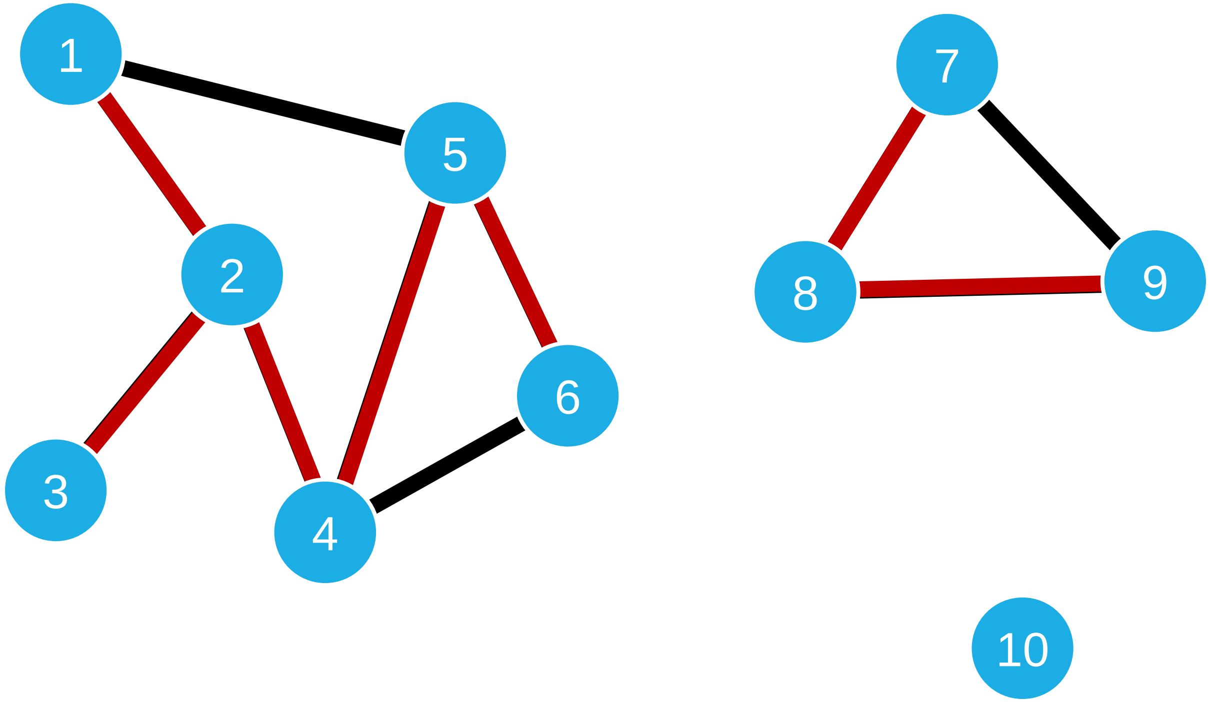
Connected Components Problem

- Input: a graph $G = (V, E)$
- Output: a connected component of G
  - a **maximal** subset U of V s.t. any two nodes in U are connected in G



Why must the connected components of a graph be disjoint?

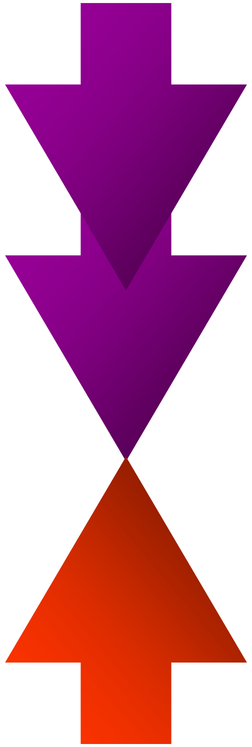
Connected Components



Time Complexity: $O(n + m)$

BFS and DSF both find the connected components with the same complexity

Problem Complexity



$$\text{Upper bound} = O(m+n)$$

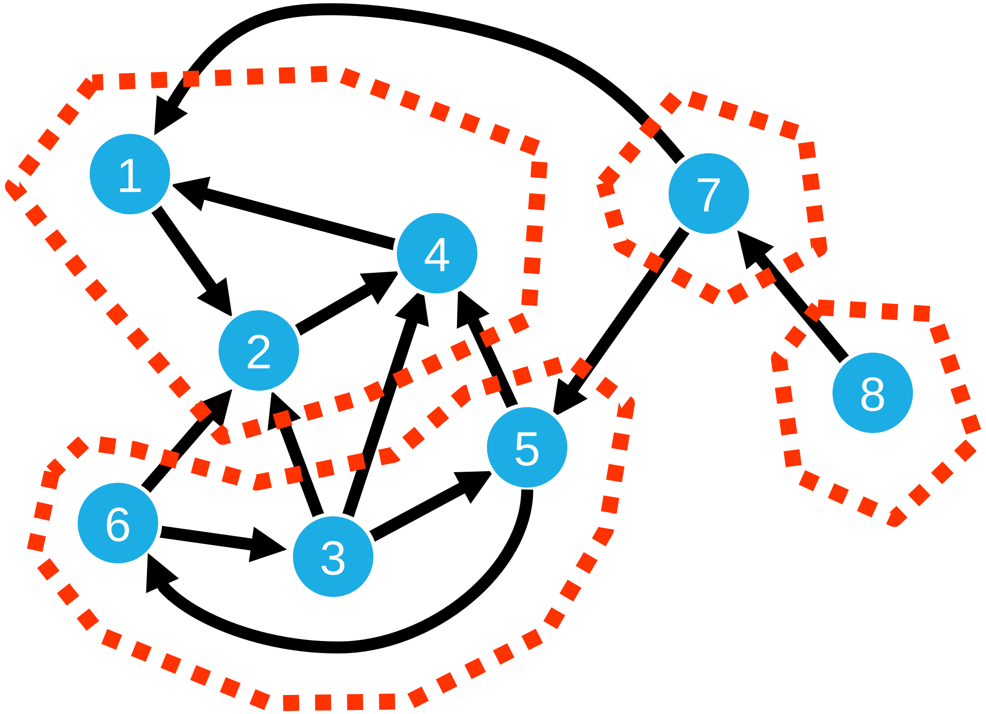
$$\text{Lower bound} = \Omega(m+n)$$

Strongly Connected Components

Textbook Chapter 22.5 – Strongly connected components

Strongly Connected Components

- Input: a directed graph $G = (V, E)$
- Output: a connected component of G
  - a **maximal** subset U of V s.t. any two nodes in U are reachable in G

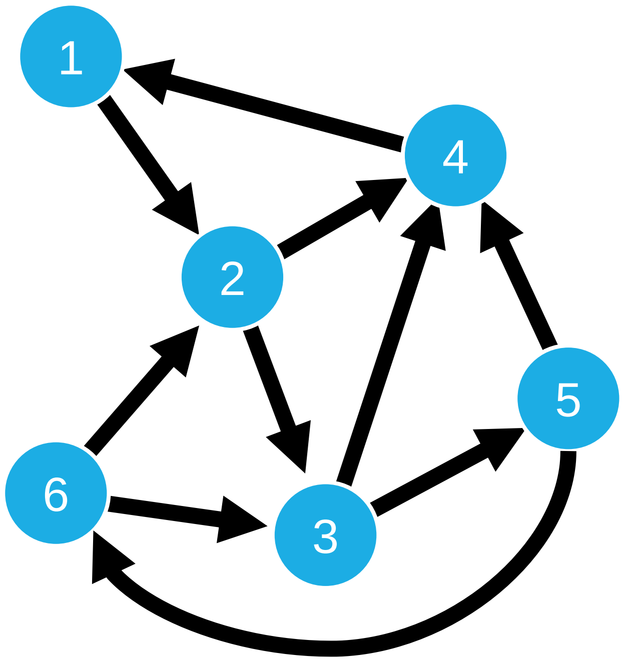


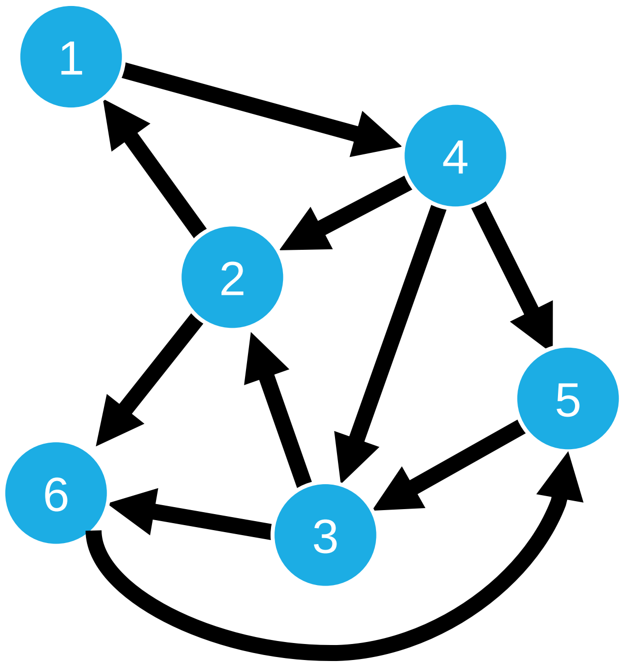
Why must the strongly connected components of a graph be disjoint?

Algorithm

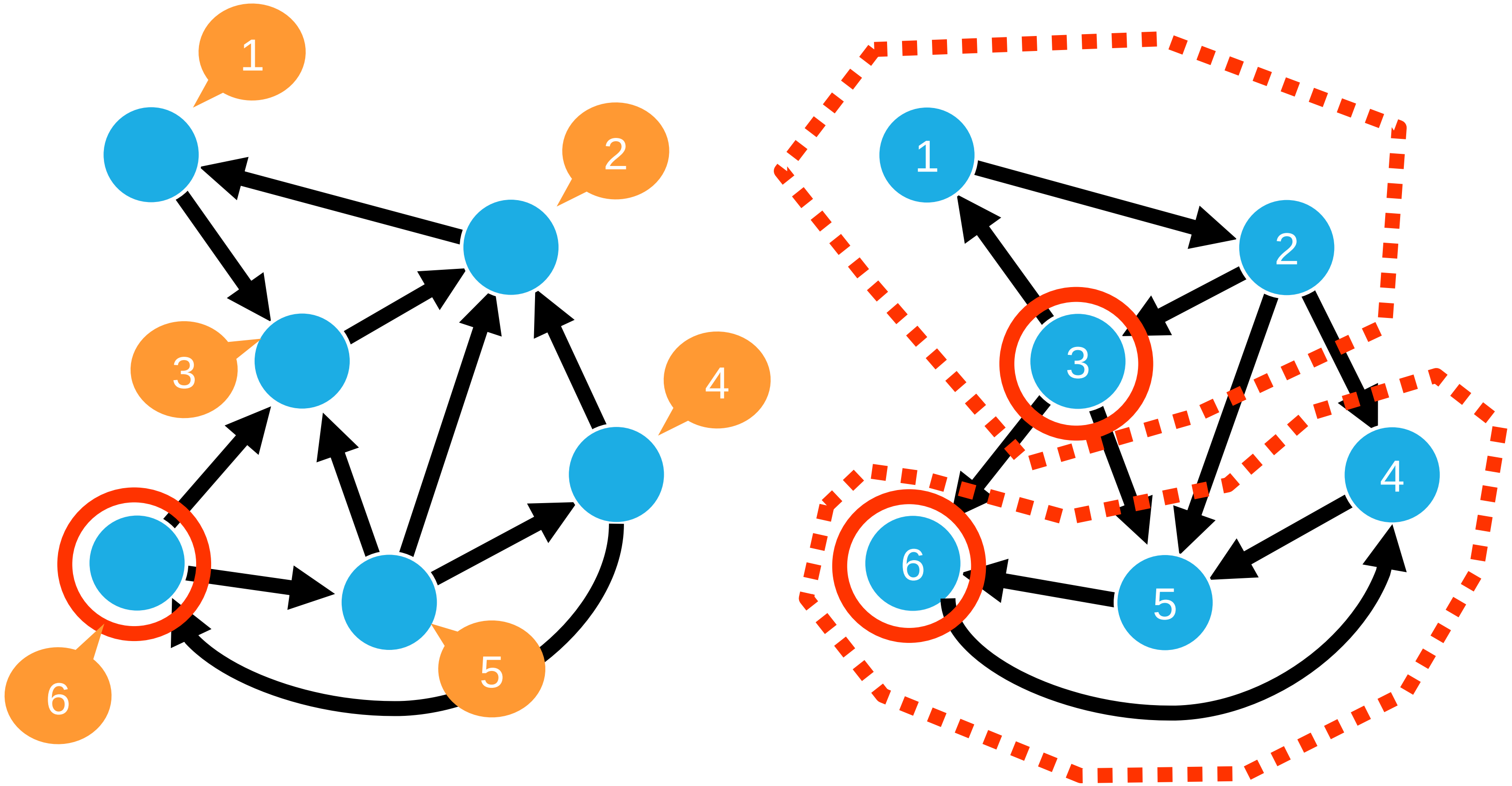
- Step 1: Run DFS on G to obtain the finish time $v.f$ for $v \in V$.
- Step 2: Run DFS on the transpose of G where the vertices V are processed in the decreasing order of their finish time.
- Step 3: output the vertex partition by the second DFS

Transpose of A Graph

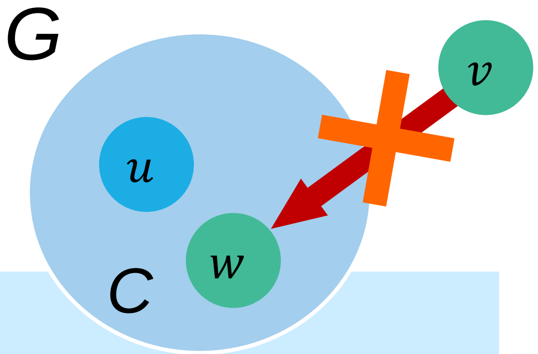




Example Illustration



Algorithm Correctness



<u>Lemma</u>

Let C be the strongly connected component of G (and G^T) that contains the node u with the largest finish time $u.f$. Then C cannot have any incoming edge from any node of G not in C.

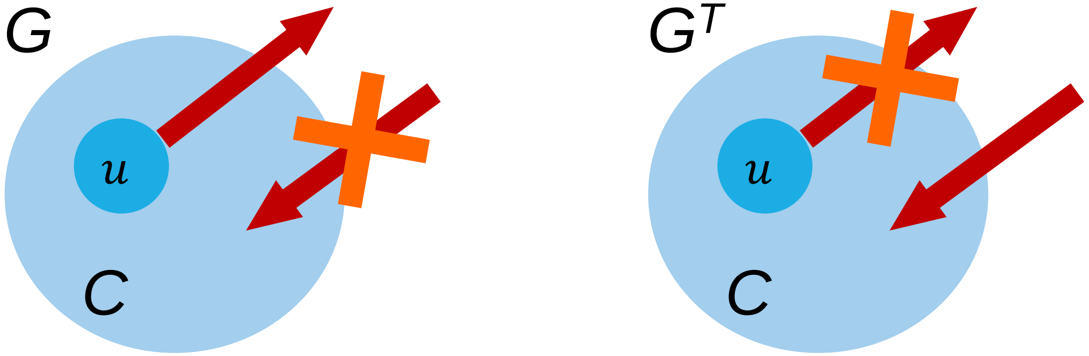
- Proof by contradiction
  - Assume that (v, w) is an incoming edge to C.
  - Since C is a strongly connected component of G, there cannot be any path from any node of C to v in G.
  - Therefore, the finish time of v has to be larger than any node in C, including u. → $v.f > u.f$, contradiction

Algorithm Correctness

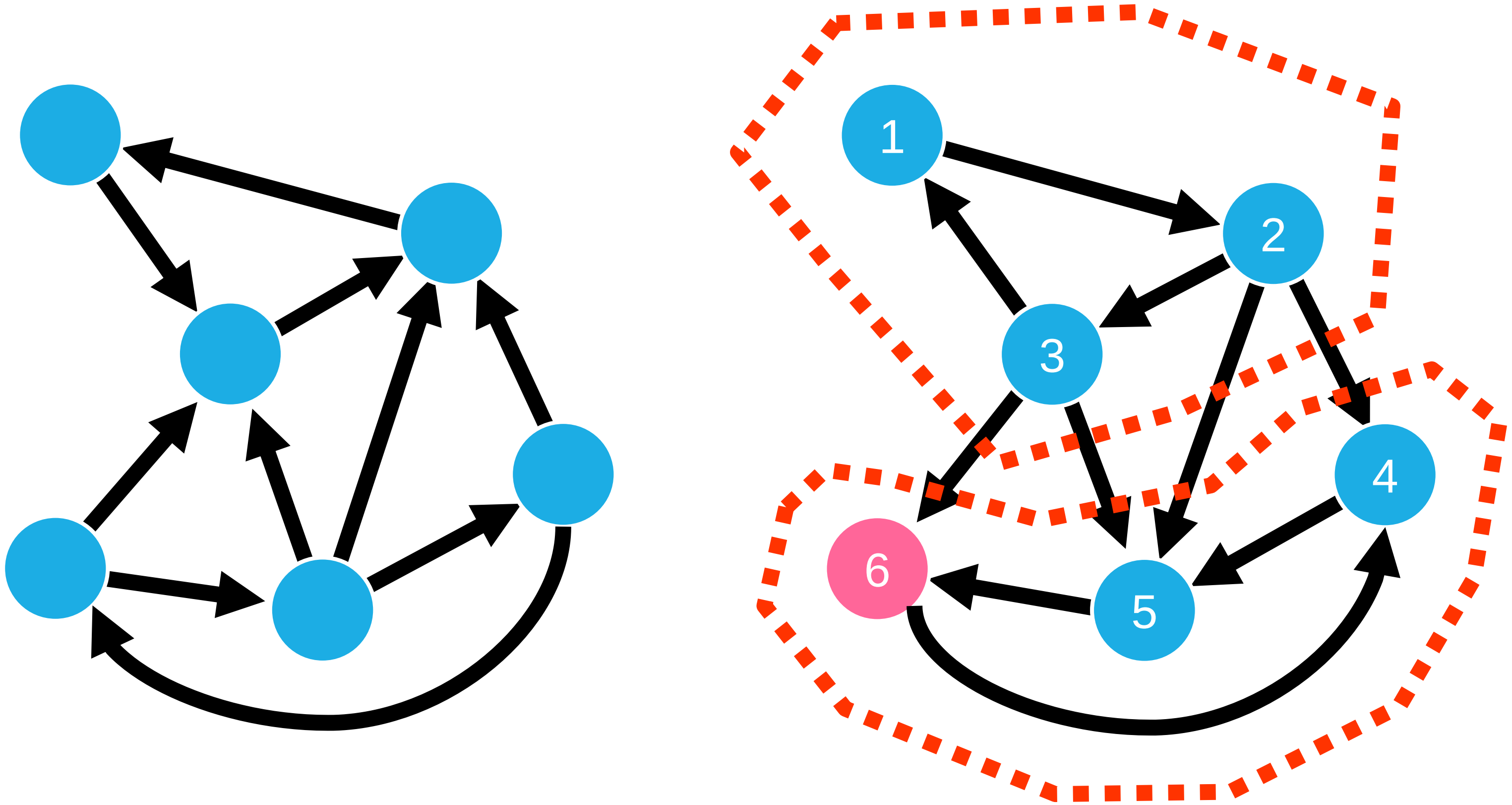
Theorem

By continuing the process from the vertex u^* whose finish time $u^*.f$ is the largest excluding those in C, the algorithm returns the strongly connected components.

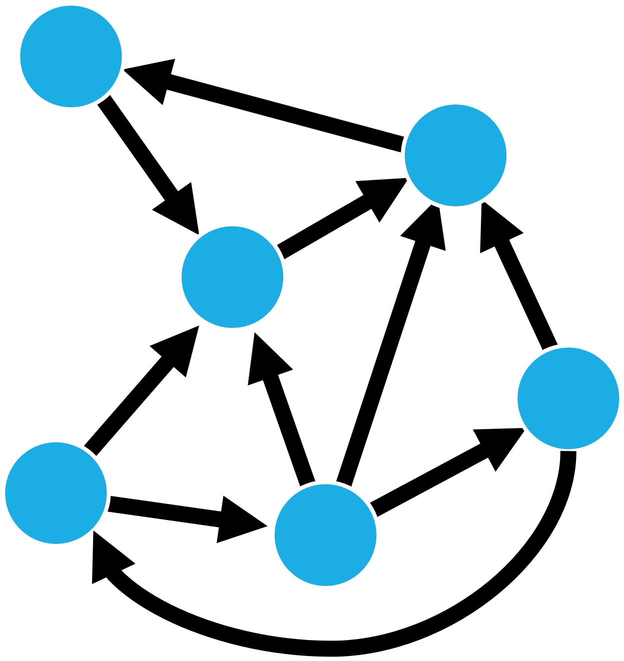
- Practice to prove using induction

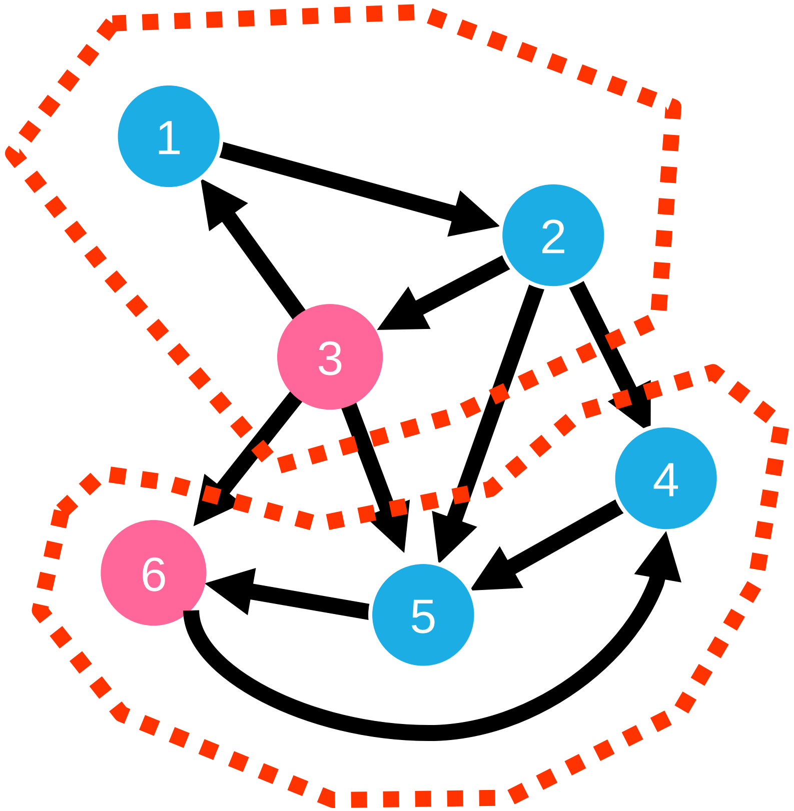


Example



Example



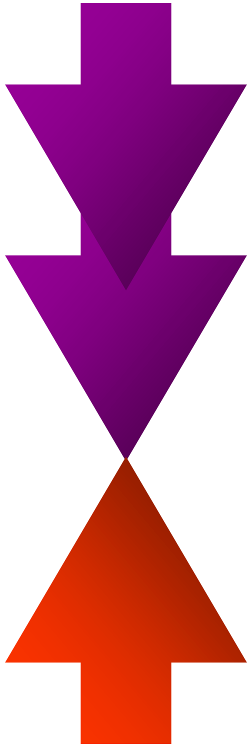


Time Complexity

- Step 1: Run DFS on G to obtain the finish time $v.f$ for $v \in V$.
- Step 2: Run DFS on the transpose of G where the vertices V are processed in the decreasing order of their finish time.
- Step 3: output the vertex partition by the second DFS

Time Complexity: $\Theta(n+m)$

Problem Complexity



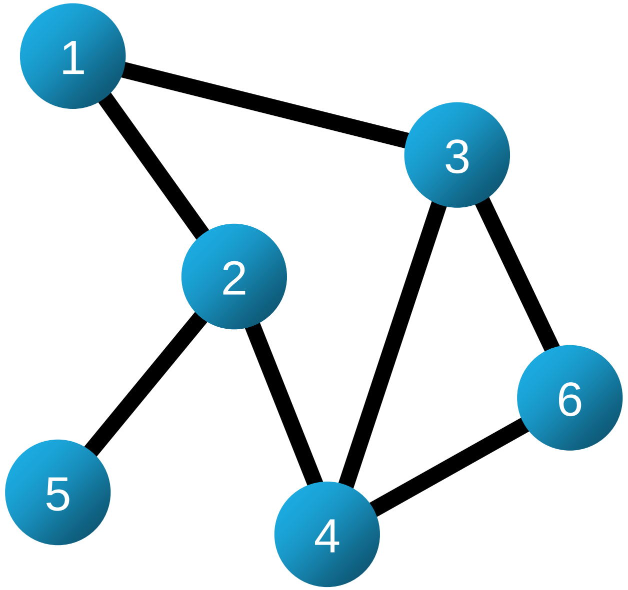
$$\text{Upper bound} = O(m+n)$$

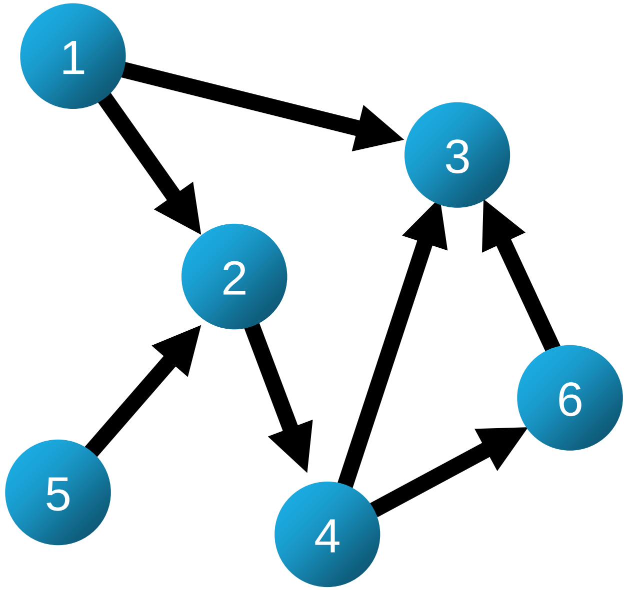
$$\text{Lower bound} = \Omega(m+n)$$

Topological Sort

Textbook Chapter 22.4 – Topological sort

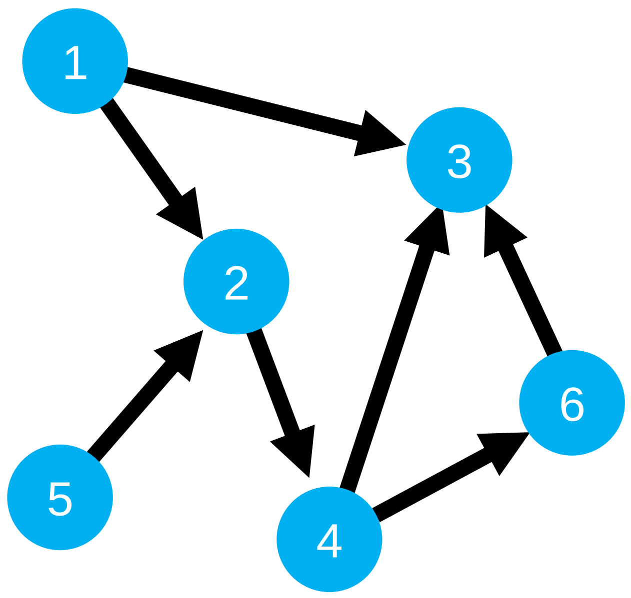
Directed Graph





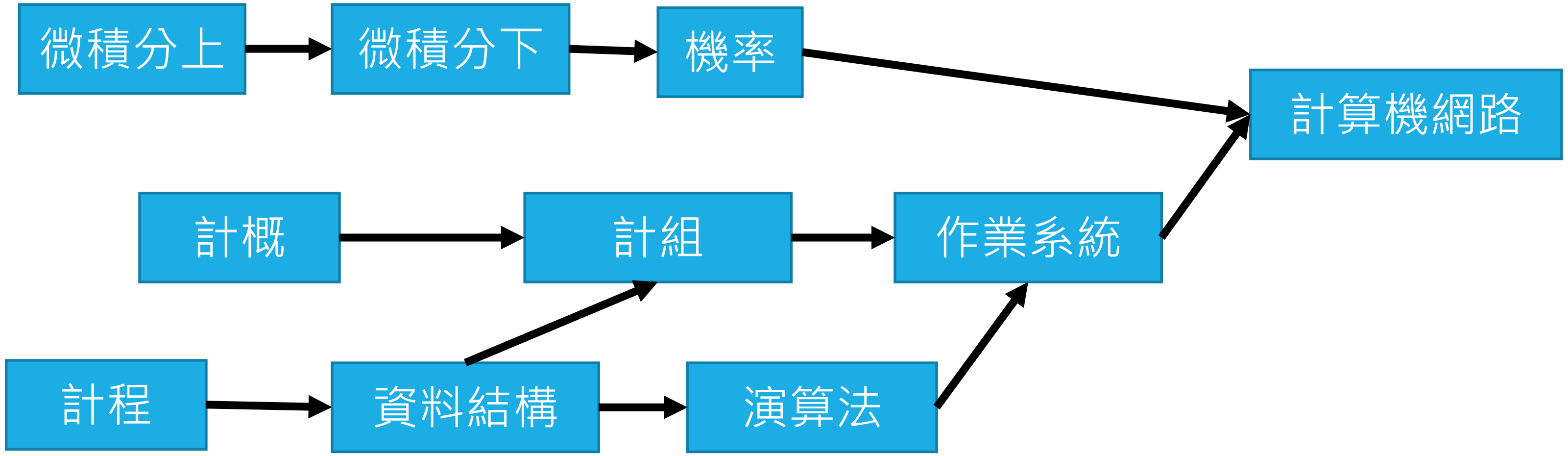
Directed Acyclic Graph (DAG)

- Definition
  - a directed graph without any directed cycle



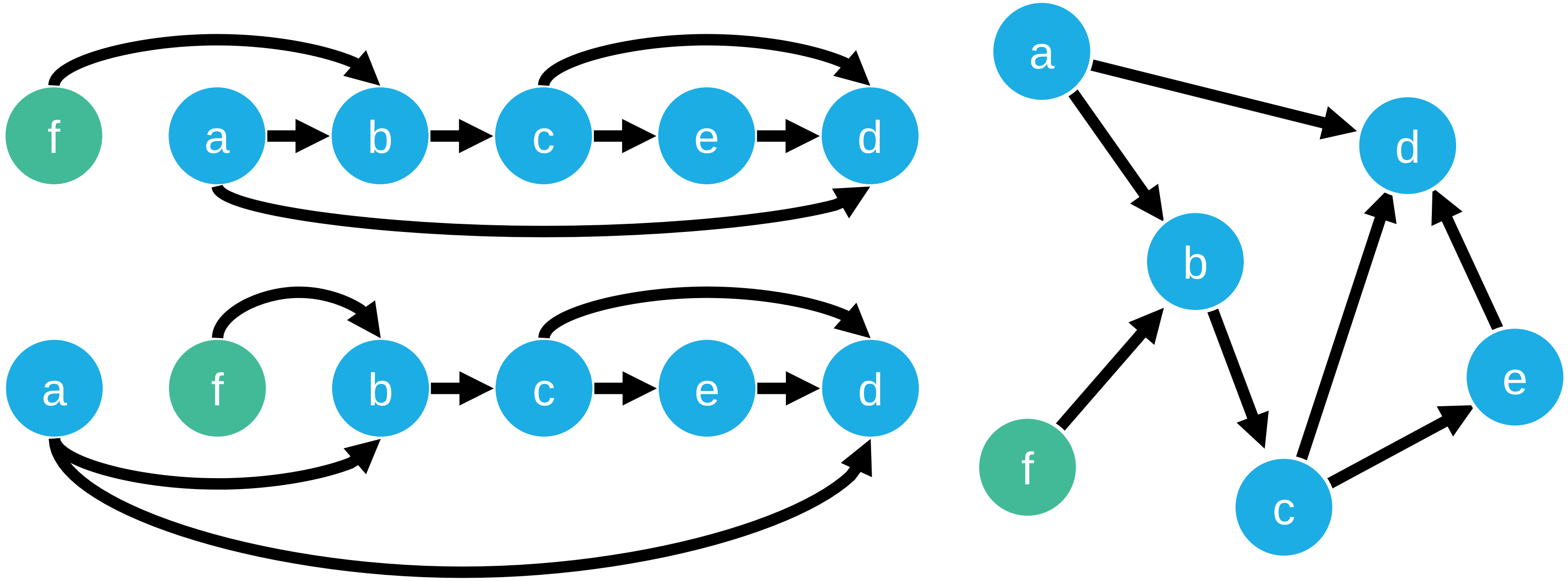
Topological Sort Problem

- Taking courses should follow the specific order
- How to find a course taking order?



Topological Sort Problem

- Input: a directed acyclic graph $G = (V, E)$
- Output: a linear order of V s.t. all edges of G going from lower-indexed nodes to higher-indexed nodes (左→右)



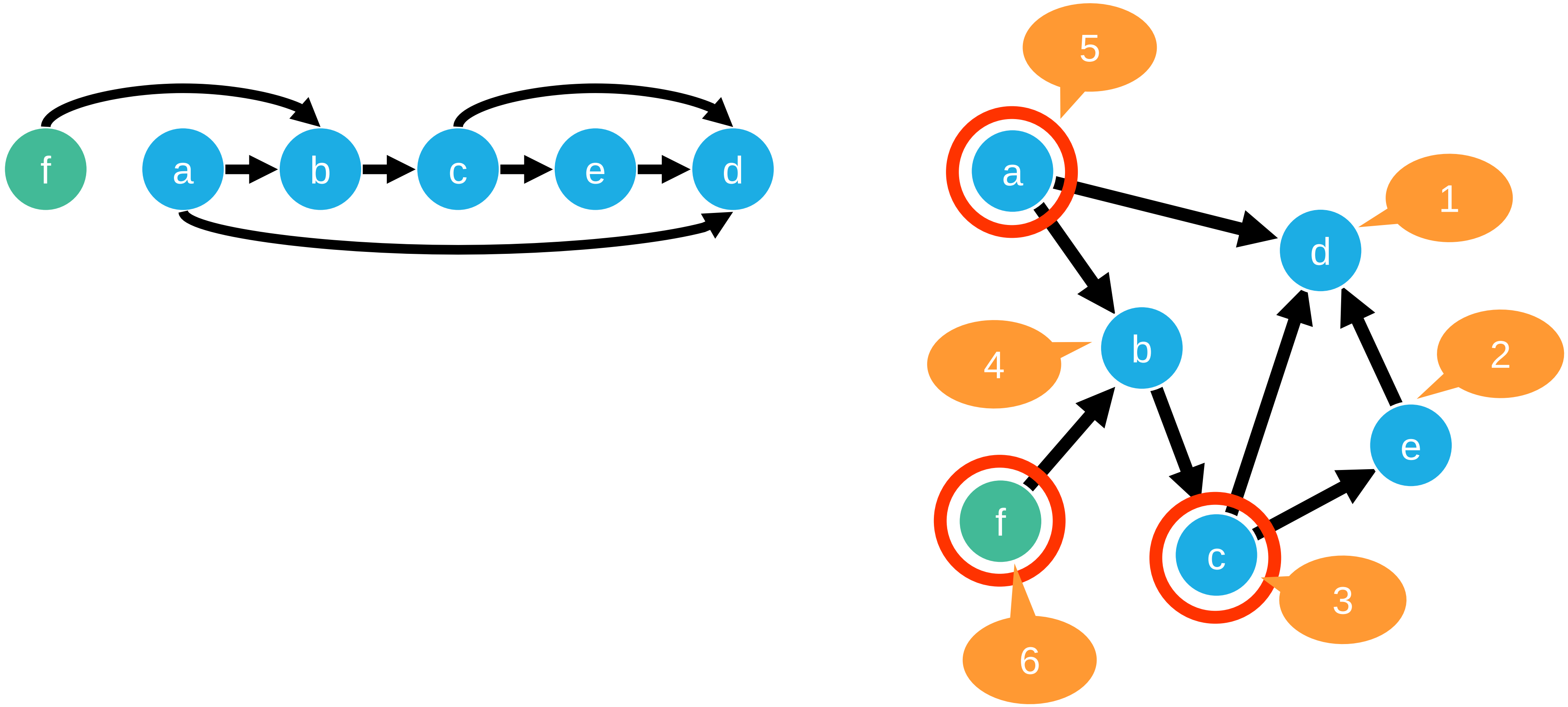
Algorithm

- Run DFS on the input DAG *G*.
- Output the nodes in decreasing order of their finish time.

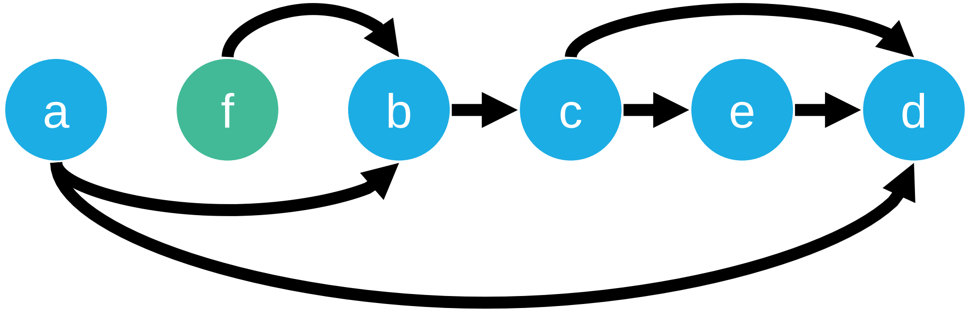
```
DFS(G)
  for each vertex u in G.V
    u.color = WHITE
    u.pi = NIL
  time = 0
  for each vertex u in G.V
    if u.color == WHITE
      DFS-VISIT(G, u)
```

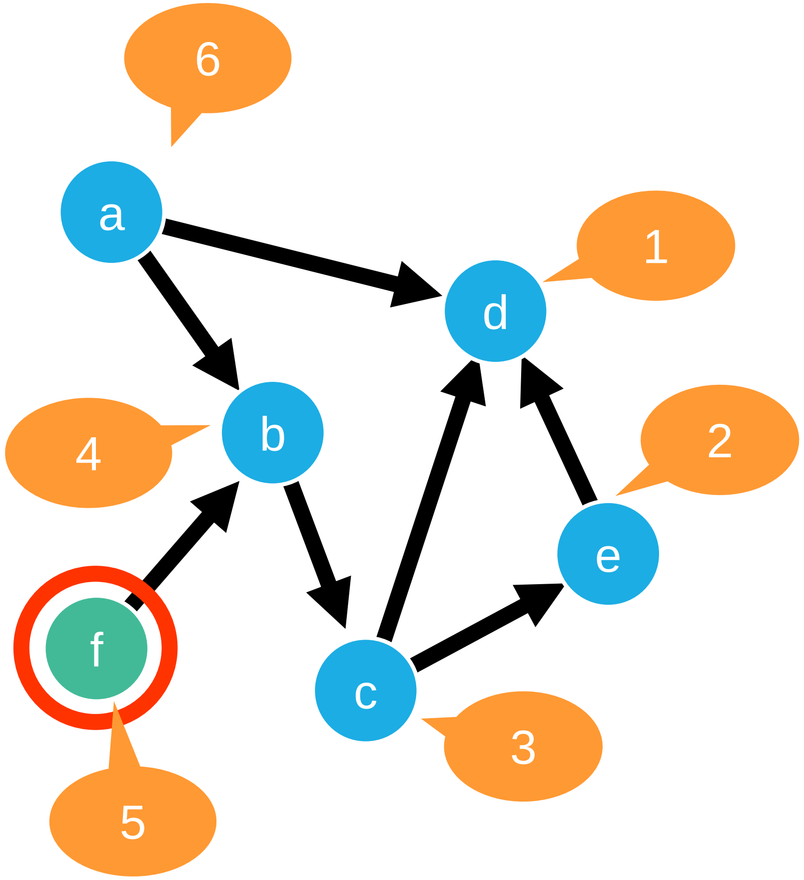
```
DFS-Visit(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY
  for each v in G.Adj[u] (outgoing)
    if v.color == WHITE
      v.pi = u
      DFS-VISIT(G, v)
  u.color = BLACK
  time = time + 1
  u.f = time // finish time
```

Example Illustration



Example Illustration





Time Complexity

- Run DFS on the input DAG *G*. $\Theta(n+m)$
- Output the nodes in decreasing order of their finish time.
  - As each vertex is finished, insert it onto the front of a linked list $\Theta(n)$
  - Return the linked list of vertices

Time Complexity: $\Theta(n+m)$

```
DFS(G)
  for each vertex u in G.V
    u.color = WHITE
    u.pi = NIL
  time = 0
  for each vertex u in G.V
    if u.color == WHITE
      DFS-VISIT(G, u)
```

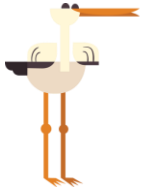
```
DFS-Visit(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY
  for each v in G.Adj[u]
    if v.color == WHITE
      v.pi = u
      DFS-VISIT(G, v)
  u.color = BLACK
  time = time + 1
  u.f = time // finish time
```

Algorithm Correctness

Lemma 22.11

A directed graph is acyclic ⇔ a DFS yields no back edges.

- Proof
  - →: suppose there is a back edge (u, v)
    - v is an ancestor of u in DFS forest
    - There is a path from v to u in G and (u, v) completes the cycle
  - ← : suppose there is a cycle c
    - Let v be the first vertex in c to be discovered and u is a predecessor of v in c
    - Upon discovering v the whole cycle from v to u is WHITE
    - At time $v.d$, the vertices of c form a path of white vertices from v to u
    - By the white-path theorem, vertex u becomes a descendant of v in the DFS forest
    - Therefore, (u, v) is a back edge



White Path Theorem: In a DFS forest of G, v is a descendant of u in the forest ⇔ at the time $u.d$ that the search discovers u, there is a path from u to v in G consisting entirely of WHITE vertices

Algorithm Correctness

Theorem 22.12

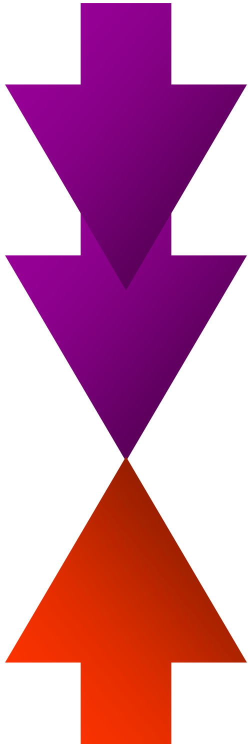
The algorithm produces a topological sort of the input DAG. That is, if (u, v) is a directed edge (from u to v) of G, then $u.f > v.f$.

- Proof
  - When (u, v) is being explored, u is GRAY and there are three cases for v:
    - Case 1 – GRAY
      - (u, v) is a back edge (contradicting Lemma 22.11), so v cannot be GRAY
    - Case 2 – WHITE
      - v becomes descendant of u
      - v will be finished before u

      ➡ $v.f < u.f$
    - Case 3 – BLACK
      - v is already finished

      ➡ $v.f < u.f$

Problem Complexity



$$\text{Upper bound} = O(m+n)$$

$$\text{Lower bound} = \Omega(m+n)$$

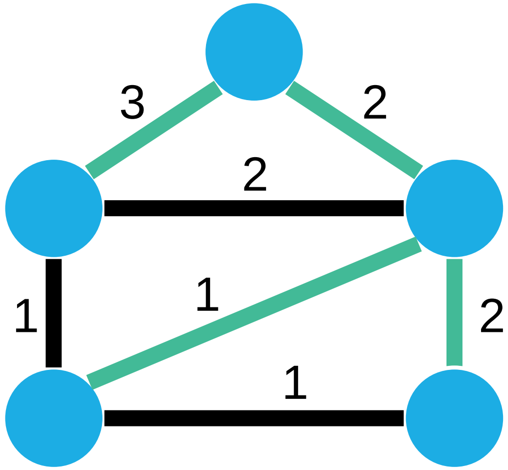
Discussion

- Since cycle detection becomes back edge detection (Lemma 22.11), DFS can be used to test whether a graph is a DAG
- Is there a topological order for cyclic graphs?
- Given a topological order, is there always a DFS traversal that produces such an order?

Minimal Spanning Tree (MST)

Textbook Chapter 23 – Minimal Spanning Trees

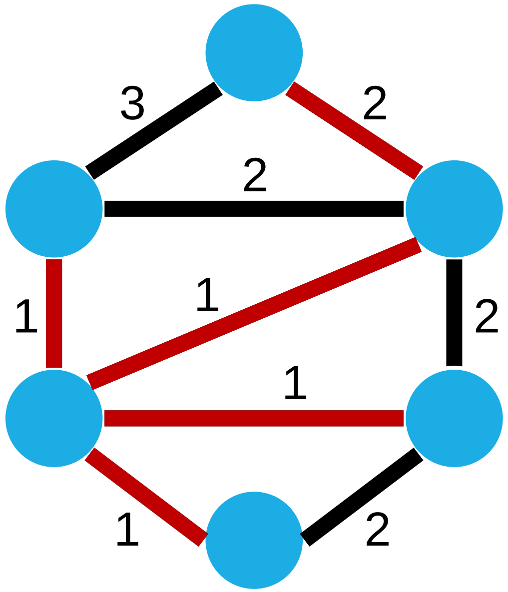
Spanning Tree



- Definition
  - a subgraph that is a tree and connects all vertices
    - Exactly $n - 1$ edges
    - Acyclic
  - There can be many spanning trees of a graph
- BFS and DFS also generate spanning trees
  - BFS tree is typically “short and bushy”
  - DFS tree is typically “long and stringy”

Minimal Spanning Tree Problem

- Input: a connected n-node m-edge graph G with edge weights w
- Output: a spanning tree T of G with minimum $w(T)$



WLOG: we may assume that all edge weights are distinct

Minimal Spanning Tree Problem

- Q: What if the graph is unweighted?

  Trivial

- Q: What if the graph contains edges with negative weights?

  Add a large constant to every edge; a MST remains the same

Uniqueness of MST

Theorem: MST is unique if all edge weights are distinct

- Proof by contradiction
  - Suppose there are two MSTs A and B
  - Let e be the least-weight edge in $A \cup B$ and e is not in both
  - WLOG, assume e is in A
  - Add e to B; $\{e\} \cup B$ contains a cycle C
  - B includes at least one edge e' that is not in A but on C
  - Replacing e' with e yields a MST with less cost

If edge weights are not all distinct, then the (multi-)set of weights in MST is unique

Borůvka’s Algorithm

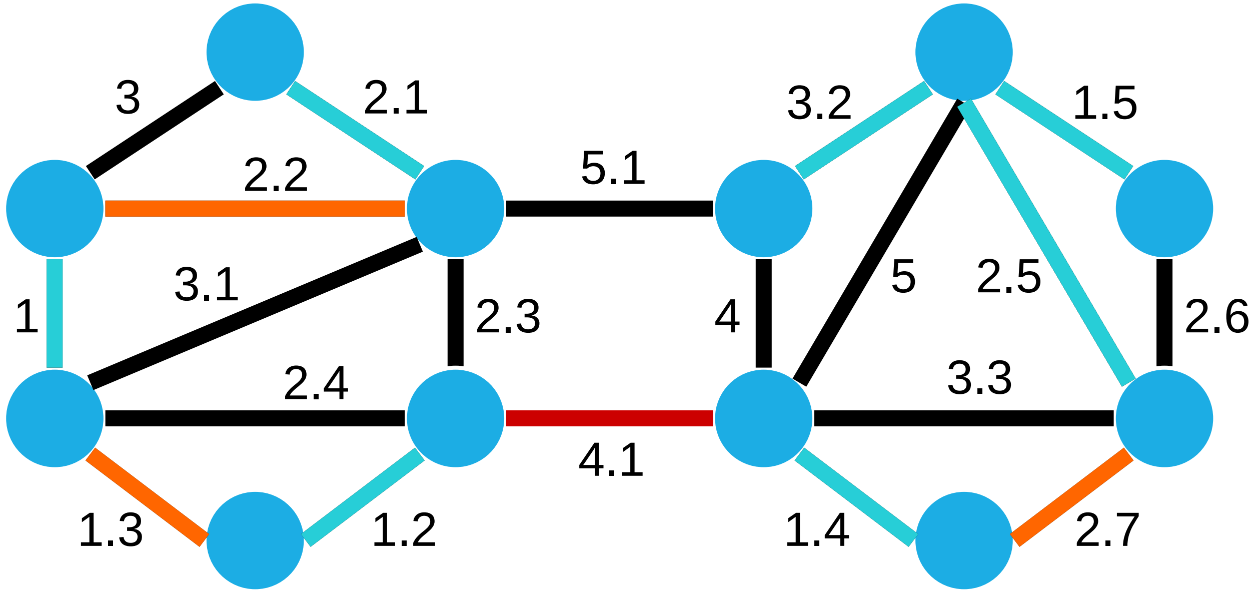
Inventor of MST

- Otakar Borůvka
    - Czech scientist
    - Introduced the problem
    - Gave an $O(m \log n)$ time algorithm
        - The original paper was written in Czech in 1926
        - The purpose was to efficiently provide electric coverage of Bohemia

Borůvka's Algorithm

- Repeat the below procedure until the resulting graph becomes a single node
  - For each node u, mark its lightest incident edge
  - From the marked edges form a forest F, add the edges of F into the set of edges to be reported
  - Contract each maximal subtree of F into a single node

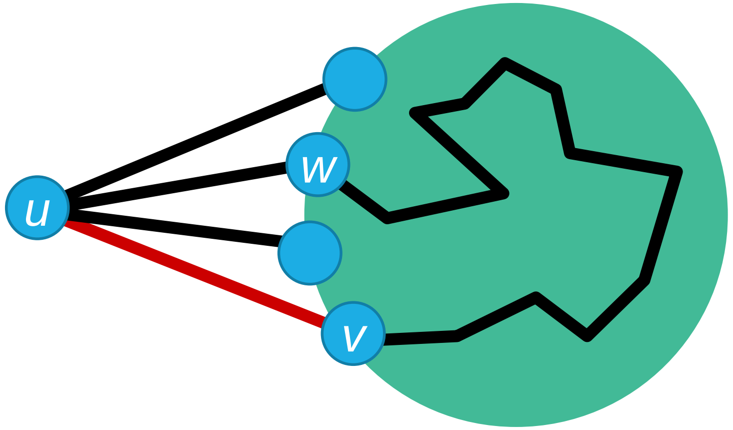
Borůvka’s Algorithm Illustration



Algorithm Correctness

Claim: If (u, v) is the lightest edge incident to u in G, (u, v) must belong to any MST of G

- Proof via contradiction
  - An MST T of G that does not contain (u, v)
  - A cycle $C = T \cup (u, v)$ contains an edge (u, w) in C that has larger weight than (u, v)
  - $T' = T \cup (u, v) \backslash (u, w)$ must be a spanning tree of G lighter than T



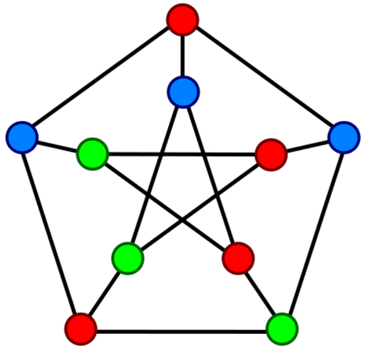
Time Complexity

- The recurrence relation

$$T(m,n) \leq T(m,n/2) + O(m)$$

  - We check all edges in each phase ➡ $O(m)$
  - After each contraction phase, the number of nodes is reduced by at least one half
- Time complexity: $O(m \log n)$

Cycle Property

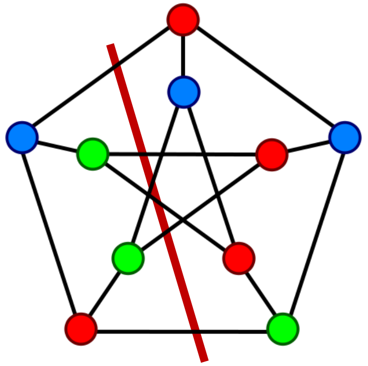


Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.

- For simplicity, assume all edge weights are distinct

- Proof by contradiction
  - Suppose e is in the MST
  - Removing e disconnects the MST into two components T1 and T2
  - There exists another edge e' in C that can reconnect T1 and T2
  - Since $w(e') < w(e)$, the new tree has a lower weight
  - Contradiction!

Cut Property



Let C be a cut in the graph, and let e be the edge with the minimum cost in C. Then the MST contains e.

- Cut = a partition of the vertices
- For simplicity, assume all edge weights are distinct

- Proof by contradiction
  - Suppose e is not in the current MST
  - Adding e creates a cycle in the MST
  - There exists another edge e' in C that can break the cycle
  - Since $w(e') > w(e)$, the new tree has a lower weight
  - Contradiction!

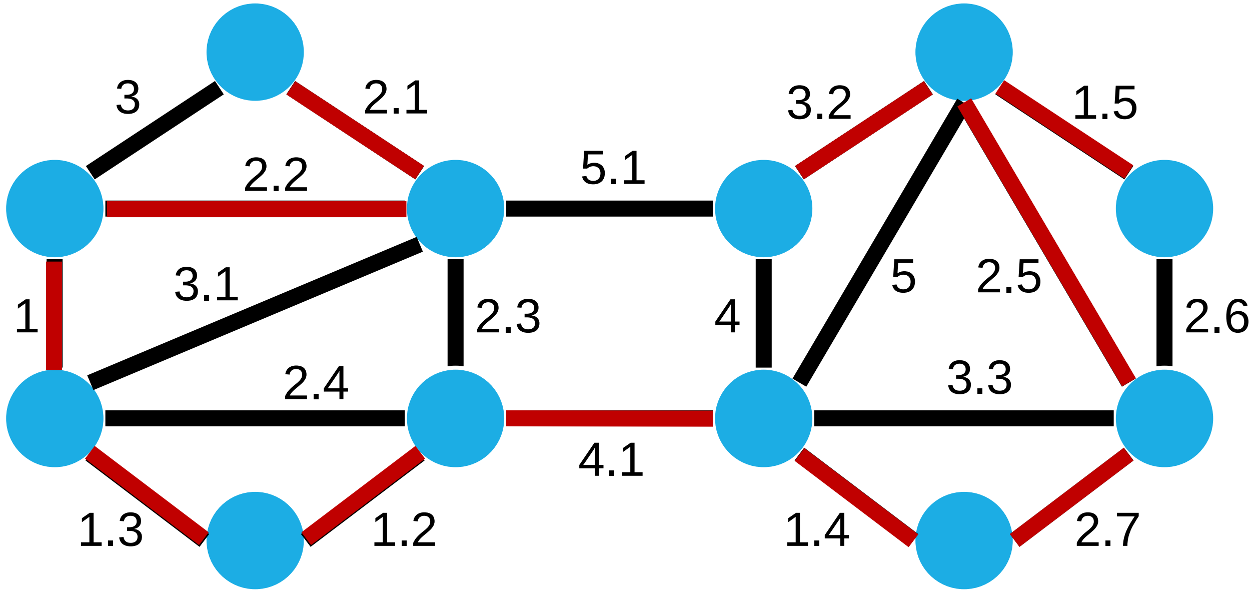
Kruskal’s Algorithm

Textbook Chapter 23.2 – The algorithms of Kruskal and Prim

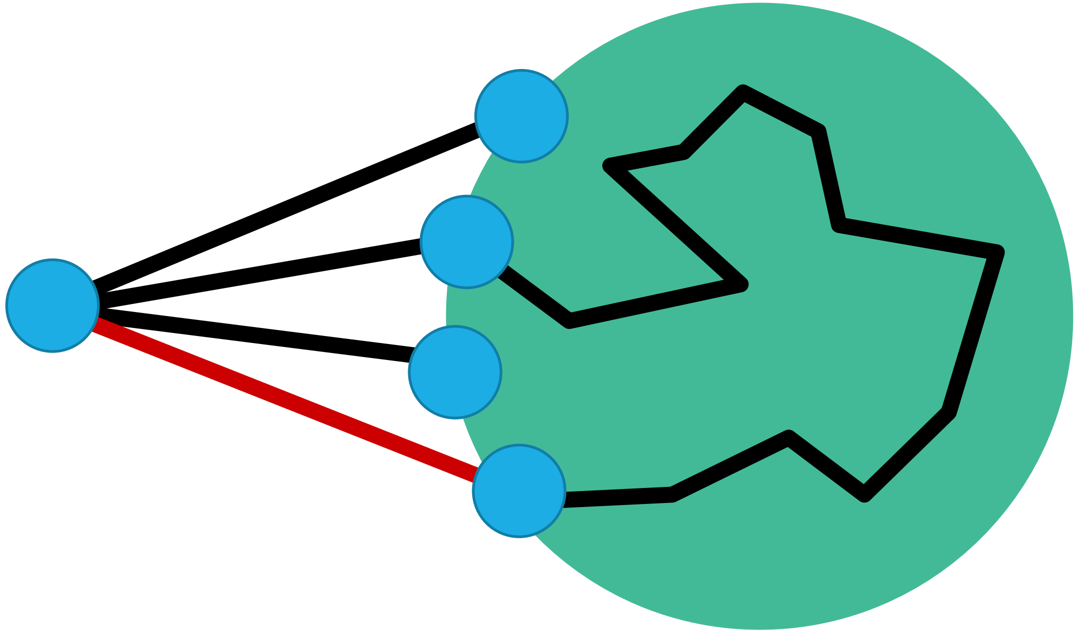
Kruskal’s Algorithm

- For each node u
  - Make-set(u): create a set consisting of u
- For each edge (u, v), taken in non-decreasing order by weights
  - if Find-set(u) ≠Find-set(v) (i.e., u and v are not in the same set) then
    - Output edge (u, v)
    - Union(u, v): union the sets containing u and v into a single set

Kruskal’s Algorithm Illustration



Kruskal’s Algorithm Correctness



The lightest edge incident to a vertex must be in the MST

Kruskal’s Algorithm Correctness

- Consider whether adding e creates a cycle:
  - If adding e to T creates a cycle C
    - Then e is the max weight edge in C
    - The cycle property ensures that e is not in the MST
  - If adding $e = (u, v)$ to T does not create a cycle
    - Before adding e, the current MST can be divided into two trees T1 and T2 such that u in T1 and V in T2
    - e is the minimum-cost edge on the cut of T1 and T2
    - The cut property ensures that e is in the MST

Kruskal's Time Complexity

```
MST-KRUSKAL(G, w) // w = weights
  A = empty // edge set of MST
  for v in G.V
    MAKE-SET(v)
  sort edges of G.E into non-decreasing order by weight w          $O(m \log m)$
  for (u, v) in G.E, taken in non-decreasing order by weight         $m$ times
    if FIND-SET(u) ≠ FIND-SET(v)
      A = A ∪ {u, v}
      UNION(u, v)
  return A
```

- Disjoint-set data structure with union-by-rank (Textbook Ch. 21)
  - `MAKE-SET`: $O(1)$
  - `FIND-SET`: $O(\log n)$
  - `UNION`: $O(\log n)$
  - The amortized cost of m operations on n elements (Exercise 21.4-4): $O(m \log n)$
- Total complexity: $O(m \log m) = O(m \log n)$

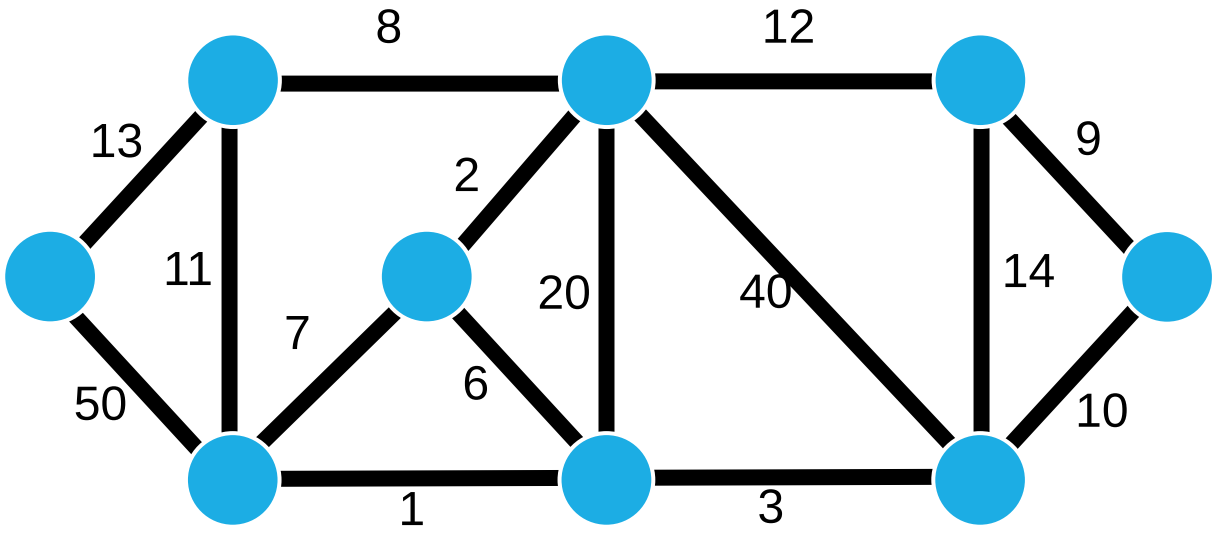
Prim’s Algorithm

Textbook Chapter 23.2 – The algorithms of Kruskal and Prim

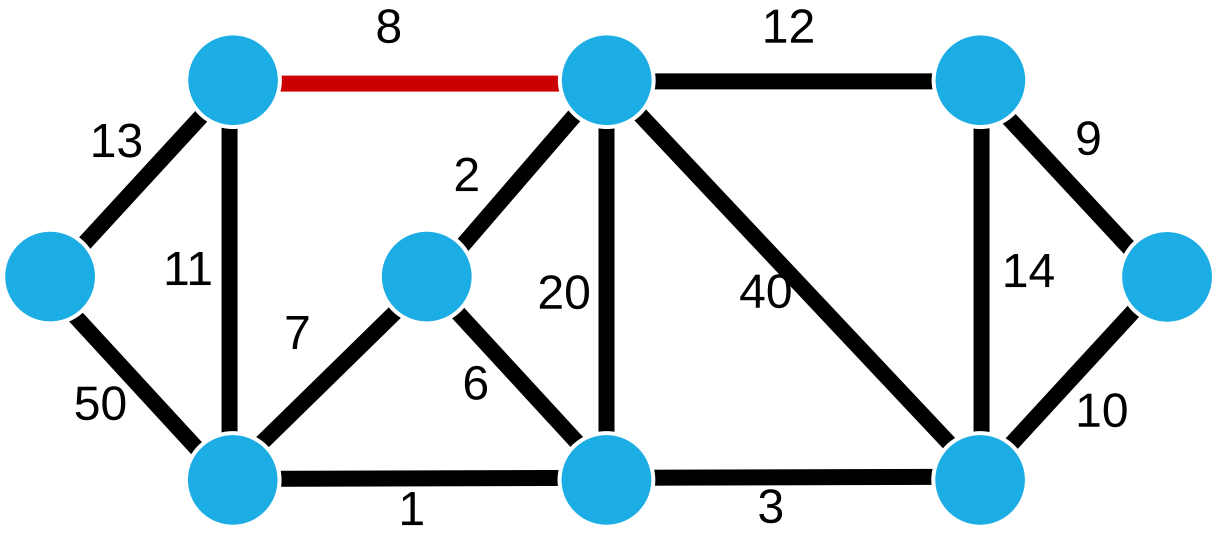
Prim’s Algorithm

- Let T consist of an arbitrary node
- For $i = 1$ to $n - 1$
    - add the least-weighted edge incident to the current subtree T that does not incur a cycle

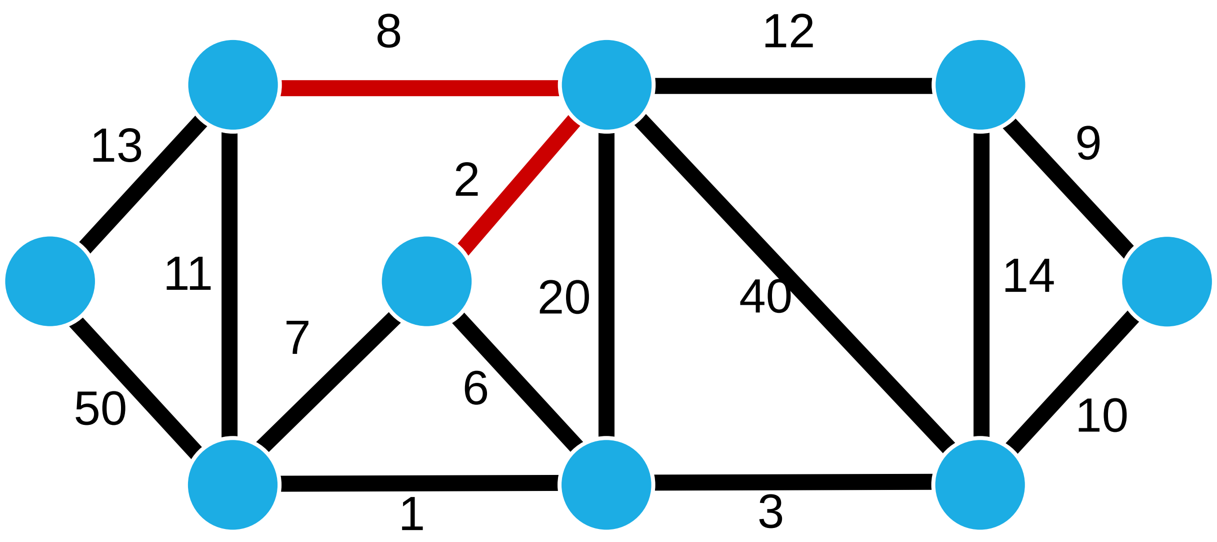
Prim’s Algorithm Illustration



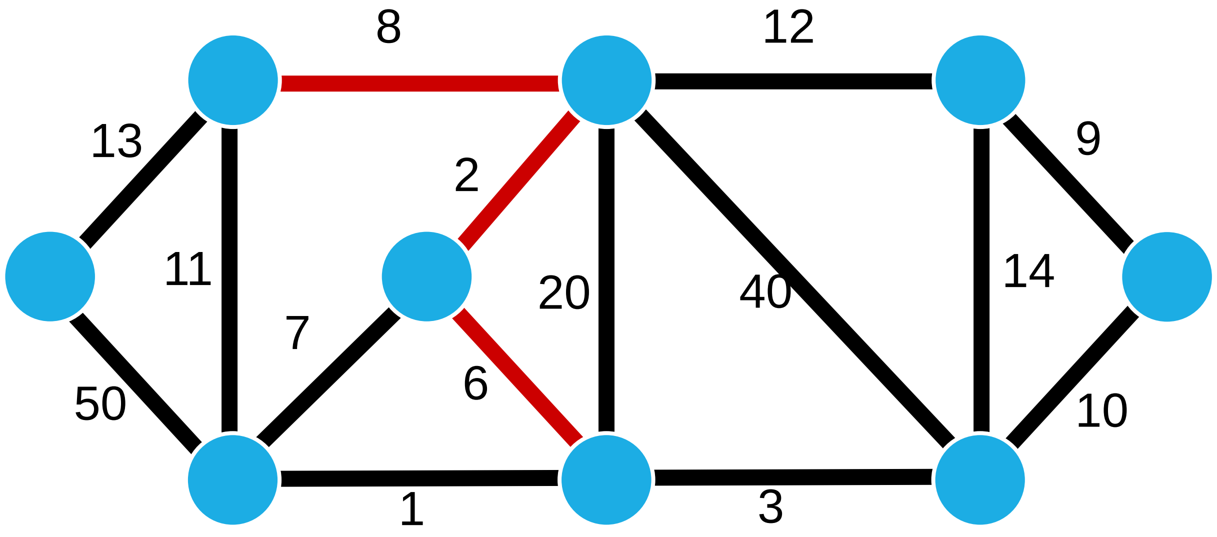
Prim’s Algorithm Illustration



Prim’s Algorithm Illustration



Prim’s Algorithm Illustration



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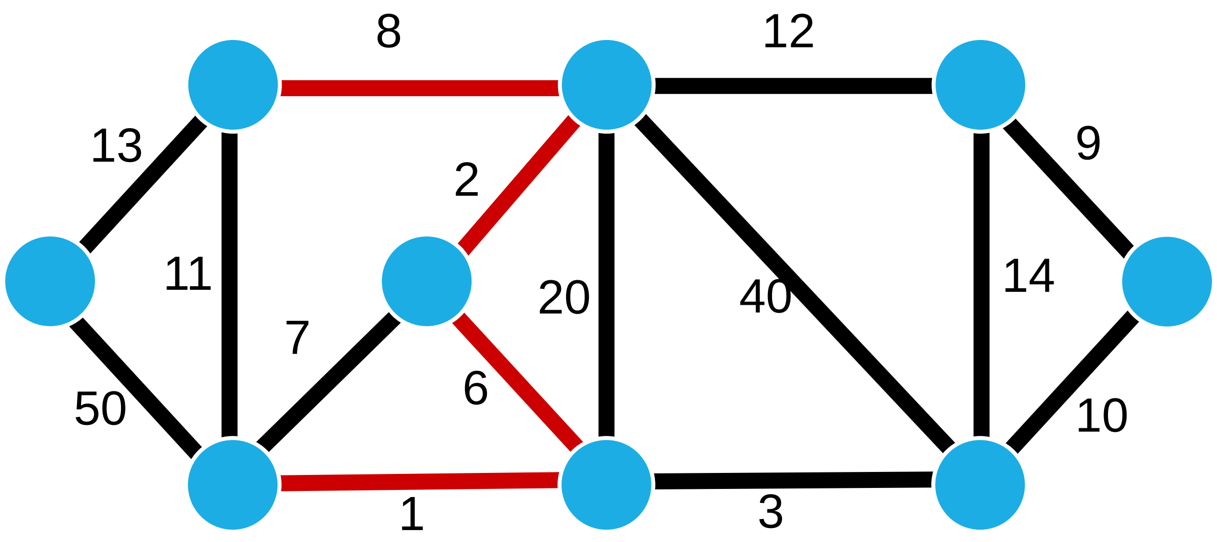

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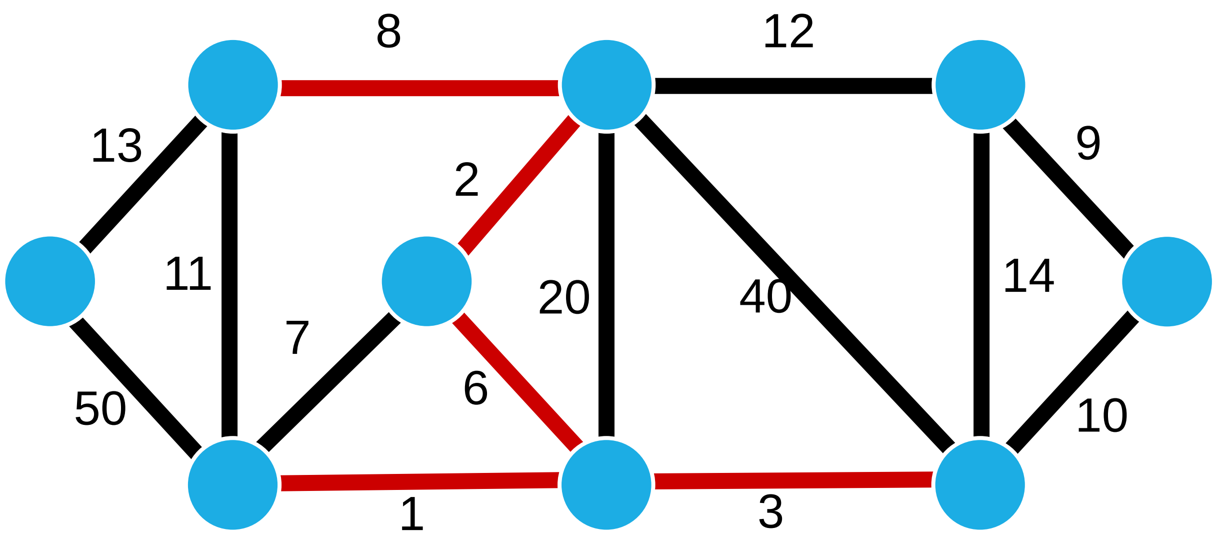
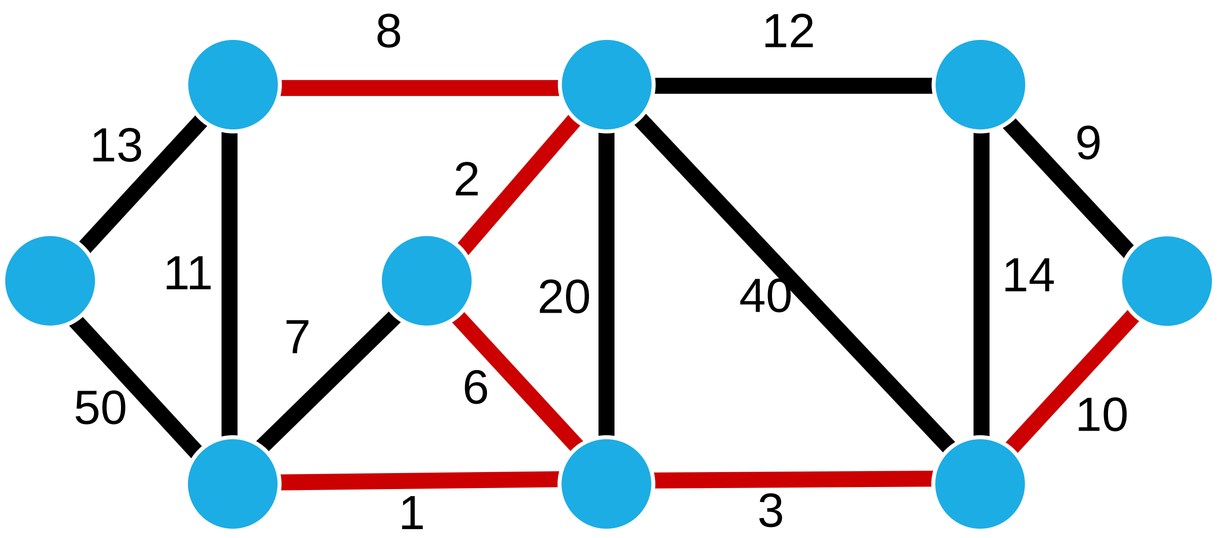
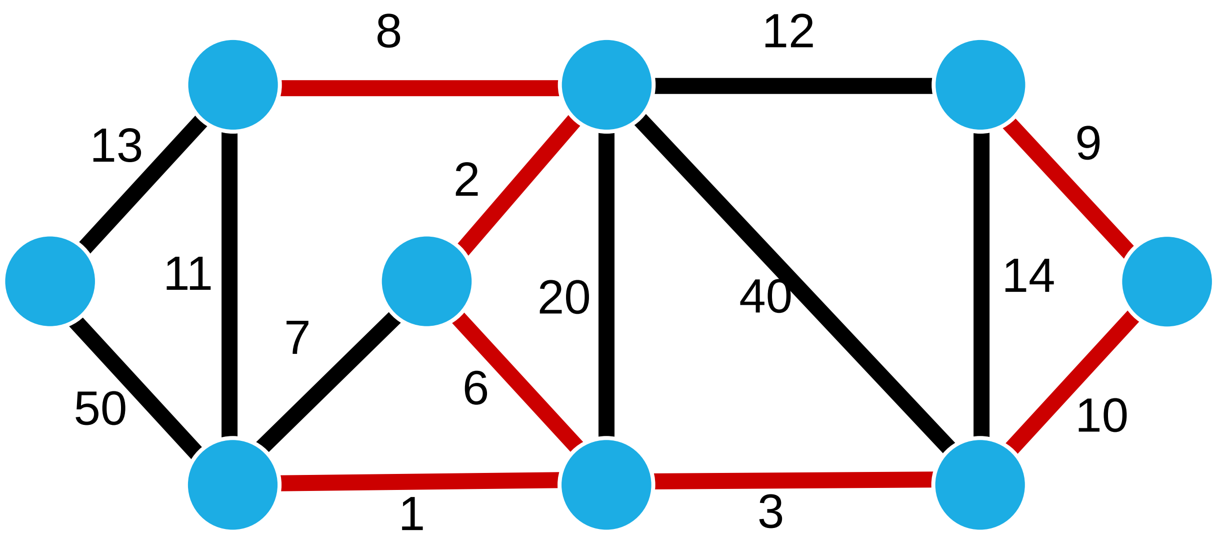


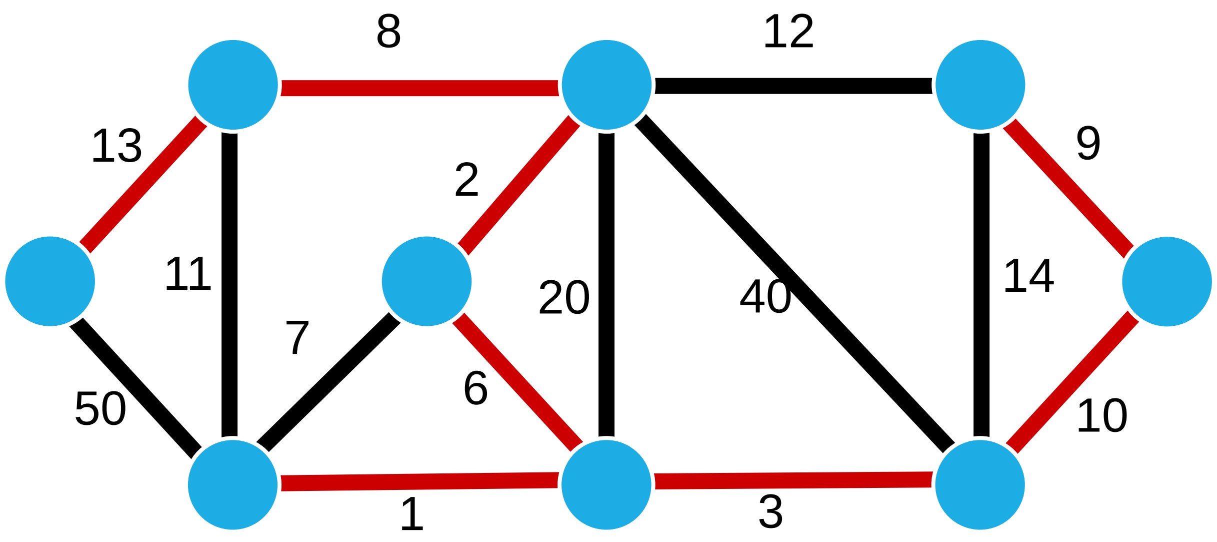
Prim’s Algorithm Illustration



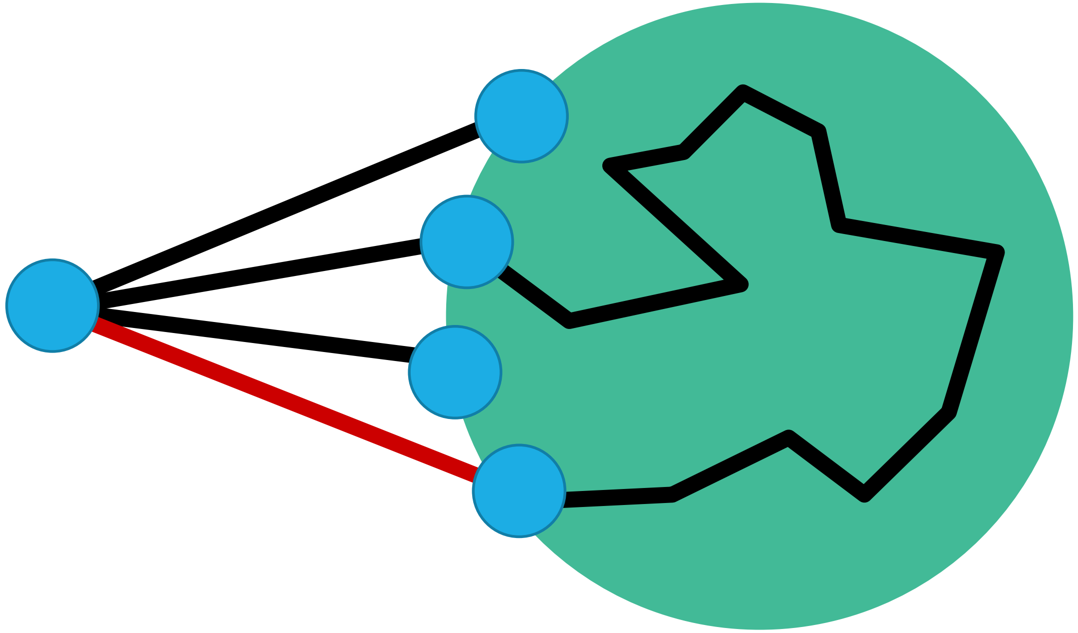
Prim’s Algorithm Illustration



Prim’s Algorithm Illustration



Prim’s Algorithm Correctness



The lightest edge incident to a vertex must be in the MST

Prim’s Time Complexity

```
MST-PRIM(G, w, r) // w = weights, r = root
  for u in G.V                                          
    u.key = ∞                                           
    u.π = NIL                                           $O(n)$
  r.key = 0                                             
  Q = G.V
  while Q ≠ empty                                       $n$ times
    u = EXTRACT-MIN(Q)                                  $O(\log n)$
    for v in G.adj[u]                                   $m$ times
      if v ∈ Q and w(u, v) < v.key
        v.π = u
        v.key = w(u, v) // DECREASE-KEY                 $O(\log n)$
```

- Binary min-heap (Textbook Ch. 6)
  - BUILD-MIN-HEAP: $O(n)$
  - EXTRACT-MIN: $O(\log n)$
  - DECREASE-KEY: $O(\log n)$
- Total complexity: $O(n \log n + m \log n) = O(m \log n)$

Prim’s Time Complexity

```
MST-PRIM(G, w, r) // w = weights, r = root
  for u in G.V
    u.key = ∞
    u.π = NIL                                        } $O(n)$
  r.key = 0
  Q = G.V
  while Q ≠ empty                                    $n$ times
    u = EXTRACT-MIN(Q)                               $O(\log n)$
    for v in G.adj[u]                                $m$ times
      if v ∈ Q and w(u, v) < v.key
        v.π = u
        v.key = w(u, v) // DECREASE-KEY              $O(1)$
```

- Fibonacci heap (Textbook Ch. 19)
  - BUILD-MIN-HEAP: $O(n)$
  - EXTRACT-MIN: $O(\log n)$ (amortized)
  - DECREASE-KEY: $O(1)$ (amortized)
- Total complexity: $O(m + n \log n)$

Concluding Remarks

- Minimal Spanning Trees (MST)
  - Boruvka’s Algorithm: $O(m \log n)$
  - Kruskal’s Algorithm: $O(m \log n)$
  - Prim’s Algorithm: $O(m \log n)$ with binary min-heap
  - Prim’s Algorithm: $O(m + n \log n)$ with Fabonacci heap

Question?

Important announcement will be sent to
@ntu.edu.tw mailbox & post to the course website

Course Website: http://ada.miulab.tw
Email: ada-ta@csie.ntu.edu.tw