Algorithm Design and Analysis
Midterm Review

http://ada.miulab.tw

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Announcement

- Homework assignment
  - HW2 due on 11/12 1pm
  - HW2作業箱已擺在R217
  - HW2作業解答會在死線當天晚上公布
- Midterm announcement
  - Next week!!!

Midterm!!!

- **Date: 11/14 (Thursday)**
- **Time: 14:20-17:20** (3 hours)
- **Location: R102 + R104** (check the seat assignment before entering the room)
- Content
  - Recurrence and Asymptotic Analysis
  - Divide and Conquer
  - Dynamic Programming
  - Greedy
- Based on slides, assignments, and some variations (practice via textbook exercises)
- Format: Yes/No, Multiple-Choice, Short Answer, Prove/Explanation
- Easy: ~65%, Medium: ~25%, Hard: ~10%
- Close book

Tip: exam questions are not all of equal difficulty, move on if you get stuck!

Algorithm Design & Analysis Process

1) Formulate a **problem**
2) Develop an **algorithm**
3) Prove the **correctness**
4) Analyze **running time/space** requirement

Design Step

Analysis Step

Algorithm Analysis

- Analysis Skills
  - Prove by contradiction
  - Induction
  - Asymptotic analysis
  - Problem instance
- Algorithm Complexity
  - In the worst case, what is the growth of function <u>an algorithm</u> takes
- Problem Complexity
  - In the worst case, what is the growth of the function <u>the optimal algorithm of the problem</u> takes

Algorithm Design Strategy

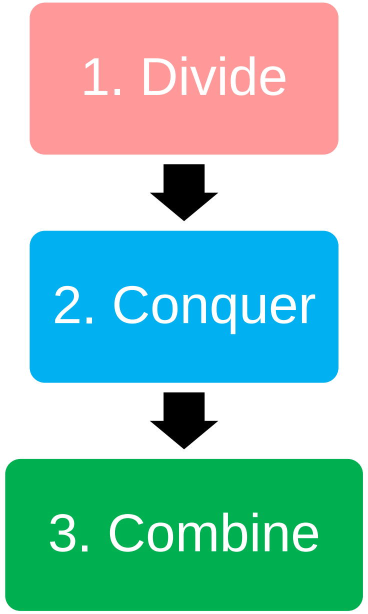
- Do not focus on "specific algorithms"
- But "some strategies" to "design" algorithms

- First Skill: Divide-and-Conquer (各個擊破)
- Second Skill: Dynamic Programming (動態規劃)
- Third Skill: Greedy (貪婪法則)

Divide-and-Conquer

What is Divide-and-Conquer?

- Solve a problem recursively
- Apply three steps at each level of the recursion
  1. **Divide** the problem into a number of subproblems that are smaller instances of the same problem (比較小的同樣問題)
  2. **Conquer** the subproblems by solving them recursively
     If the subproblem sizes are *small enough*
     - then solve the subproblems — base case
     - else recursively solve itself — recursive case
  3. **Combine** the solutions to the subproblems into the solution for the original problem



How to Solve Recurrence Relations?

1. **Substitution Method** (取代法)
   - Guess a bound and then prove by induction
2. **Recursion-Tree Method** (遞迴樹法)
   - Expand the recurrence into a tree and sum up the cost
3. **Master Method** (套公式大法/大師法)
   - Apply Master Theorem to a specific form of recurrences

Master Theorem

The proof is in Ch. 4.6

divide a problem of size n into a subproblems, each of size $\frac{n}{b}$ is solved in time $T\left(\frac{n}{b}\right)$ recursively

Let $T(n)$ be a positive function satisfying the following recurrence relation

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ a \cdot T(\frac{n}{b}) + f(n) & \text{if } n > 1, \end{cases}$$

Should follow this format

where $a \geq 1$ and $b > 1$ are constants.

- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and
  - $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

compare $f(n)$ with $n^{\log_b a}$

When to Use D&C?

- Analyze the problem about
  - Whether the problem with small inputs can be solved directly
  - Whether subproblem solutions can be combined into the original solution
  - Whether the overall complexity is better than naïve
- If no, then
  - Try to modify it or add more information
  - Try another way for dividing
  - Do not use D&C

Pseudo-Polynomial Time

- Polynomial: polynomial in the **length of the input** (#bits for the input)
- Pseudo-polynomial: polynomial in the **numeric value**

- The time complexity of 0-1 knapsack problem is $\Theta(nW)$
  - n: number of objects
  - W: knapsack's capacity (non-negative integer)
  - polynomial in the numeric value

  = pseudo-polynomial in input size

  = exponential in the length of the input
- Note: the size of the representation of W is $\log_2 W$

  $= 2^m$ $= m$

Dynamic Programming

What is Dynamic Programming?

- Dynamic programming, like the divide-and-conquer method, solves problems by <u>combining the solutions to subproblems</u>
  - 用空間換取時間
  - 讓走過的留下痕跡
- “Dynamic”: time-varying
- “Programming”: a *tabular* method

Dynamic Programming: planning over time

Algorithm Design Paradigms

- Divide-and-Conquer
  - partition the problem into **independent** or **disjoint** subproblems
  - repeatedly solving the common subsubproblems

  → more work than necessary

- Dynamic Programming
  - partition the problem into **dependent** or **overlapping** subproblems
  - avoid recomputation
    - ✓ Top-down with memoization
    - ✓ Bottom-up method

Dynamic Programming Procedure

- Apply four steps
  1. Characterize the structure of an optimal solution
  2. **Recursively** define the value of an optimal solution
  3. Compute the value of an optimal solution, typically in a **bottom-up** fashion
  4. Construct an optimal solution from computed information

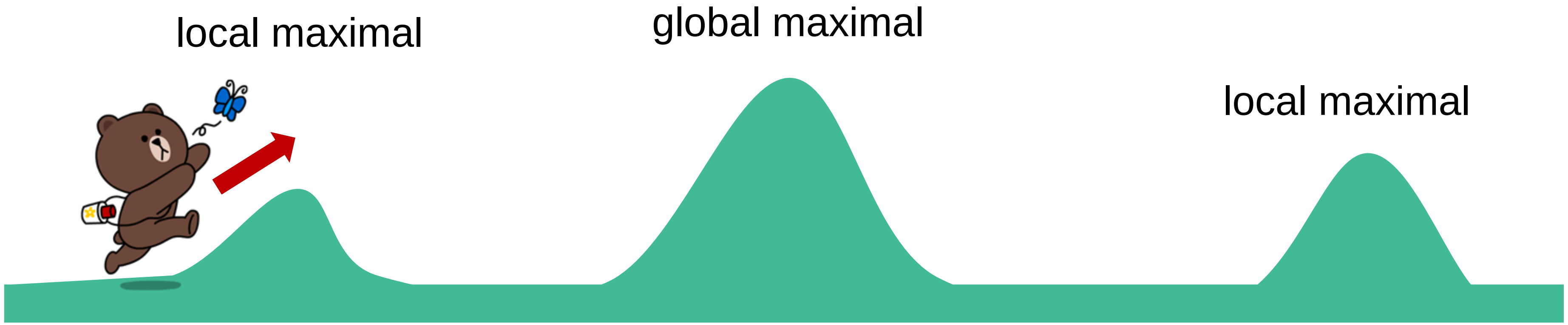
When to Use DP?

- Analyze the problem about
  - Whether subproblem solutions can combine into the original solution
  - When subproblems are overlapping
  - Whether the problem has optimal substructure
  - Common for optimization problem
- Two ways to avoid recomputation
  - Top-down with memoization
  - Bottom-up method
- Complexity analysis
  - Space for tabular filling
  - Size of the subproblem graph

Greedy Algorithms

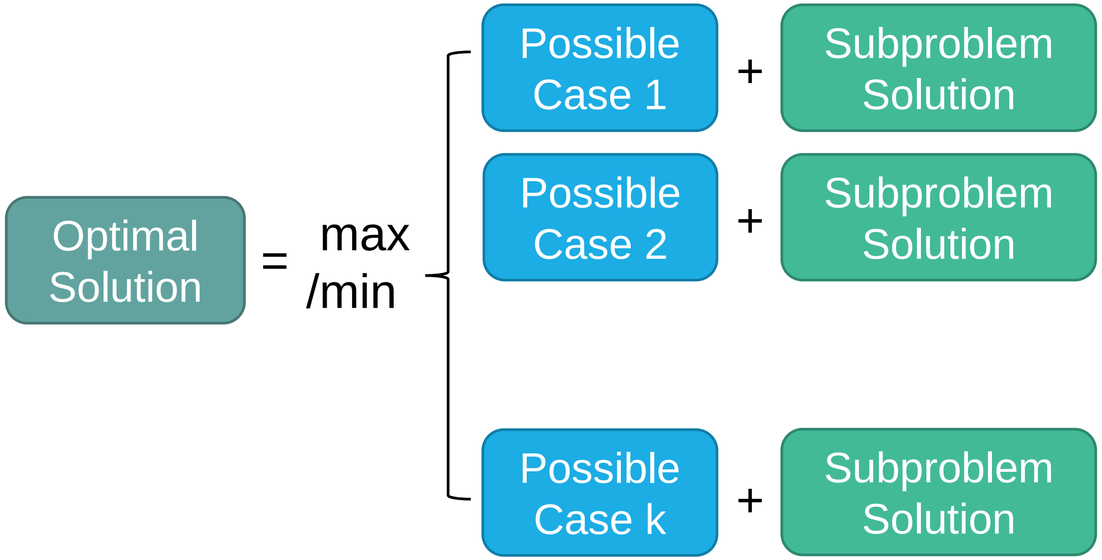
What is Greedy Algorithms?

- always makes the choice that looks best at the moment
- makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution
  - not always yield optimal solution; may end up at local optimal



Algorithm Design Paradigms

- Dynamic Programming
  - has **optimal substructure**
  - make an informed choice after getting optimal solutions to subproblems
  - **dependent** or **overlapping** subproblems



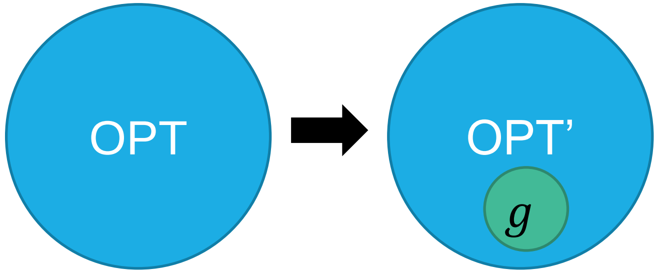
- Greedy Algorithms
  - has **optimal substructure**
  - make a greedy choice before solving the subproblem
  - **no overlapping** subproblems
    - ✓ Each round selects only one subproblem
    - ✓ The subproblem size decreases

Greedy Procedure

1. Cast the optimization problem as one in which we make a choice and remain one subproblem to solve
2. Demonstrate the optimal substructure
   - ✓ Combining an optimal solution to the subproblem via greedy can arrive an optimal solution to the original problem
3. Prove that there is always an optimal solution to the original problem that makes the greedy choice

Proof of Correctness Skills

- Optimal Substructure : an optimal solution to the problem contains within it optimal solutions to subproblems
- Greedy-Choice Property : making locally optimal (greedy) choices leads to a globally optimal solution
  - Show that it exists an optimal solution that "contains" the greedy choice using **exchange argument**
  - For any optimal solution OPT, the greedy choice g has two cases
    - g is in OPT: done
    - g not in OPT: modify OPT into OPT' s.t. OPT' contains g and is at least as good as OPT



✓ If OPT' is better than OPT, the property is proved by contradiction
✓ If OPT' is as good as OPT, then we showed that there exists an optimal solution containing g by construction

When to Use Greedy?

- Analyze the problem about
  - Whether the problem has optimal substructure
  - Whether we can make a greedy choice and remain only one subproblem
  - Common for optimization problem

Exercises

Short Answer Questions

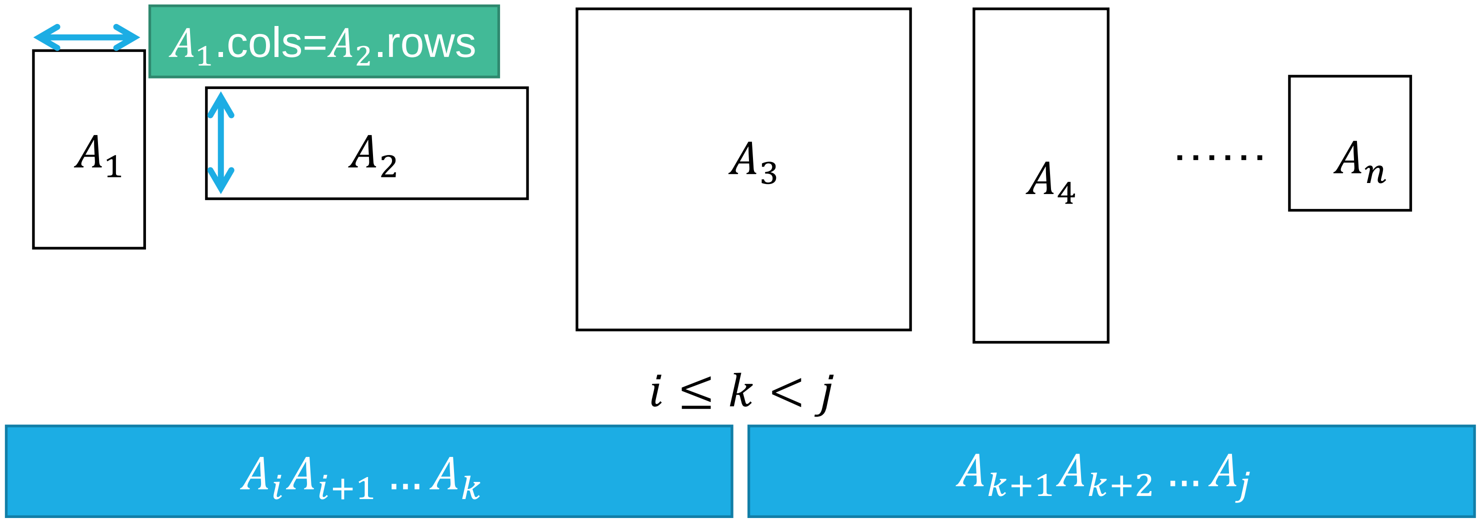
- True or False: To prove the correctness of a greedy algorithm, we must prove that every optimal solution contains our greedy choice.

- Given the following recurrence relation, provide a valid traversal order to fill the DP table or justify why no valid traversal exists.

$$A(i, j) = F(A(i-2, j+1), A(i+1, j-2))$$

Matrix-Chain Multiplication

- Input: a sequence of integers $l_0, l_1, \dots, l_n$
  - l_{i-1} is the number of rows of matrix A_i
  - l_i is the number of columns of matrix A_i
- Output: an order of performing $n-1$ matrix multiplications in the maximum number of operations to obtain the product of $A_1 A_2 \dots A_n$



Q: Does optimal substructure still hold?

Painting

- Put stickers in a single row on each tube to indicate its color.
- There are k types of stickers.
- Tubes with the same color should have the same sticker pattern and should be prefix free.

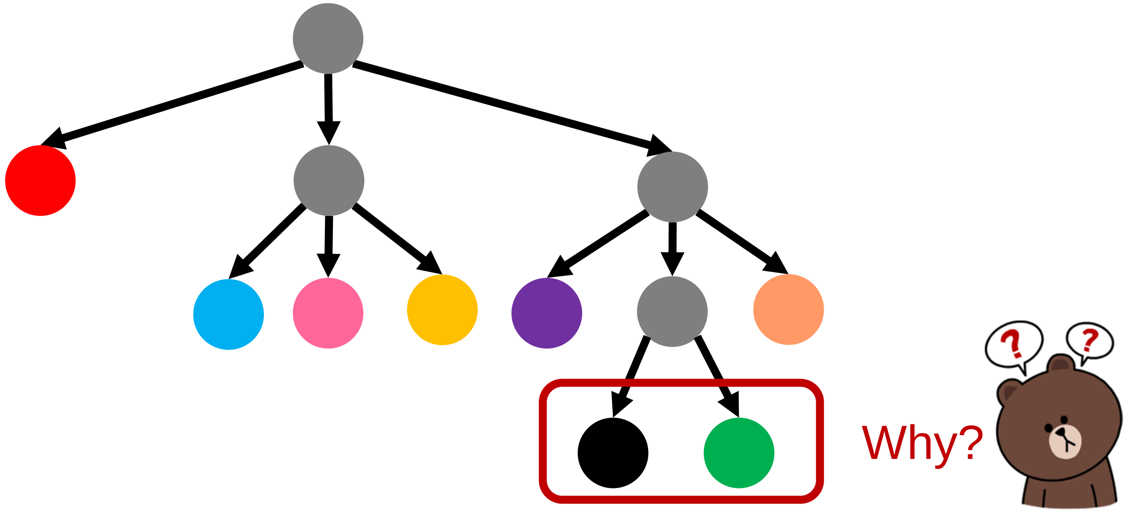
Color	red	pink	orange	yellow	green	blue	purple	black
#Tubes	25	15	12	19	7	12	8	2

- Minimize the total number of stickers put on all tubes
- 3-ary prefix tree (each node can have at most k children).

3-arry Huffman Coding

- The total length is

Color	red	pink	orange	yellow	green	blue	purple	black
#Tubes	25	15	12	19	7	12	8	2



$$25 \cdot 1 + (12 + 15 + 19 + 8 + 12) \cdot 2 + (7 + 2) \cdot 3 = 184$$

T/F Question

- Given a file containing a sequence of 8-bit characters (256 characters), if the maximum character frequency is less than *k* of the minimum character frequency in the file, then a *binary Human code* is always worse than or equal to an 8-bit fixed length code (in terms of the length of the encoded file).
- What is the minimal value of *k*?
  - https://stackoverflow.com/questions/8960698/huffman-coding-prove-on-a-8-bit-sequence

考古題 Practice 1

1. Maximum Subarray of a Circular Infinite Sequence (2015 midterm) Recall that a maximum subarray of A is a contiguous subarray $a_s, \cdots, a_t$ of A such that $\sum_{s \le i \le t} a_i$ is maximized over all s and t, $0 \le s \le t$.

Given a *circular infinite sequence* $A = \langle a_0, a_1, a_2, \cdots \rangle$ in which $a_i = a_j$ if $i = j \mod n$, please answer the following questions.

1. Suppose $\sum_{0 \le i < n} a_i > 0$. What is the length of the maximum subarray of A? Briefly explain your answer.
2. Suppose $\sum_{0 \le i < n} a_i < 0$. Please briefly explain why the length of any maximum subarray is at most n.
3. Please design an algorithm to find a maximum subarray of the circular infinite sequence A in $O(n \log n)$ time. Can you reduce the running time of your algorithm to $O(n)$? Please justify the correctness and running time of your algorithm.

考古題 Practice 2

2. Fair Division of Christmas Gifts (2014 midterm) Christmas is approaching. You're helping Santa Clauses to distribute gifts to children.

For ease of delivery, you are asked to divide n gifts into two groups such that the weight difference of these two groups is minimized. The weight of each gift is a positive integer. Please design an algorithm to find an optimal division minimizing the value difference. The algorithm should find the minimal weight difference as well as the groupings in $O(nS)$ time, where S is the total weight of these n gifts. Briefly justify the correctness of your algorithm.

Hint: This problem can be converted into making one set as close to $S/2$ as possible.

考古題 Practice 3

3. Zombie Apocalypse (2016 midterm) Due to a zombie virus outbreak, some cities have been occupied by zombies and are no longer safe. You and your survivor team need to travel through several cities to get to a far away shelter.

There are n cities forming a line topology. You are at city 1 now and the shelter is at city n. The location of city i is $L[i]$, and $L[i] < L[j]\ \forall 1 \leq i < j \leq n$. $z[i] = 1$ indicates city i has been occupied by zombies; otherwise, $z[i] = 0$ indicates the city is still safe to stop at night.

If you plan to move at most 100km a day, and you need to rest at a safe city at night, please design a greedy algorithm to pick the cities for resting at night so that you can arrive at the shelter as soon as possible. Your algorithm should run in $O(n)$ time. Please show that your algorithm has the greedy choice property.

Question?

Important announcement will be sent to @ntu.edu.tw mailbox & post to the course website

Course Website: http://ada.miulab.tw
Email: ada-ta@csie.ntu.edu.tw