Algorithm Design and Analysis
Divide and Conquer (1)

http://ada.miulab.tw

Yun-Nung (Vivian) Chen

國立臺灣大學
National Taiwan University

Algorithm Design Strategy

- Do not focus on “specific algorithms”
- But “some strategies” to “design” algorithms

- First Skill: Divide-and-Conquer (各個擊破/分治)

Outline

- Recurrence (遞迴)
- Divide-and-Conquer
- D&C #1: Tower of Hanoi (河內塔)
- D&C #2: Merge Sort
- D&C #3: Bitonic Champion
- D&C #4: Maximum Subarray

Divide-and-Conquer 首部曲

- Solving Recurrences
  - Substitution Method
  - Recursion-Tree Method
  - Master Method
- D&C #5: Matrix Multiplication
- D&C #6: Selection Problem
- D&C #7: Closest Pair of Points Problem

Divide-and-Conquer
之神乎奇技

What is Divide-and-Conquer?

- Solve a problem recursively
- Apply three steps at each level of the recursion
  1. **Divide** the problem into a number of subproblems that are smaller instances of the same problem (比較小的同樣問題)
  2. **Conquer** the subproblems by solving them recursively
     If the subproblem sizes are *small enough*
     - then solve the subproblems base case
     - else recursively solve itself recursive case
  3. **Combine** the solutions to the subproblems into the solution for the original problem

Divide-and-Conquer Benefits

- Easy to solve difficult problems
  - Thinking: solve easiest case + combine smaller solutions into the original solution
- Easy to find an efficient algorithm
  - Better time complexity
- Suitable for parallel computing (multi-core systems)
- More efficient memory access
  - Subprograms and their data can be put in cache in stead of accessing main memory

Recurrence (遞迴)

Recurrence Relation

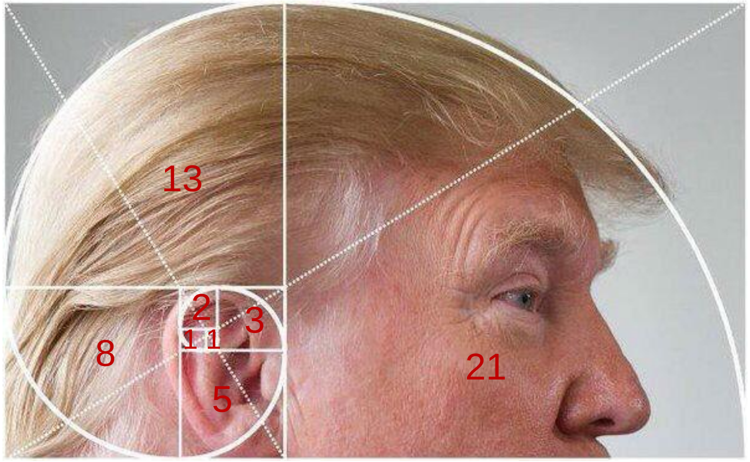
- Definition

  A ***recurrence*** is an equation or inequality that describes a function in terms of its value on smaller inputs.

- Example

  Fibonacci sequence (費波那契數列)

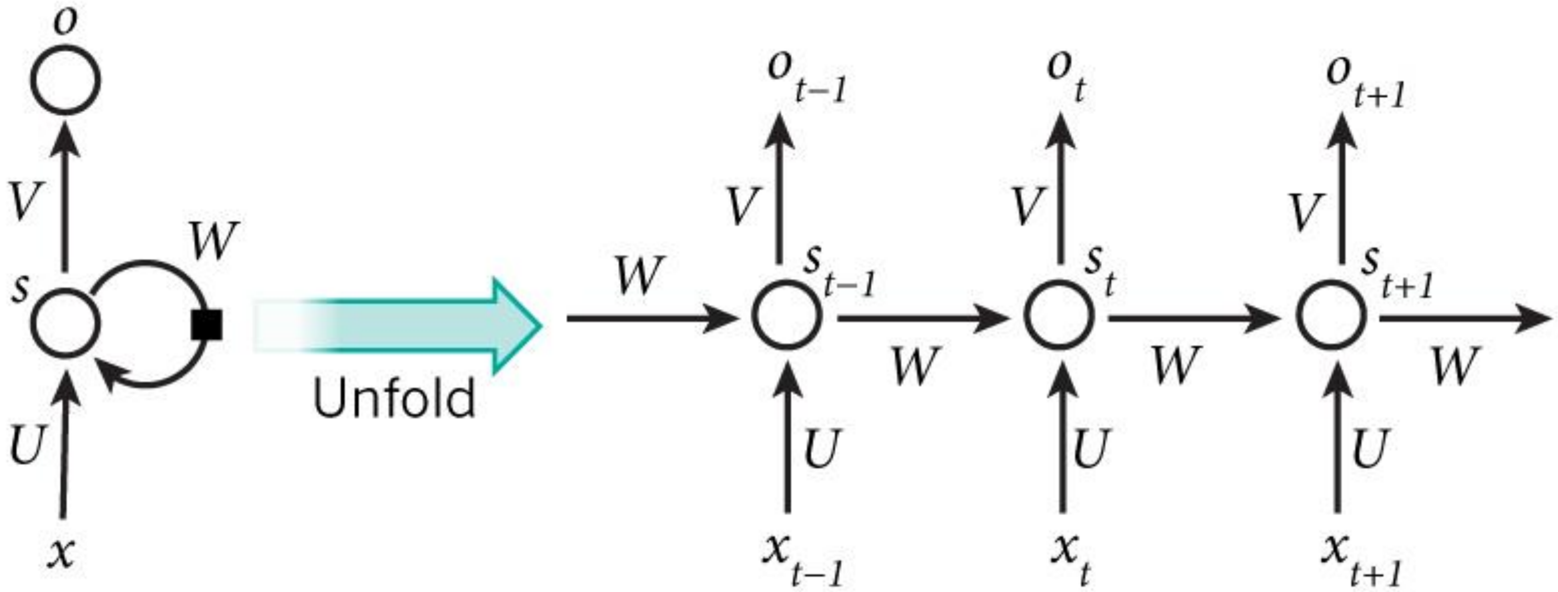
  - Base case: F(0) = F(1) = 1
  - Recursive case: F(n) = F(n-1) + F(n-2)



n	0	1	2	3	4	5	6	7	8	…
F(n)	1	1	2	3	5	8	13	21	34	…

Recurrent Neural Network (RNN)

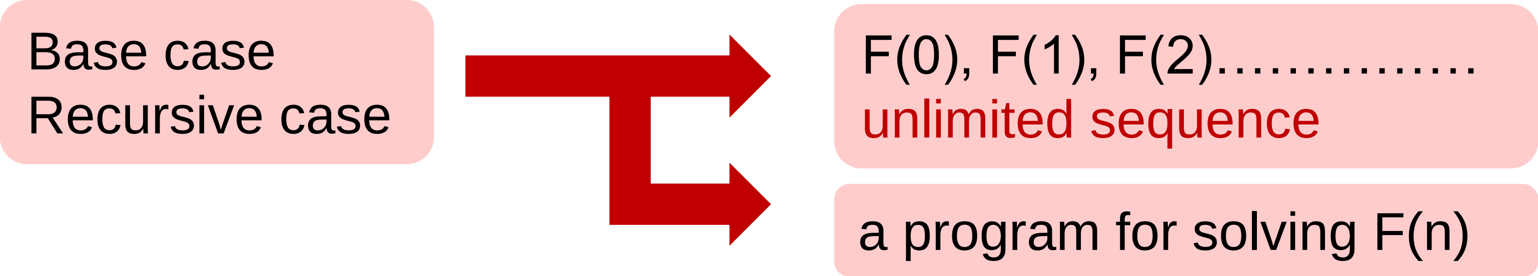
$$s_t = \sigma(W s_{t-1} + U x_t)$$
$$o_t = \mathrm{softmax}(V s_t)$$



http://www.wildml.com/2015/09/recurrent-neural-networks-tutorial-part-1-introduction-to-rnns/

Recurrence Benefits

- Easy & Clear
  - Define base case and recursive case
  - Define a long sequence



Base case
Recursive case

F(0), F(1), F(2)…………….
unlimited sequence

a program for solving F(n)

```
Fibonacci(n) // recursive function:程式中會呼叫自己的函數
  if n < 2 // base case: termination condition
    return 1

  // recursive case: call itself for solving subproblems
  return Fibonacci(n-1) + Fibonacci(n-2)
```

important otherwise the program cannot stop

Recurrence v.s. Non-Recurrence

```
Fibonacci(n)
    if n < 2 // base case
        return 1
    // recursive case
    return Fibonacci(n-1) + Fibonacci(n-2)
```

Recursive function

- Clear structure 👍
- Poor efficiency 👎

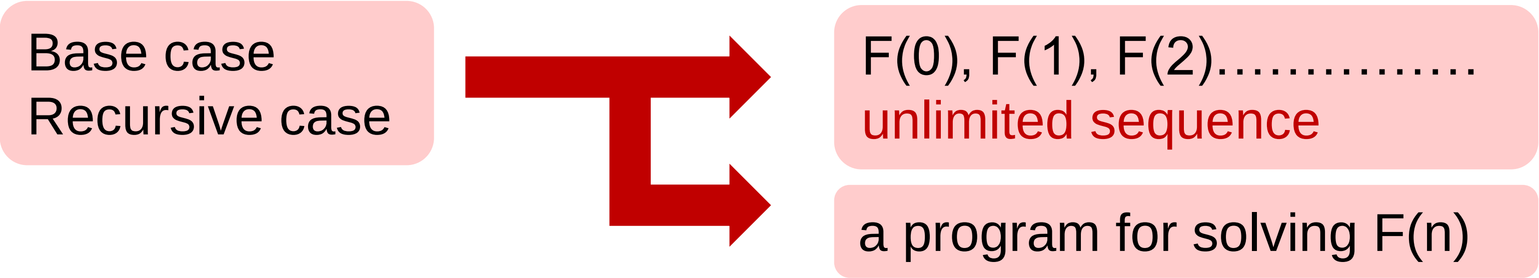
```
Fibonacci(n)
    if n < 2
        return 1
    a[0] <- 1
    a[1] <- 1
    for i = 2 … n
        a[i] = a[i-1] + a[i-2]
    return a[n]
```

Non-recursive function

- Better efficiency 👍
- Unclear structure 👎

Recurrence Benefits

- Easy & Clear
  - Define base case and recursive case
  - Define a long sequence

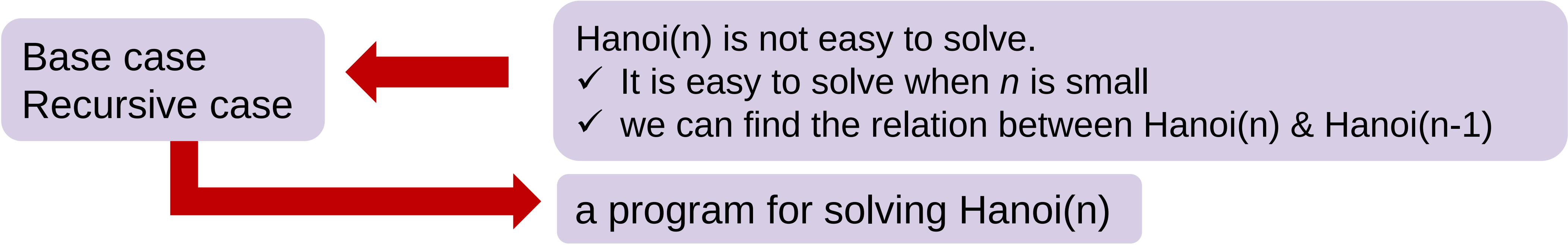


Base case
Recursive case

F(0), F(1), F(2)……………
unlimited sequence

a program for solving F(n)

If a problem can be simplified into a **base case** and a **recursive case**, then we can find an algorithm that solves this problem.



Base case
Recursive case

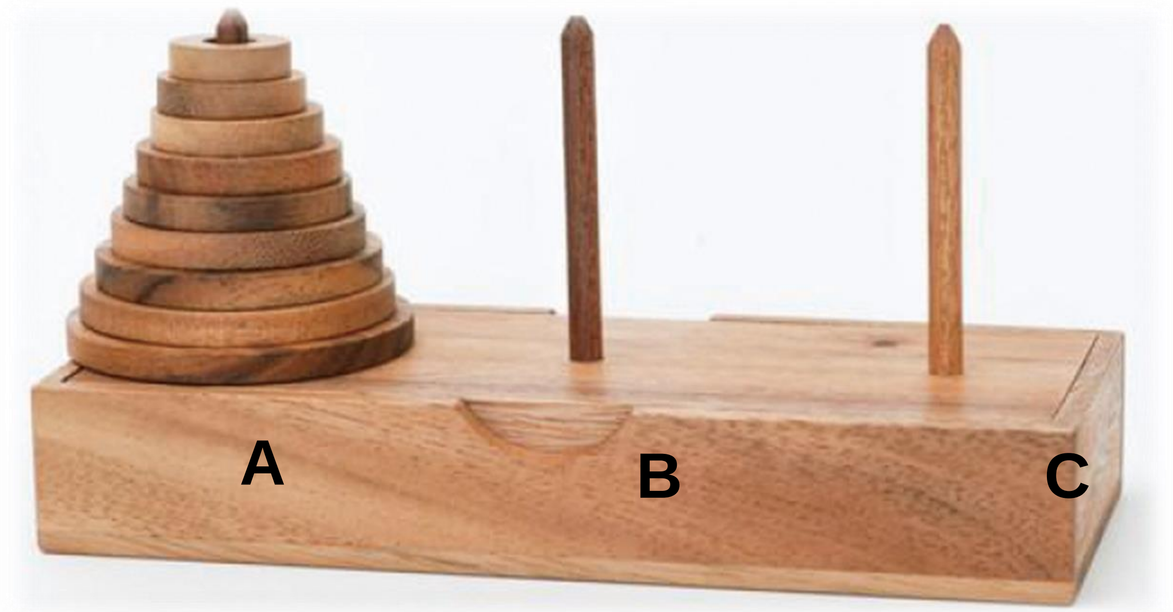
Hanoi(n) is not easy to solve.
✓ It is easy to solve when *n* is small
✓ we can find the relation between Hanoi(n) & Hanoi(n-1)

a program for solving Hanoi(n)

D&C #1: Tower of Hanoi

Tower of Hanoi (河內塔)

- Problem: move *n* disks from A to C
- Rules
  - Move one disk at a time
  - Cannot place a larger disk onto a smaller disk

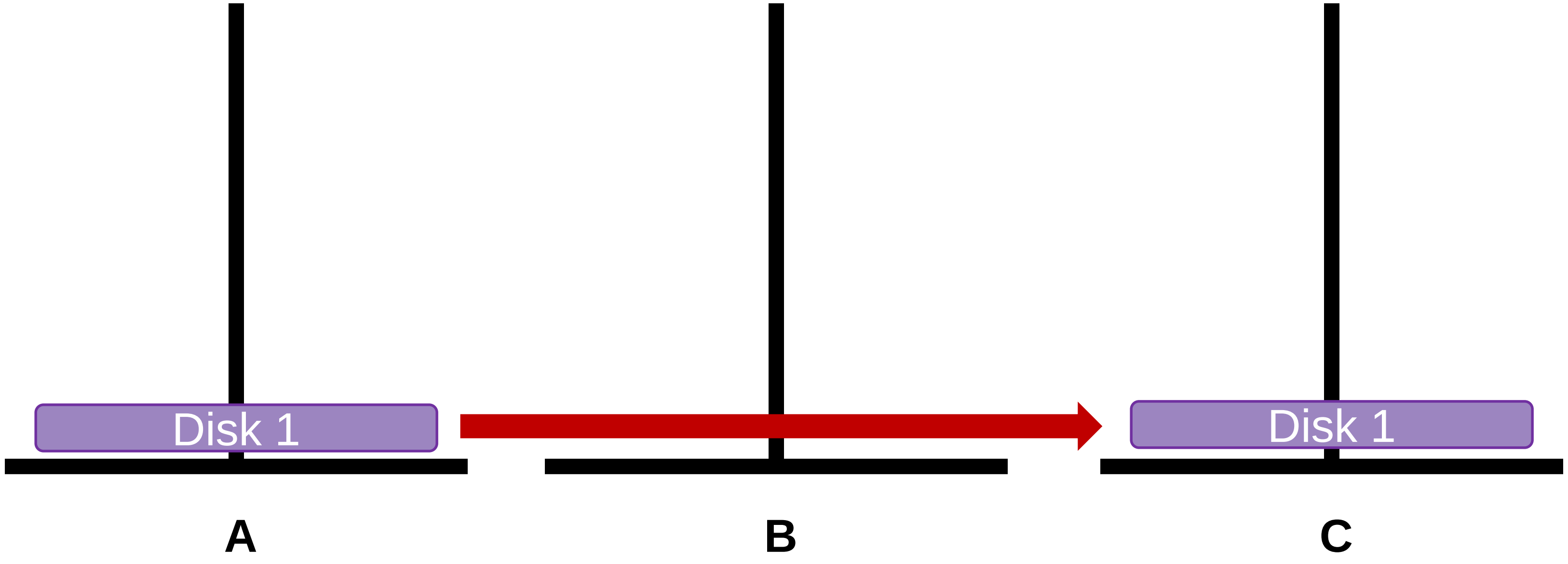


Play online: https://www.mathsisfun.com/games/towerofhanoi.html

Hanoi(1)

- Move 1 from A to C

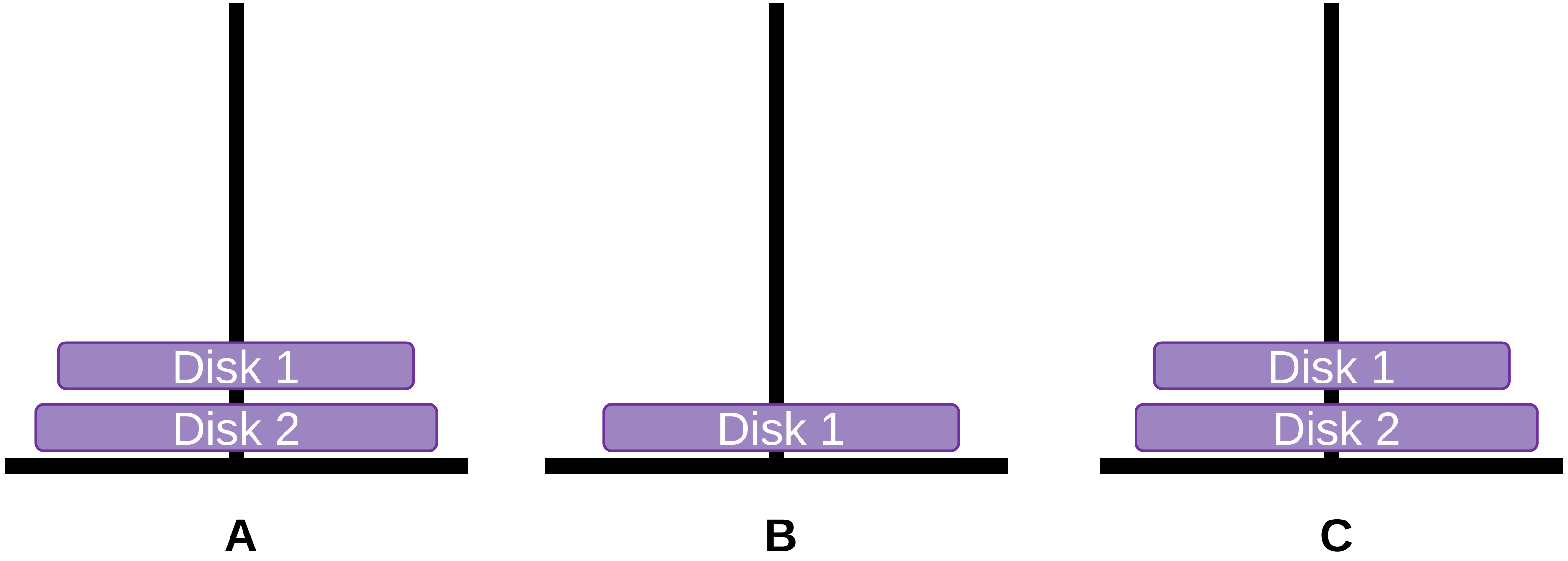
→ 1 move in total
Base case



Hanoi(2)

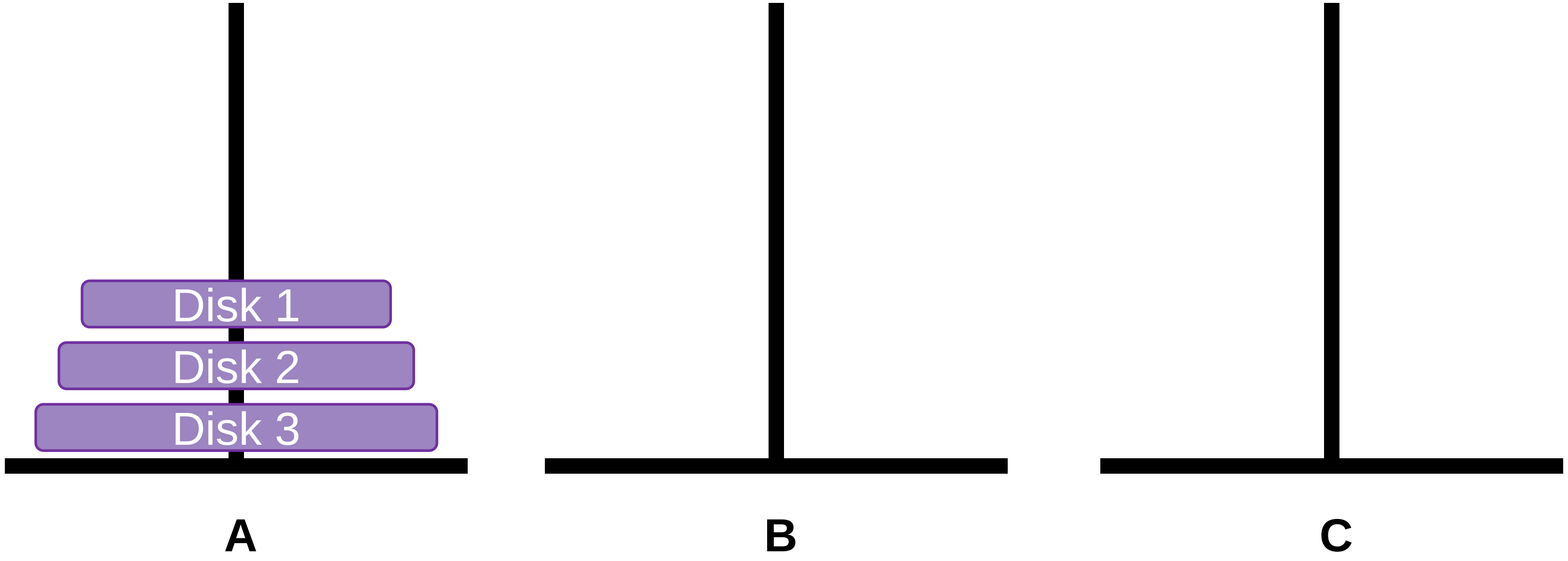
- Move 1 from A to B
- Move 2 from A to C
- Move 1 from B to C

→ 3 moves in total



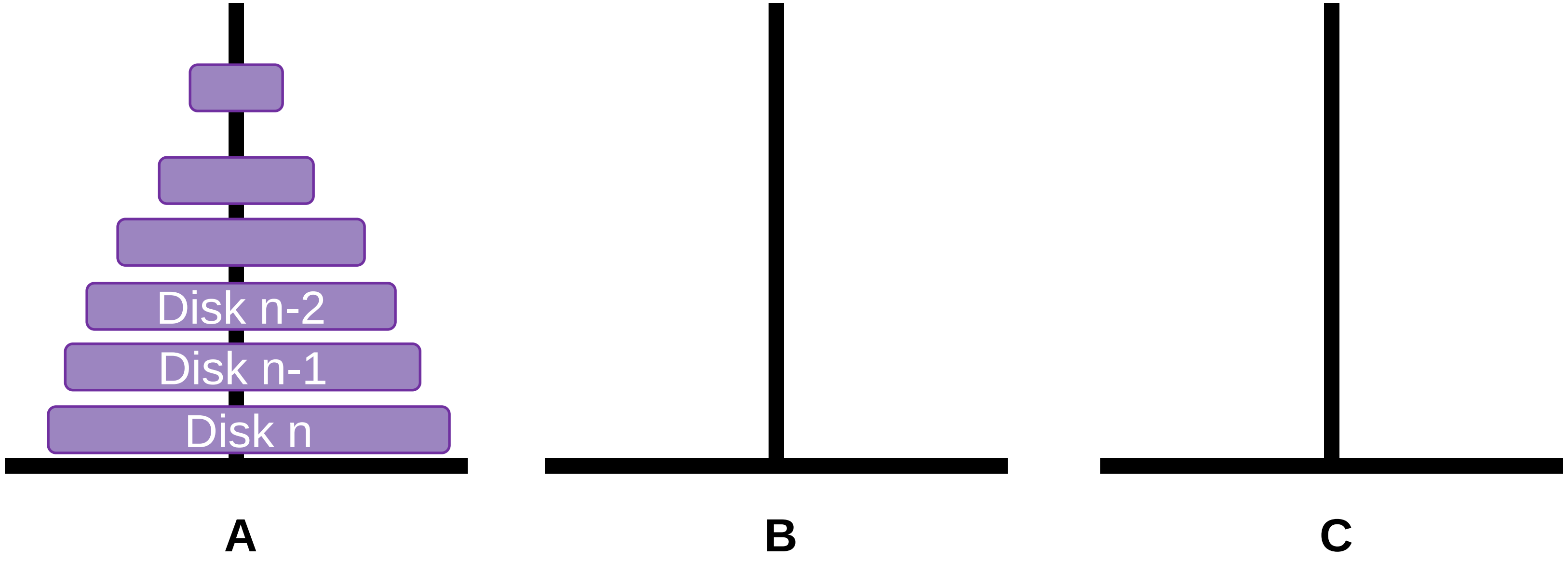
Hanoi(3)

- How to move 3 disks?
- How many moves in total?



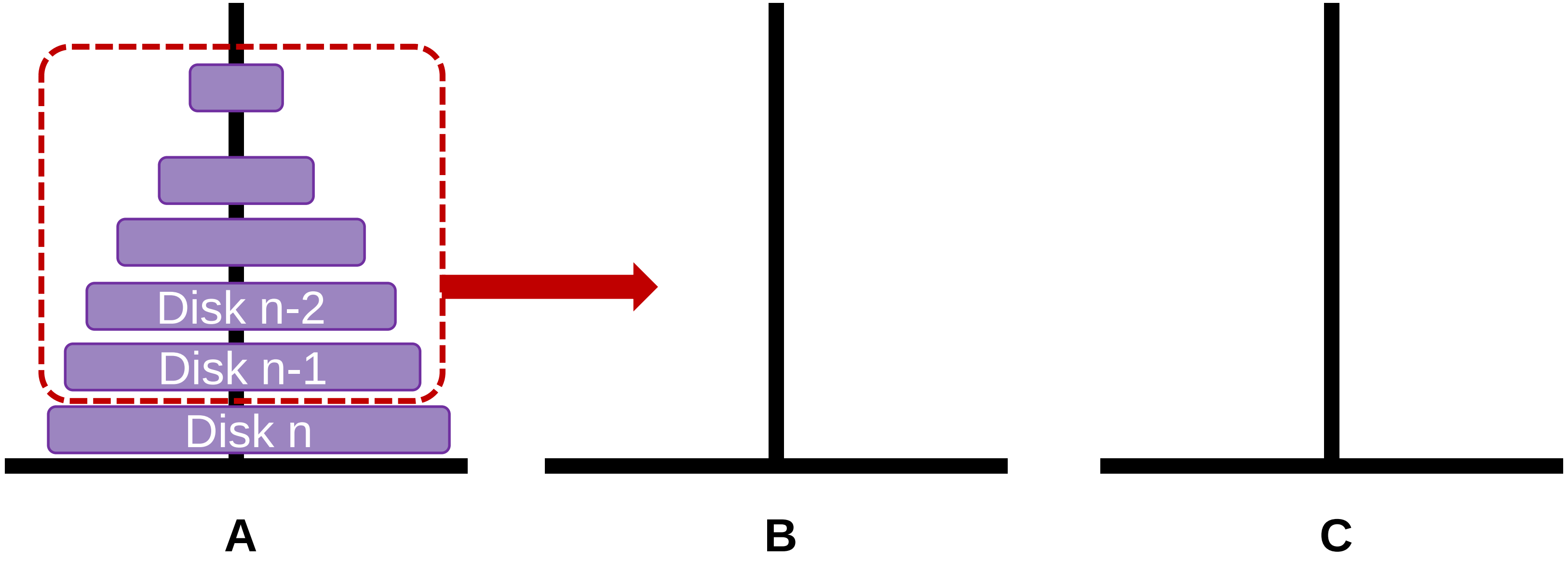
Hanoi(n)

- How to move n disks?
- How many moves in total?



Hanoi(n)

- To move n disks from A to C (for n > 1):
  1. Move Disk 1~n-1 from A to B



Hanoi(n)

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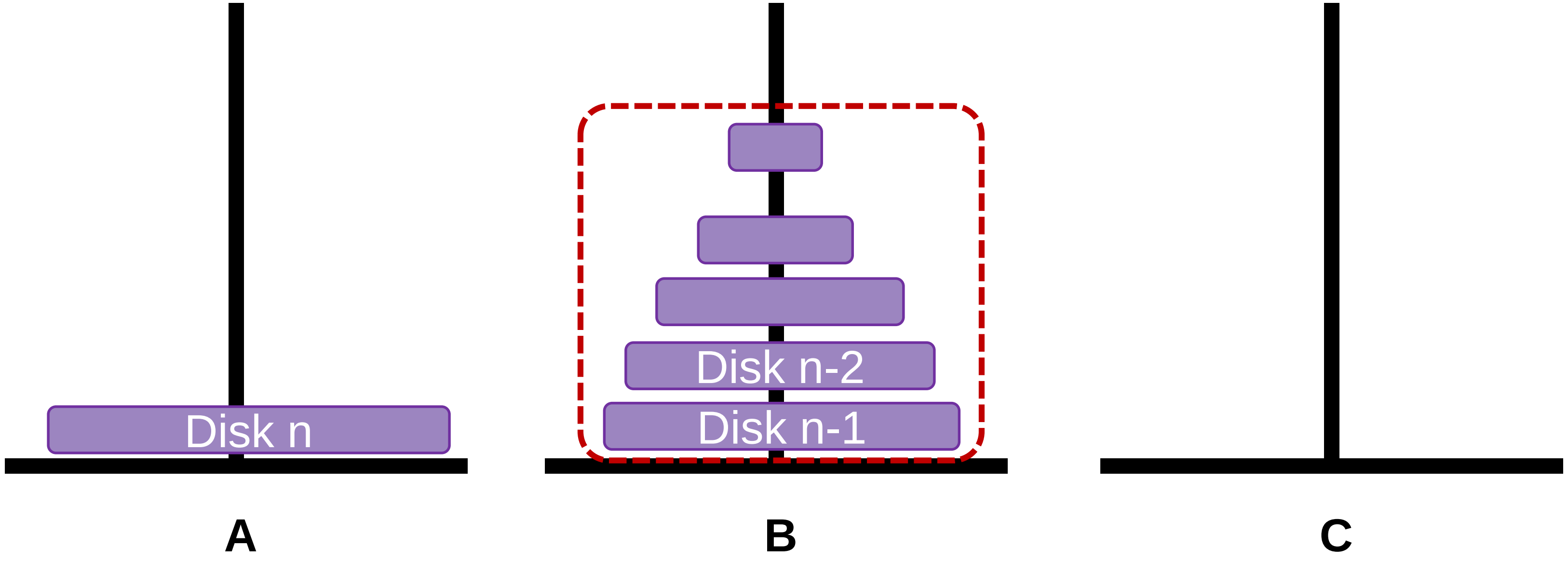
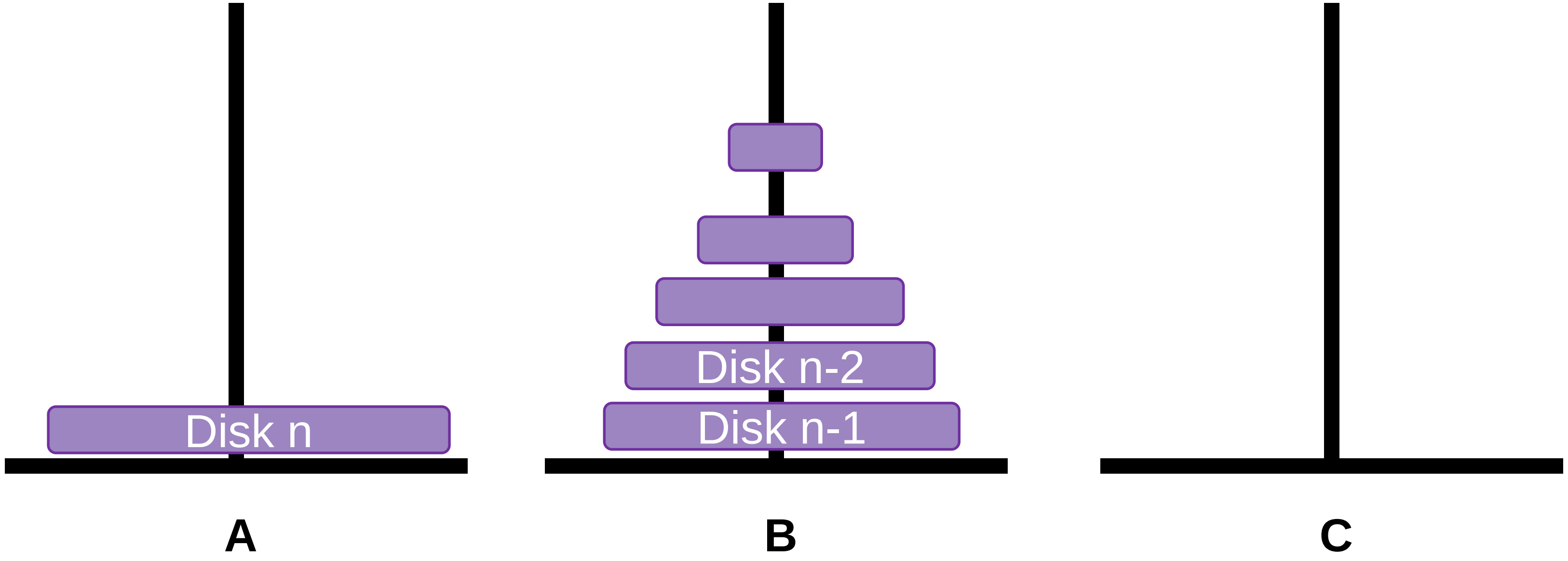


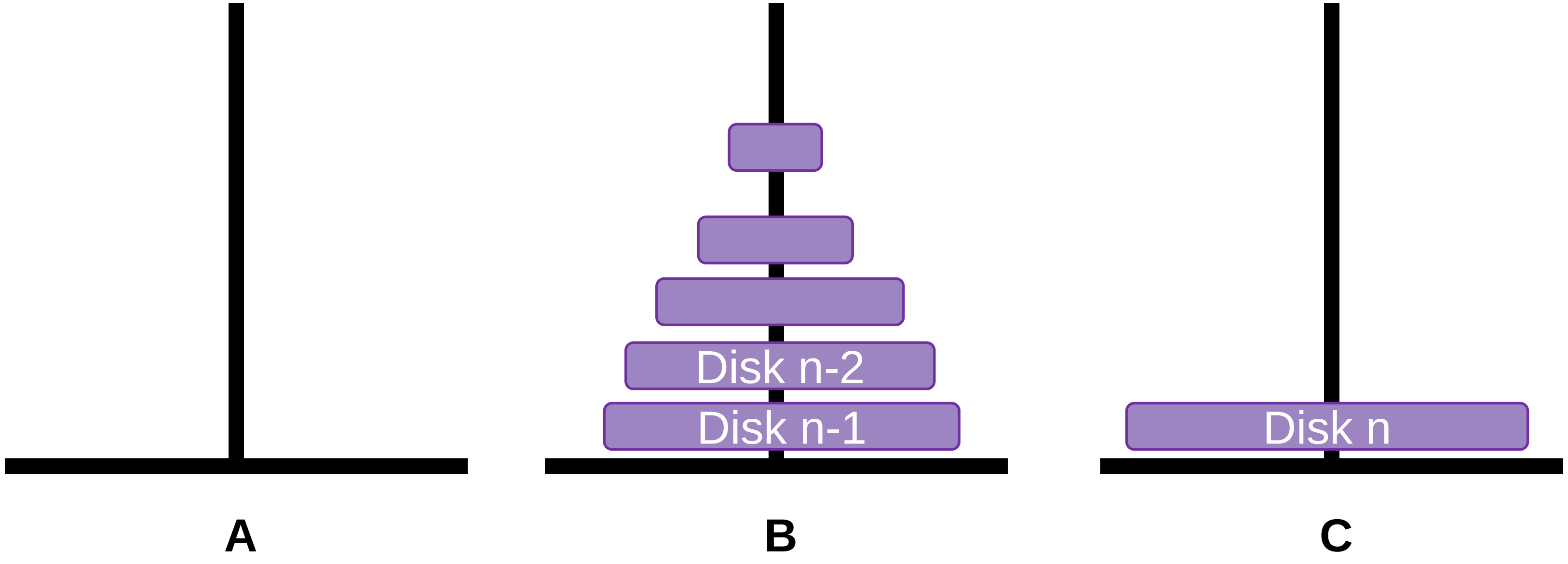
Hanoi(n)

- To move n disks from A to C (for n > 1):
  1. Move Disk 1~n-1 from A to B
  2. Move Disk n from A to C



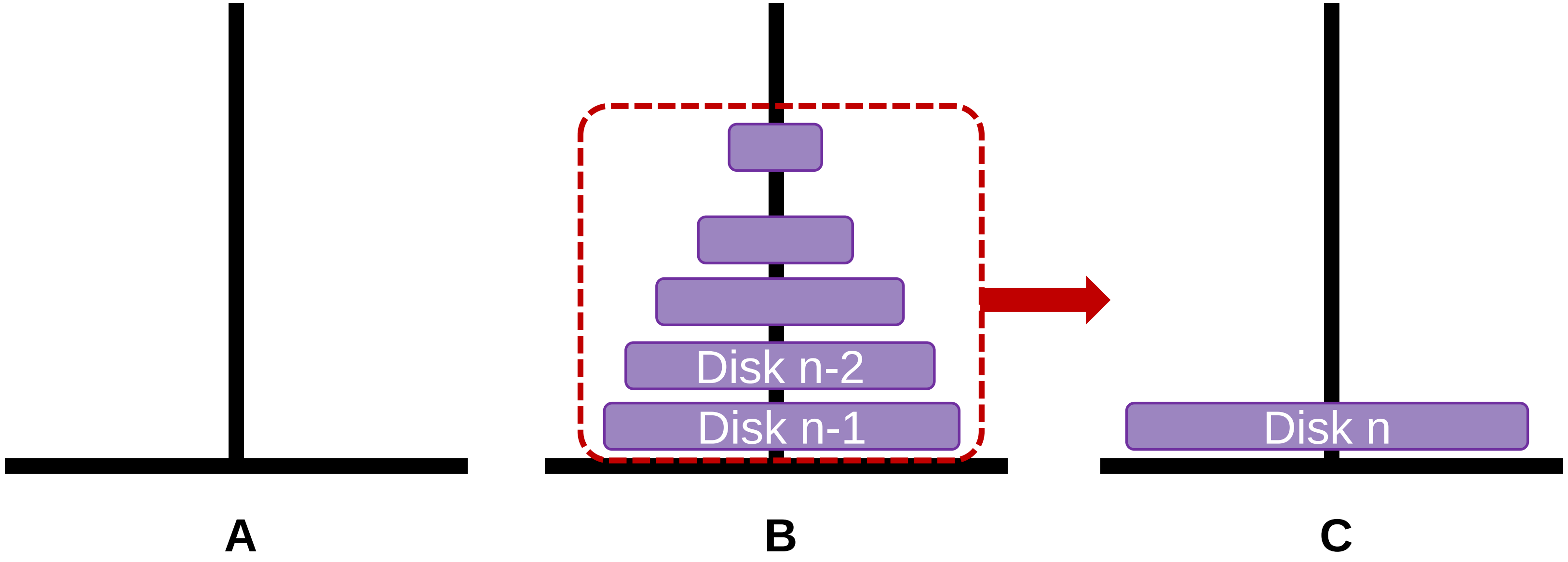
Hanoi(n)

- To move n disks from A to C (for n > 1):
  1. Move Disk 1~n-1 from A to B
  2. Move Disk n from A to C



Hanoi(n)

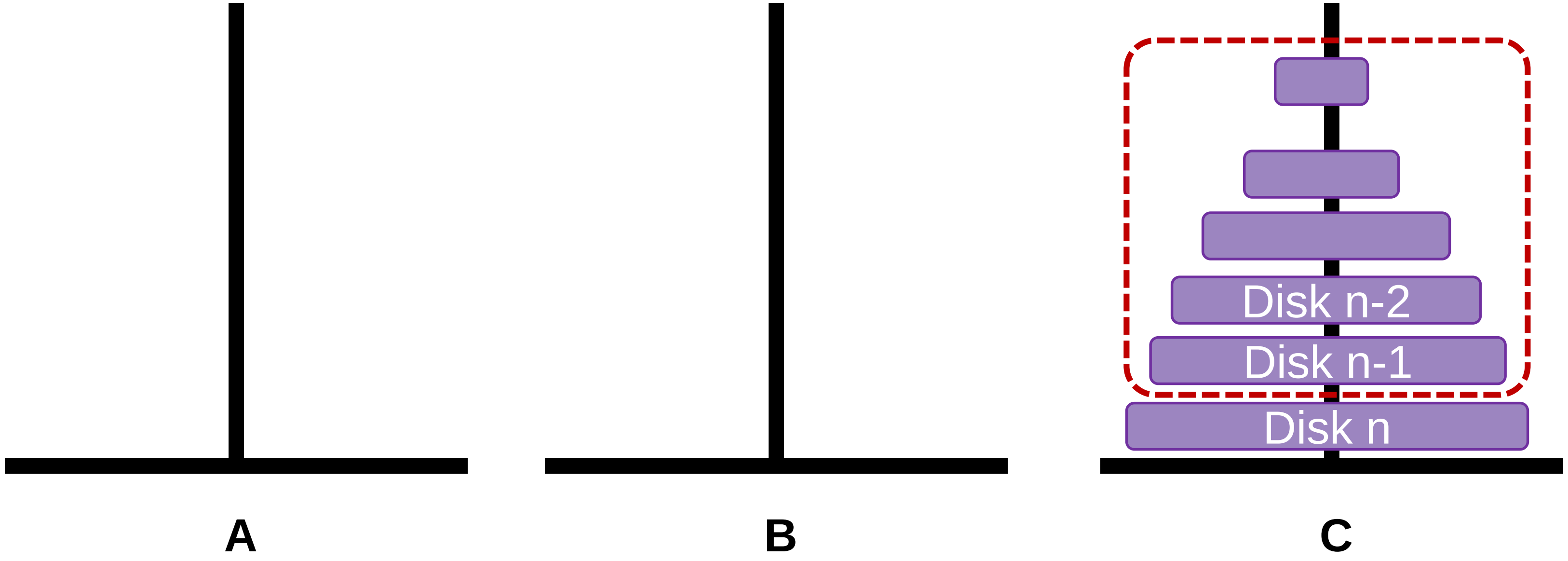
- To move n disks from A to C (for n > 1):
  1. Move Disk 1~n-1 from A to B
  2. Move Disk n from A to C
  3. Move Disk 1~n-1 from B to C



Hanoi(n)

- To move n disks from A to C (for n > 1):
  1. Move Disk 1~n-1 from A to B
  2. Move Disk n from A to C
  3. Move Disk 1~n-1 from B to C

→ 2Hanoi(n-1) + 1 moves in total
recursive case

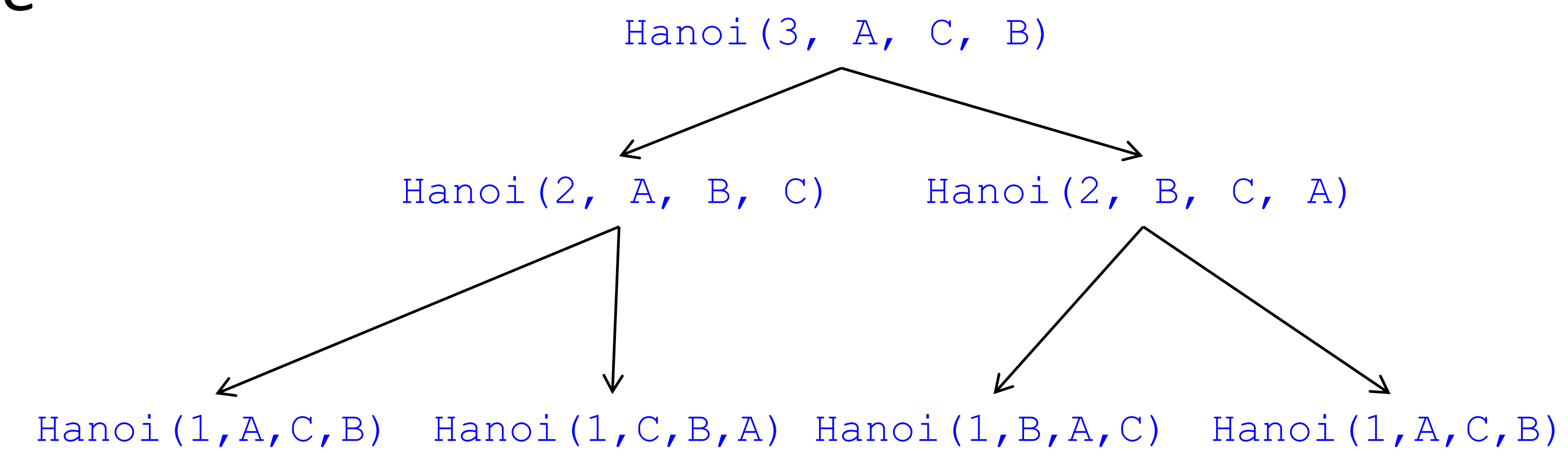


Pseudocode for Hanoi

```
Hanoi(n, src, dest, spare)
  if n==1 // base case
    Move disk from src to dest
  else // recursive case
    Hanoi(n-1, src, spare, dest)
    Move disk from src to dest
    Hanoi(n-1, spare, dest, src)
```

No need to combine the results in this case

- Call tree



Hanoi(3, A, C, B)

Hanoi(2, A, B, C) Hanoi(2, B, C, A)

Hanoi(1,A,C,B) Hanoi(1,C,B,A) Hanoi(1,B,A,C) Hanoi(1,A,C,B)

Algorithm Time Complexity

- $T(n)$ = #moves with n disks
  - Base case: $T(1) = 1$
  - Recursive case ($n > 1$): $T(n) = 2T(n-1) + 1$
- We will learn how to derive $T(n)$ later

```
Hanoi(n, src, dest, spare)
  if n==1 // base case
    Move disk from src to dest
  else // recursive case
    Hanoi(n-1, src, spare, dest)
    Move disk from src to dest
    Hanoi(n-1, spare, dest, src)
```

$$T(n) = 2^n - 1 = O(2^n)$$

Further Questions

- Q1: Is $O(2^n)$ tight for Hanoi? Can $T(n) < 2^n - 1$?
- Q2: What about more than 3 pegs?
- Q3: Double-color Hanoi problem
  - Input: 2 interleaved-color towers
  - Output: 2 same-color towers

D&C #2: Merge Sort

Textbook Chapter 2.3.1 – The divide-and-conquer approach

Sorting Problem

Input: unsorted list of size *n*

6 3 5 1 8 7 2 4

What are the **base case** and **recursive case**?

1 2 3 4 5 6 7 8

Output: sorted list of size *n*

Divide-and-Conquer

- Base case (n = 1)
  - Directly output the list
- Recursive case (n > 1)
  - Divide the list into two sub-lists
  - Sort each sub-list recursively
  - Merge the two sorted lists

How?

2 sublists of size ***n/2***

# of comparisons = $\Theta(n)$

Illustration for n = 10

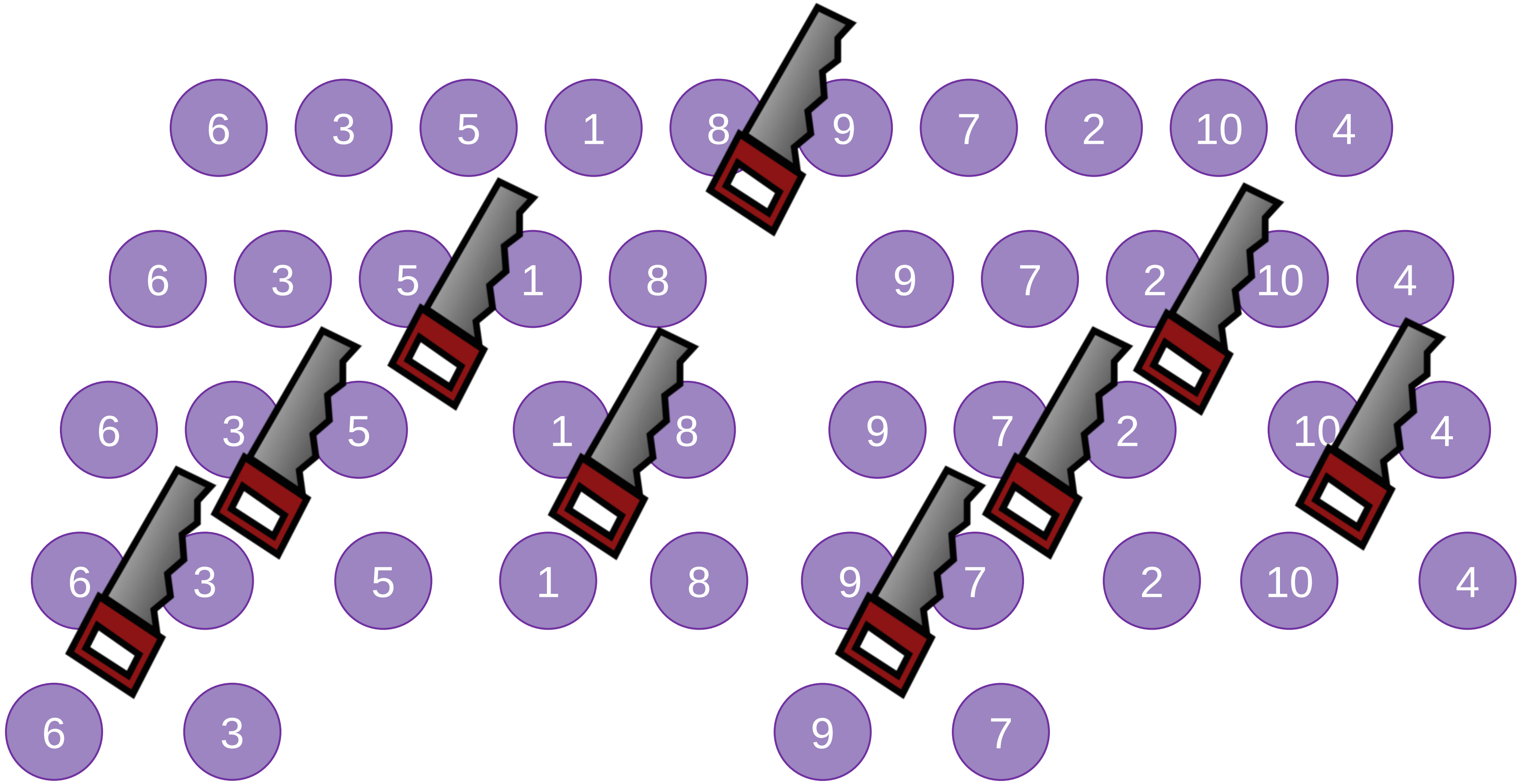
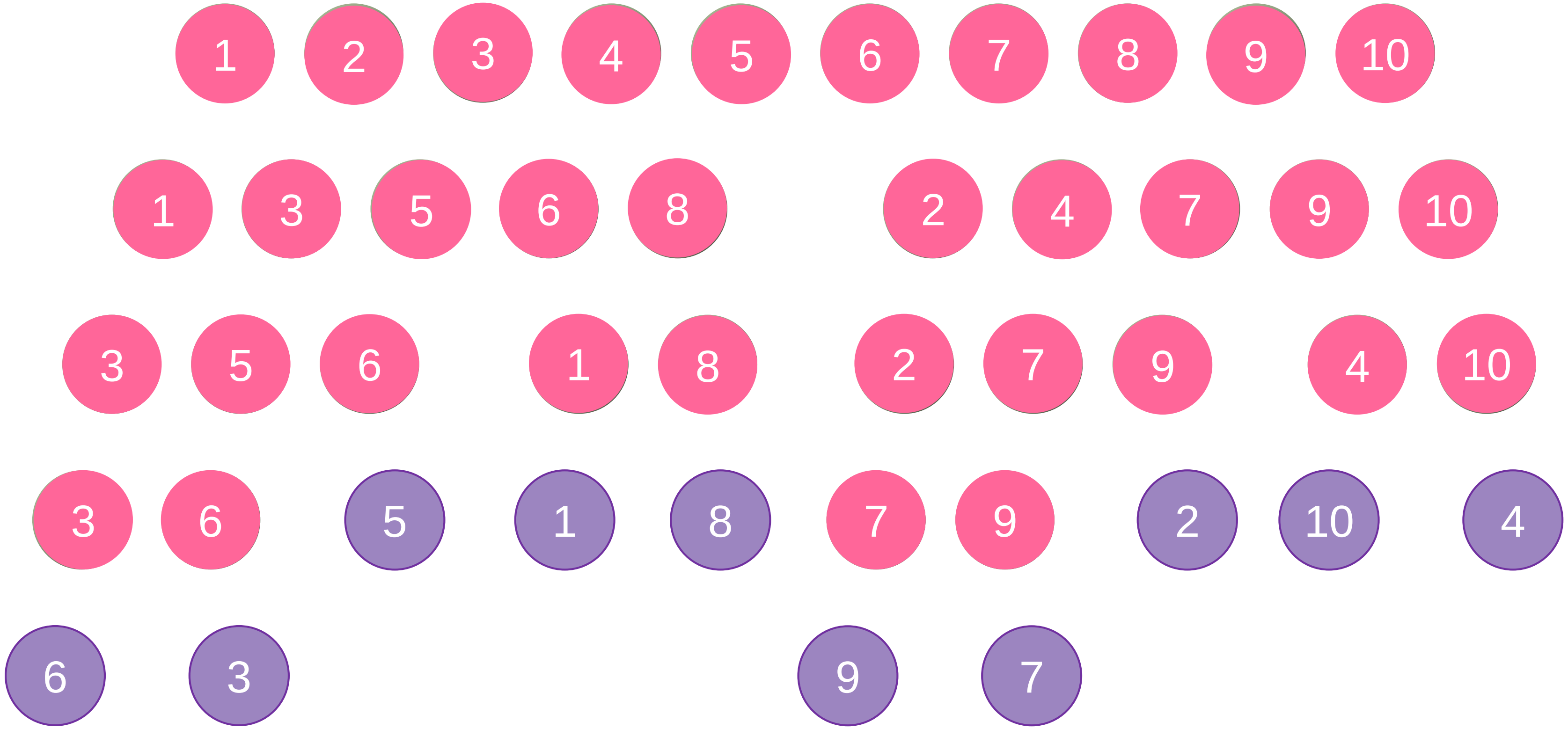
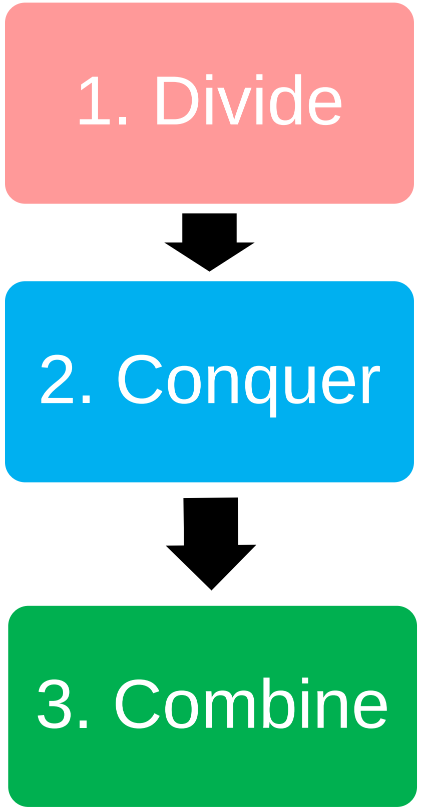


Illustration for *n* = 10



Pseudocode for Merge Sort

```
MergeSort(A, p, r)
  // base case
  if p == r
   return
  // recursive case
  // divide
  q = ⌊(p+r-1)/2]
  // conquer
  MergeSort(A, p, q)
  MergeSort(A, q+1, r)
  // combine
  Merge(A, p, q, r)
```



1. Divide

2. Conquer

3. Combine

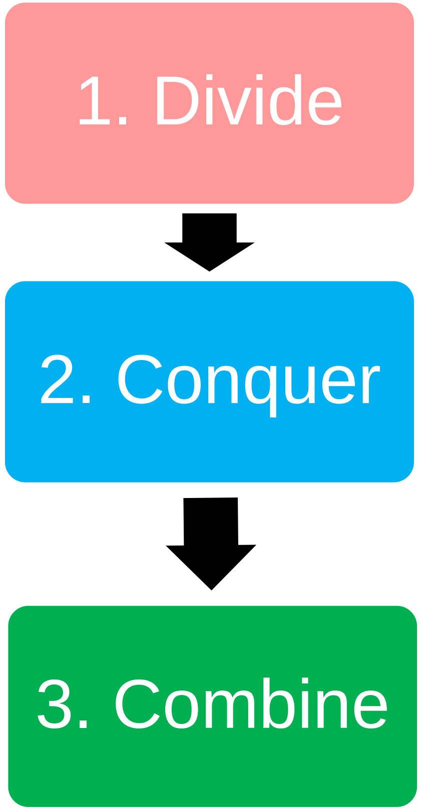
- Divide a list of size *n* into 2 sublists of size *n/2*

- Recursive case $(n > 1)$
    - Sort 2 sublists ***recursively*** using ***merge sort***
- Base case $(n = 1)$
    - Return itself

- Merge 2 sorted sublists into one sorted list in **linear** time

Time Complexity for Merge Sort

```
MergeSort(A, p, r)
  // base case
  if p == r
   return
  // recursive case
  // divide
  q = ⌊(p+r-1)/2⌋
  // conquer
  MergeSort(A, p, q)
  MergeSort(A, q+1, r)
  // combine
  Merge(A, p, q, r)
```



1. Divide

2. Conquer

3. Combine

- Divide a list of size *n* into 2 sublists of size *n/2*

$\Theta(1)$

- <u>Recursive case ($n > 1$)</u>
  - Sort 2 sublists ***recursively*** using ***merge sort***

$T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$

- <u>Base case ($n = 1$)</u>
  - Return itself

$\Theta(1)$

- Merge 2 sorted sublists into one sorted list in **linear** time

$\Theta(n)$

- $T(n)$ = time for running `MergeSort(A, p, r)` with $r - p + 1 = n$

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) & \text{if } n \geq 2 \end{cases}$$

Time Complexity for Merge Sort

- Simplify recurrences
- Ignore floors and ceilings (boundary conditions)
- Assume base cases are constant (for small *n*)

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

$$\begin{aligned}
T(n) &\leq 2T(\frac{n}{2}) + cn \\
&\leq 2[2T(\frac{n}{4}) + c\frac{n}{2}] + cn = 4T(\frac{n}{4}) + 2cn && \text{1st expansion} \\
&\leq 4[2T(\frac{n}{8}) + c\frac{n}{4}] + 2cn = 8T(\frac{n}{8}) + 3cn && \text{2nd expansion} \\
&\vdots \\
&\leq 2^k T(\frac{n}{2^k}) + kcn && \text{kth expansion}
\end{aligned}$$

The expansion stops when $2^k = n$

$$\begin{aligned}
T(n) &\leq nT(1) + cn\log_2 n \\
&= O(n) + O(n\log n) \\
&= O(n\log n)
\end{aligned}$$

Theorem 1

- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n \log n)$$

- Proof
  - There exists positive constant a, b s.t. $T(n) \leq \begin{cases} a & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b \cdot n & \text{if } n \geq 2 \end{cases}$
  - Use induction to prove $T(n) \leq 2b \cdot n \log_2 n + a \cdot n$
    - n = 1, trivial
    - n > 1, $\lceil \frac{n}{2} \rceil \leq \frac{n}{\sqrt{2}}$

$$\begin{aligned} T(n) &\leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b \cdot n \\ &\leq 2b \cdot (\lceil n/2 \rceil \log_2 \lceil n/2 \rceil) + a \cdot \lceil n/2 \rceil + 2b \cdot (\lfloor n/2 \rfloor \log_2 \lfloor n/2 \rfloor) + a \cdot \lfloor n/2 \rfloor + b \cdot n \\ &\leq 2b \cdot (\lceil n/2 \rceil \log_2 \frac{n}{\sqrt{2}}]) + a \cdot \lceil n/2 \rceil + 2b \cdot (\lfloor n/2 \rfloor \log_2 \frac{n}{\sqrt{2}}) + a \cdot \lfloor n/2 \rfloor + b \cdot n \\ &= 2b \cdot n(\log n - \log_2 \sqrt{2}) + a \cdot n + b \cdot n = 2b \cdot n \log_2 n + a \cdot n \end{aligned}$$

Inductive hypothesis

How to Solve Recurrence Relations?

1. **Substitution Method** (取代法)
   - Guess a bound and then prove by induction
2. **Recursion-Tree Method** (遞迴樹法)
   - Expand the recurrence into a tree and sum up the cost
3. **Master Method** (套公式大法/大師法)
   - Apply Master Theorem to a specific form of recurrences

Let’s see more examples first and come back to this later

D&C #3: Bitonic Champion Problem

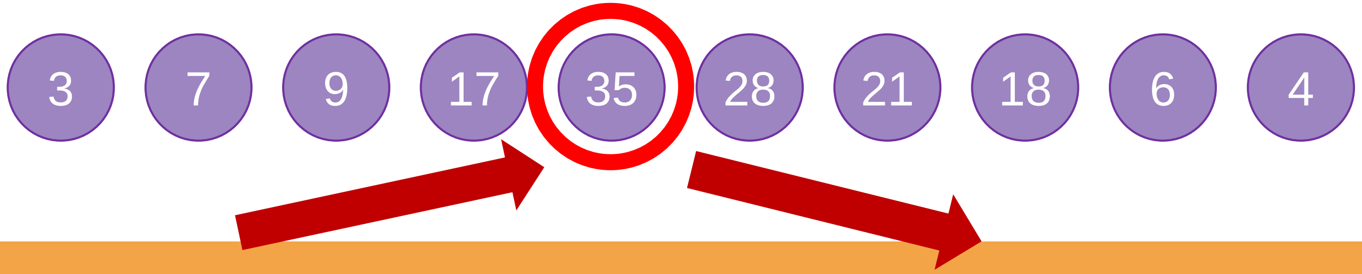
Bitonic Champion Problem

The bitonic champion problem

- Input: A bitonic sequence $A[1], A[2], \ldots, A[n]$ of distinct positive integers.
- Output: the index i with $1 \le i \le n$ such that

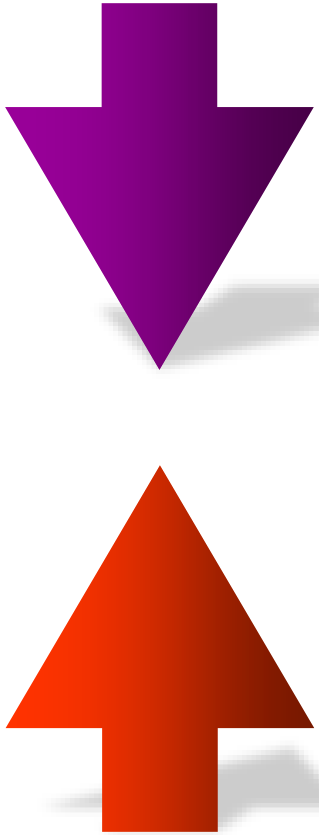
$$A[i] = \max_{1 \le j \le n} A[j].$$

The bitonic sequence means “increasing before the champion and decreasing after the champion” (冠軍之前遞增、冠軍之後遞減)



3 7 9 17 35 28 21 18 6 4

Bitonic Champion Problem Complexity



Upper bound $= O(n)$

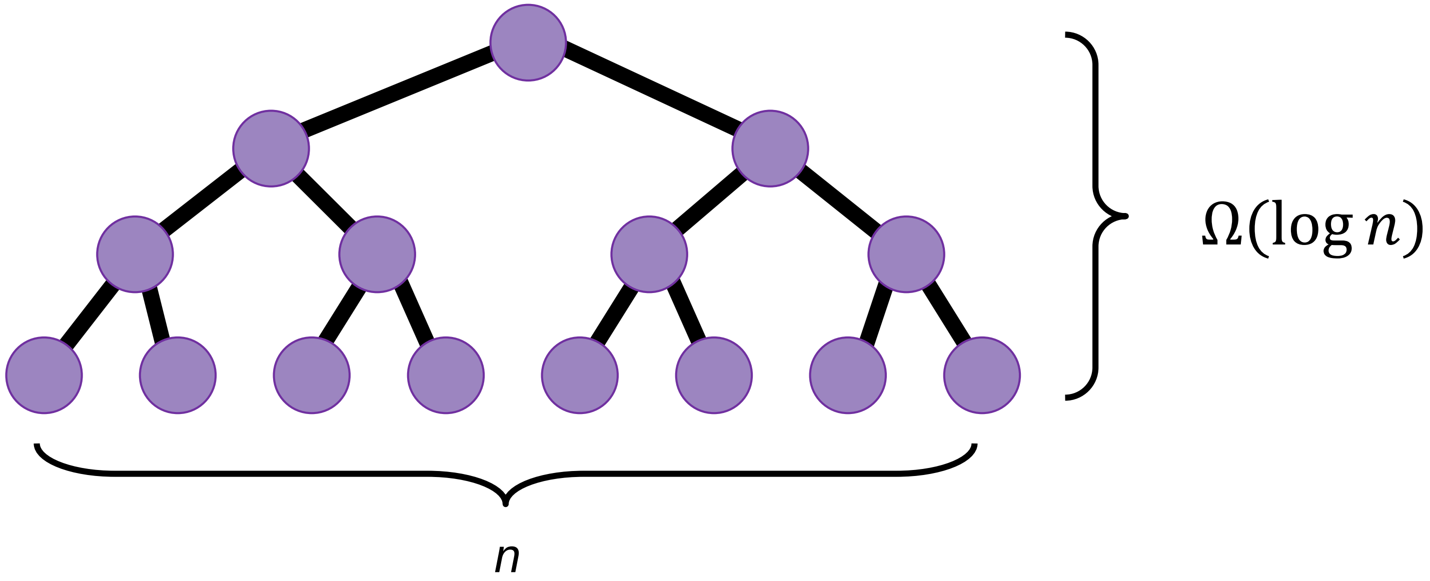
Why?

Lower bound $= \Omega(1)$

Why not Ω(n)?

Bitonic Champion Problem Complexity

- When there are *n* inputs, any solution has *n* different outputs
- Any comparison-based algorithm needs $\Omega(\log n)$ time in the worst case



Bitonic Champion Problem Complexity

Upper bound $= O(n)$

Lower bound $= \Omega(\log n)$

Lower bound $= \Omega(1)$

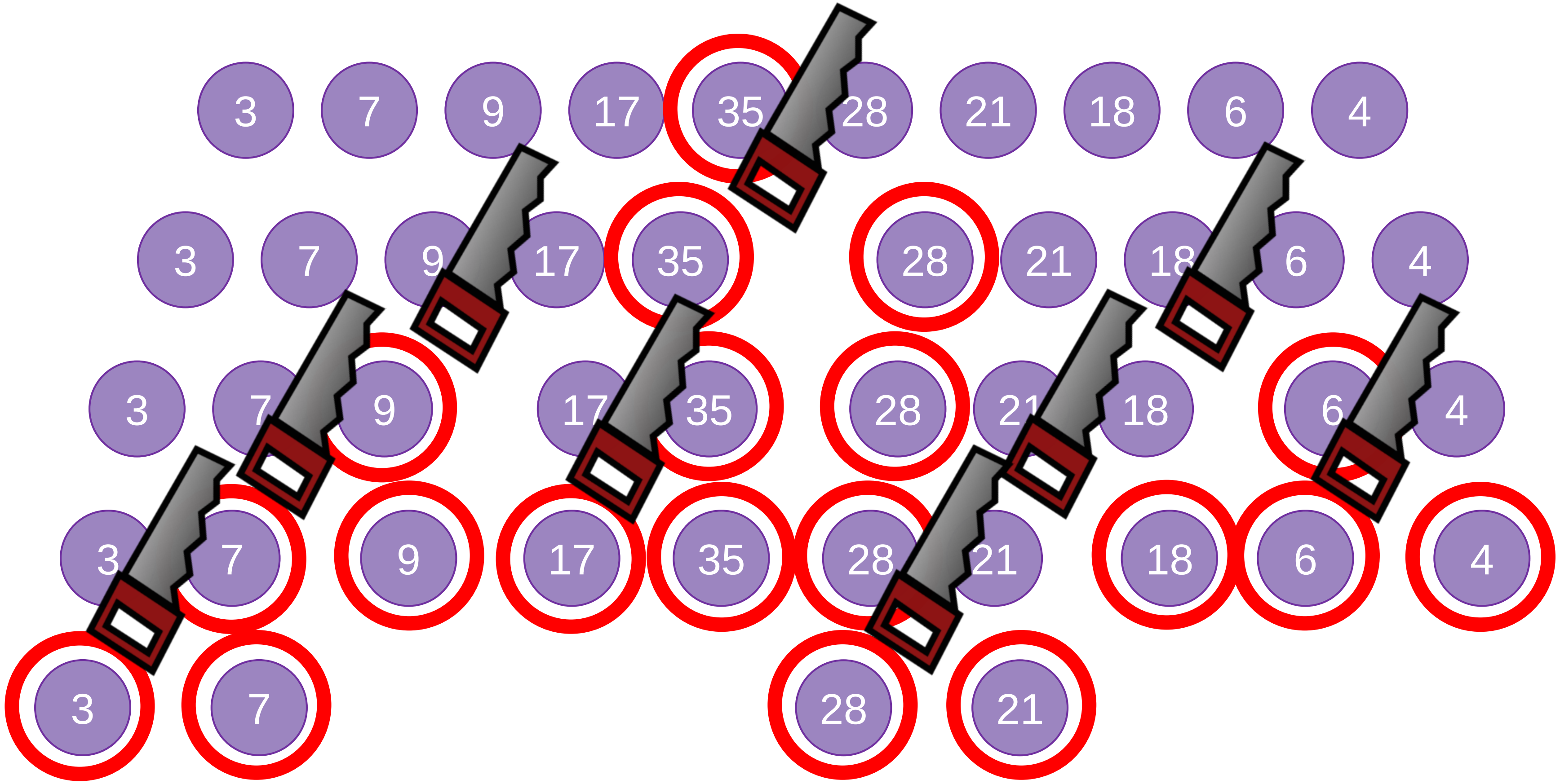
Divide-and-Conquer

- Idea: divide *A* into two subproblems and then find the final champion based on the champions from two subproblems

```
Output = Champion(1, n)
```

```
Champion(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    l = Champion(i, k)
    r = Champion(k+1, j)
    if A[l] > A[r]
      return l
    if A[l] < A[r]
      return r
```

Illustration for *n* = 10



Proof of Correctness

- Practice by yourself!

```
Output = Chamption(1, n)
```

```
Champion(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    l = Champion(i, k)
    r = Champion(k+1, j)
    if A[l] > A[r]
      return l
    if A[l] < A[r]
      return r
```

Hint: use induction on (j – i) to prove `Champion(i, j)` can return the champion from A[i … j]

Algorithm Time Complexity

- $T(n)$ = time for running `Champion(i, j)` with $j - i + 1 = n$

```
Champion(i, j)
  if i==j // base case
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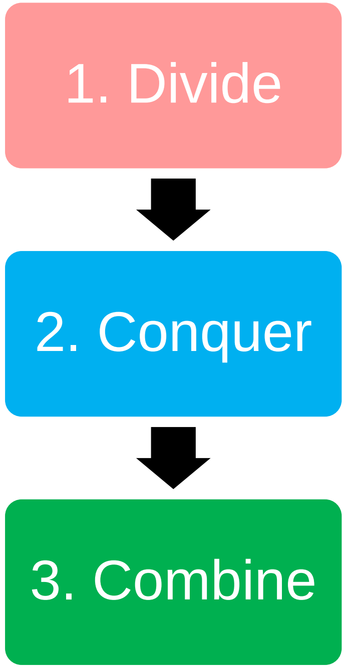


- Divide a list of size n into 2 sublists of size n/2 $\Theta(1)$
- Recursive case $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$
  - Find champions from 2 sublists ***recursively***
- Base case $\Theta(1)$
  - Return itself
- Choose the final champion by a single comparison $\Theta(1)$

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(1) & \text{if } n \geq 2 \end{cases}$$

Theorem 2

- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(1) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n)$$

- Proof
  - There exists positive constant a, b s.t. $T(n) \leq \begin{cases} a & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b & \text{if } n \geq 2 \end{cases}$
  - Use induction to prove $T(n) \leq a \cdot n + b \cdot (n - 1)$
    - n = 1, trivial
    - n > 1,

$$\begin{aligned} T(n) &\leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b \\ &\leq a \cdot \lceil n/2 \rceil + b \cdot (\lceil n/2 \rceil - 1) + a \cdot \lfloor n/2 \rfloor + b \cdot (\lfloor n/2 \rfloor - 1) + b \\ &\leq a \cdot n + b \cdot (n - 1) \end{aligned}$$

Inductive hypothesis

Bitonic Champion Problem Complexity

Upper bound $= O(n)$

Can we have a better algorithm by using the bitonic sequence property?

Lower bound $= \Omega(\log n)$

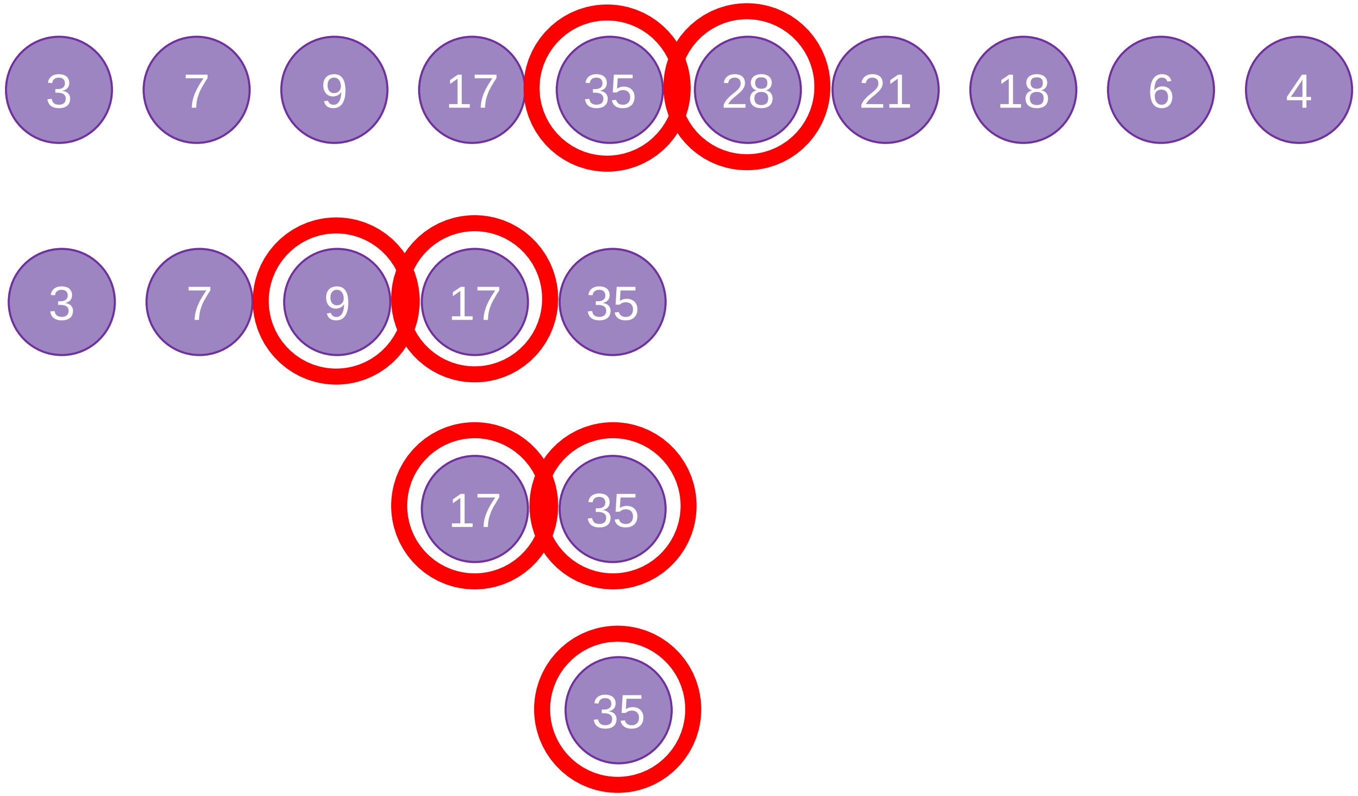
Improved Algorithm

```
Champion(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    l = Champion(i, k)
    r = Champion(k+1, j)
    if A[l] > A[r]
      return l
    if A[l] < A[r]
      return r
```

→

```
Champion-2(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    if A[k] > A[k+1]
      return Champion(i, k)
    if A[k] < A[k+1]
      return Champion(k+1, j)
```

Illustration for *n* = 10



Correctness Proof

- Practice by yourself!

```
Output = Champion-2(1, n)
```

```
Champion-2(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    if A[k] > A[k+1]
      return Champion(i, k)
    if A[k] < A[k+1]
      return Champion(k+1, j)
```

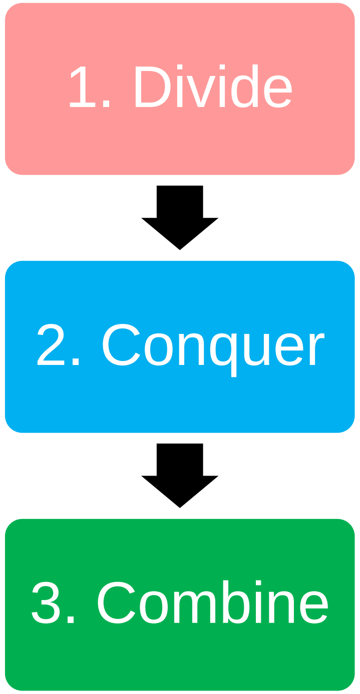
Two crucial observations:

- If $A[1 \dots n]$ is bitonic, then so is $A[i, j]$ for any indices i and j with $1 \leq i \leq j \leq n$.
- For any indices i, j, and k with $1 \leq i \leq j \leq n$, we know that $A[k] > A[k+1]$ if and only if the maximum of $A[i \dots j]$ lies in $A[i \dots k]$.

Algorithm Time Complexity

- $T(n)$ = time for running `Champion-2(i, j)` with $j - i + 1 = n$

```
Champion-2(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    if A[k] > A[k+1]
      return Champion(i, k)
    if A[k] < A[k+1]
      return Champion(k+1, j)
```



1. Divide

2. Conquer

3. Combine

- Divide a list of size *n* into 2 sublists of size *n/2* $\Theta(1)$
- <u>Recursive case</u> $T(\lceil n/2 \rceil)$
  - Find champions from 1 sublists ***recursively***
- <u>Base case</u> $\Theta(1)$
  - Return itself
- Return the champion $\Theta(1)$

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + O(1) & \text{if } n \geq 2 \end{cases}$$

Algorithm Time Complexity

- $T(n)$ = time for running `Champion-2(i, j)` with $j - i + 1 = n$

```
Champion-2(i, j)
  if i==j // base case
    return i
  else // recursive case
    k = floor((i+j)/2)
    if A[k] > A[k+1]
      return Champion(i, k)
    if A[k] < A[k+1]
      return Champion(k+1, j)
```

The algorithm time complexity is $O(\log n)$

- each recursive call reduces the size of (j - i) into half
- there are $O(\log n)$ levels
- each level takes $O(1)$

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + O(1) & \text{if } n \geq 2 \end{cases}$$

Theorem 3

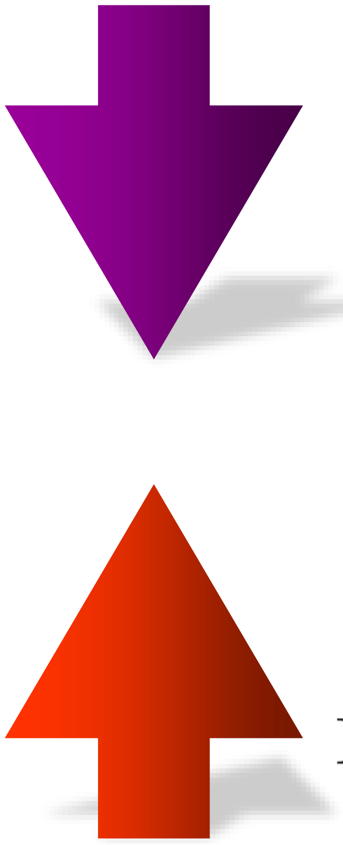
- Theorem

$$T(n) \le \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + O(1) & \text{if } n \ge 2 \end{cases} \Rightarrow T(n) = O(\log n)$$

- Proof

Practice to prove by induction

Bitonic Champion Problem Complexity



Upper bound $= O(n)$

Upper bound $= O(\log n)$

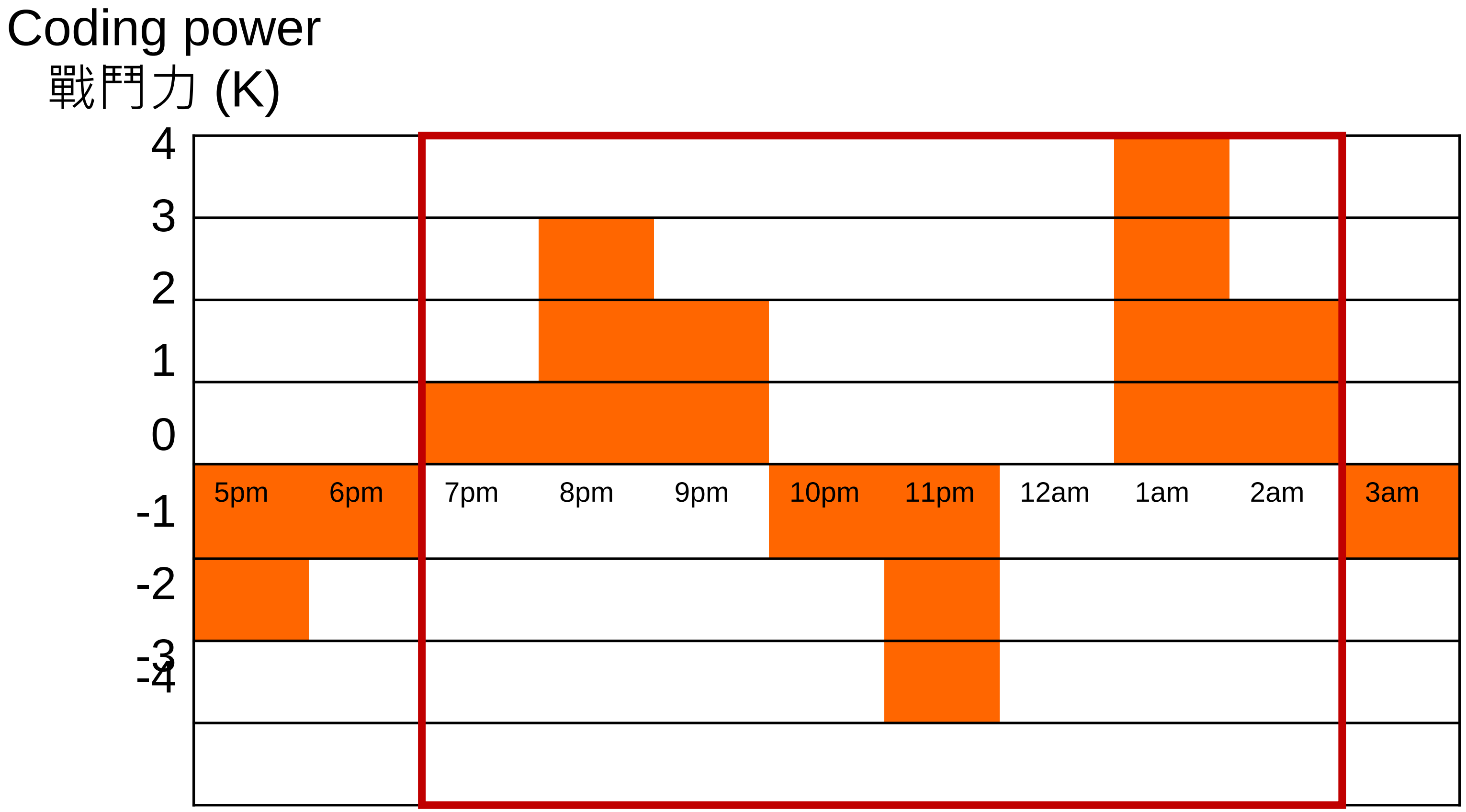
Lower bound $= \Omega(\log n)$

D&C #4: Maximum Subarray

Textbook Chapter 4.1 – The maximum-subarray problem

Coding Efficiency

- How can we find the most efficient time interval for continuous coding?

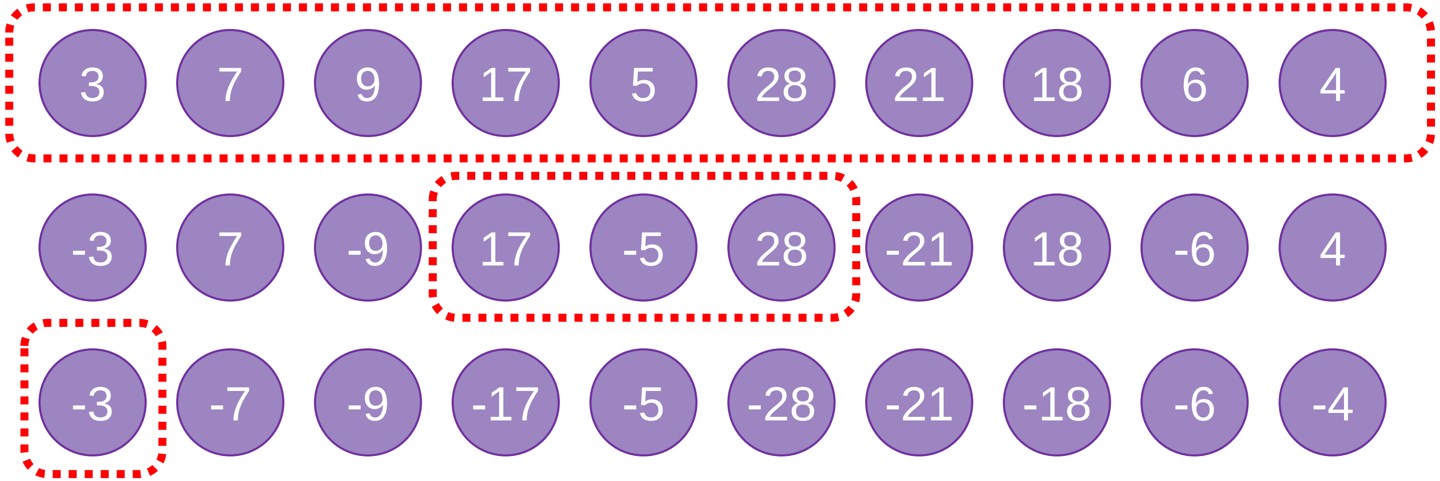


7pm-2:59am
Coding power= 8k

Maximum Subarray Problem

- Input: A sequence $A[1], A[2], \ldots, A[n]$ of integers.
- Output: Two indicex i and j with $1 \leq i \leq j \leq n$ that maximize

$$A[i] + A[i+1] + \cdots + A[j].$$



3 7 9 17 5 28 21 18 6 4

-3 7 -9 17 -5 28 -21 18 -6 4

-3 -7 -9 -17 -5 -28 -21 -18 -6 -4

$O(n^3)$ Brute Force Algorithm

```
MaxSubarray-1(i, j)
  for i = 1,…,n
    for j = 1,…,n                                      $O(n^2)$
      S[i][j] = - ∞

  for i = 1,…,n
    for j = i,i+1,…,n                                  } $O(n^3)$
      S[i][j] = A[i] + A[i+1] + … + A[j]

  return Champion(S)                                   $O(n^2)$
```

$O(n^2)$ Brute Force Algorithm

```
MaxSubarray-2(i, j)
  for i = 1,…,n
    for j = 1,…,n
      S[i][j] = - ∞                              $O(n^2)$

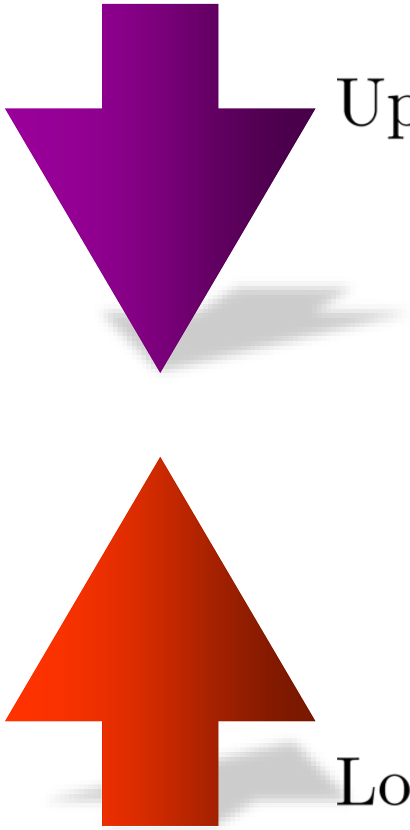
  R[0] = 0                                     } 
  for i = 1,…,n                                } $O(n)$
    R[i] = R[i-1] + A[i]                       }

  for i = 1,…,n                                }
    for j = i+1,i+2,…,n                        } $O(n^2)$
      S[i][j] = R[j] - R[i-1]

  return Champion(S)                             $O(n^2)$
```

R[n] is the sum over A[1…n]

Max Subarray Problem Complexity



Upper bound $= O(n^2)$

Lower bound $= \Omega(n)$

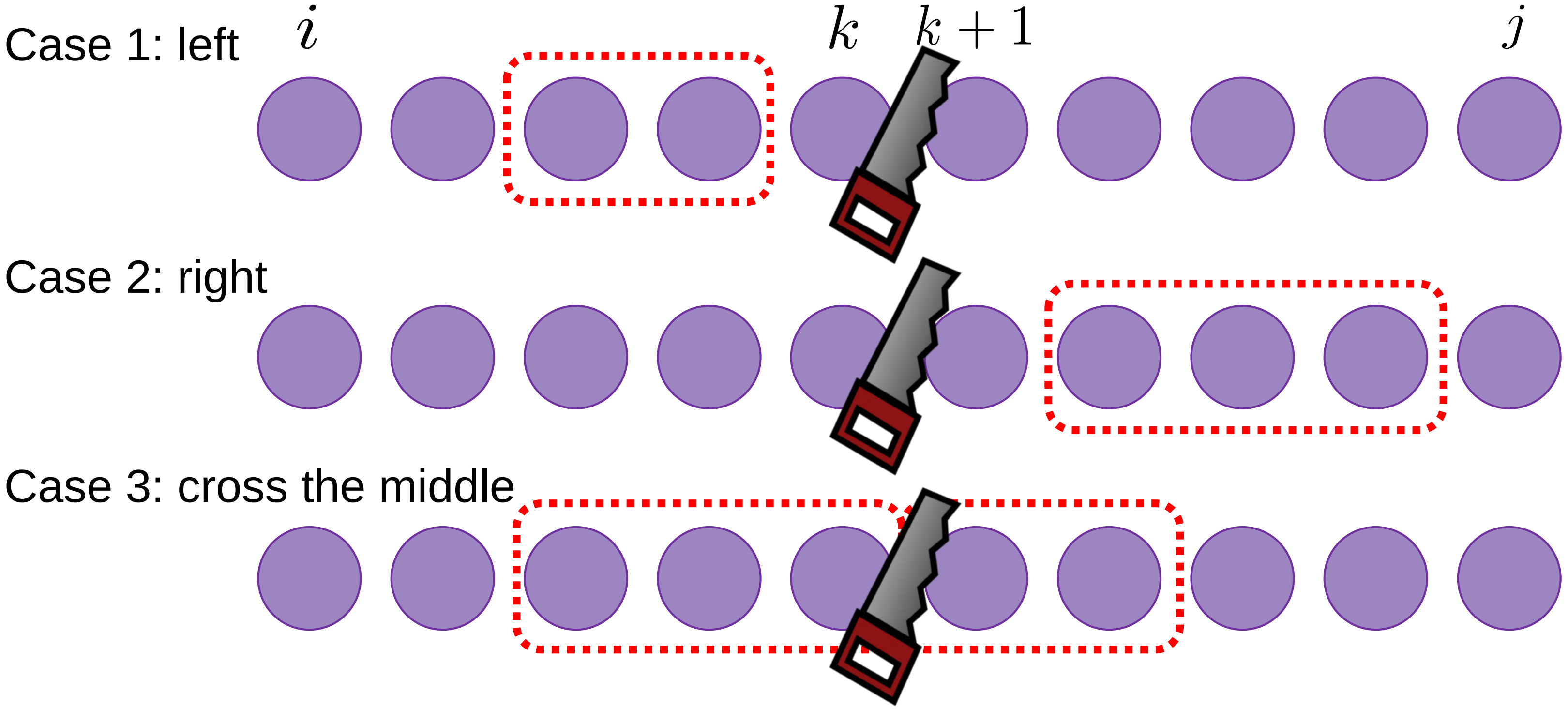
Divide-and-Conquer

- Base case (n = 1)
  - Return itself (maximum subarray)
- Recursive case (n > 1)
  - Divide the array into two sub-arrays
  - Find the maximum sub-array recursively
  - Merge the results

How?

Where is the Solution?

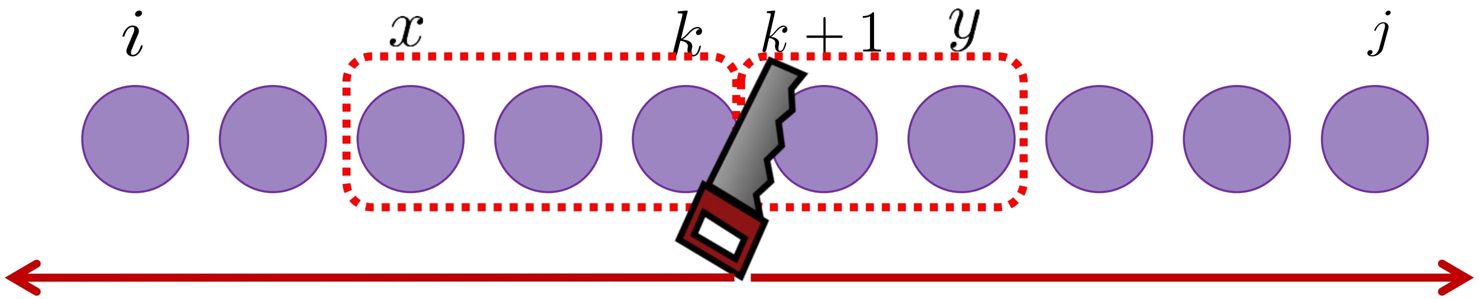
- The maximum subarray for any input must be in one of following cases:



Case 1: `MaxSub(A, i, j) = MaxSub(A, i, k)`
Case 2: `MaxSub(A, i, j) = MaxSub(A, k+1, j)`
Case 3: `MaxSub(A, i, j)` cannot be expressed using `MaxSub`!

Case 3: Cross the Middle

- Goal: find the maximum subarray that crosses the middle



(1) Start from the middle to find the left maximum subarray

(2) Start from the middle to find the right maximum subarray

The solution of Case 3 is the combination of (1) and (2)

- Observation
  - The sum of $A[x \ldots k]$ must be the maximum among $A[i \ldots k]$ (left: $i \leq k$)
  - The sum of $A[k+1 \ldots y]$ must be the maximum among $A[k+1 \ldots j]$ (right: $j > k$)
  - Solvable in linear time → $\Theta(n)$

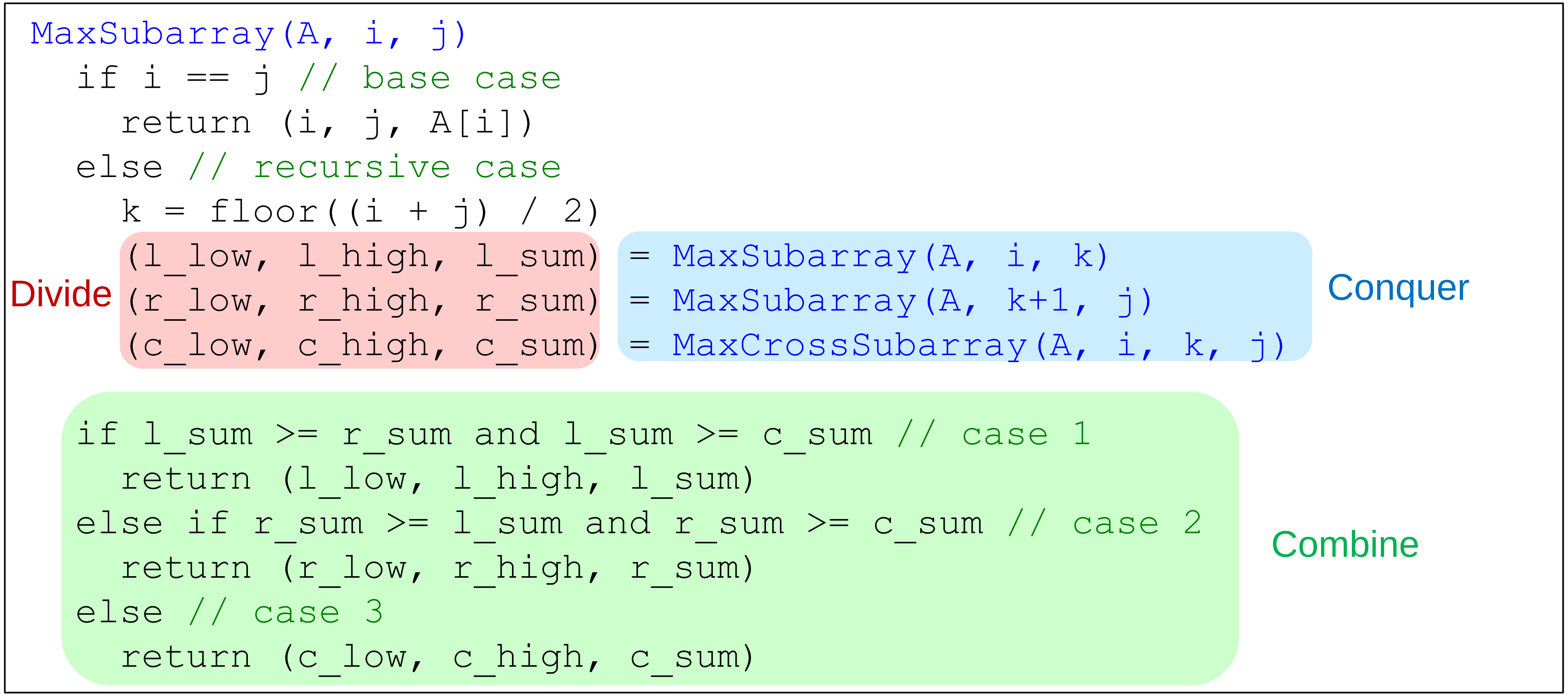
Divide-and-Conquer Algorithm

```
MaxCrossSubarray(A, i, k, j)
  left_sum = $-\infty$
  sum=0
  for p = k downto i
    sum = sum + A[p]
    if sum > left_sum
      left_sum = sum
      max_left = p

  right_sum = $-\infty$
  sum=0
  for q = k+1 to j
    sum = sum + A[q]
    if sum > right_sum
      right_sum = sum
      max_right = q
  return (max_left, max_right, left_sum + right_sum)
```

$$\left.\begin{array}{l} O(k-i+1) \\ \\ O(j-k) \end{array}\right\} = O(j-i+1)$$

Divide-and-Conquer Algorithm



```
MaxSubarray(A, i, j)
  if i == j // base case
    return (i, j, A[i])
  else // recursive case
    k = floor((i + j) / 2)
    (l_low, l_high, l_sum) = MaxSubarray(A, i, k)
    (r_low, r_high, r_sum) = MaxSubarray(A, k+1, j)
    (c_low, c_high, c_sum) = MaxCrossSubarray(A, i, k, j)

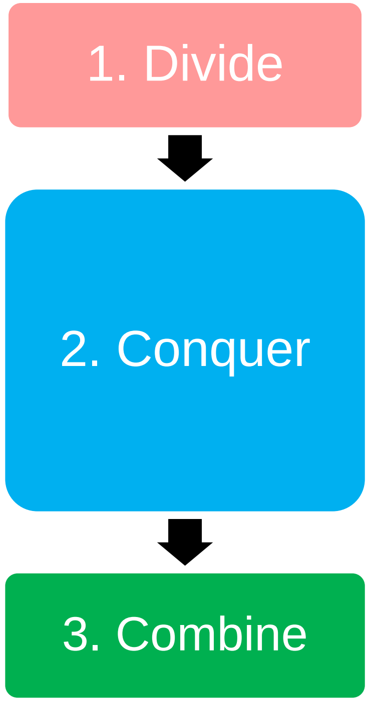
  if l_sum >= r_sum and l_sum >= c_sum // case 1
    return (l_low, l_high, l_sum)
  else if r_sum >= l_sum and r_sum >= c_sum // case 2
    return (r_low, r_high, r_sum)
  else // case 3
    return (c_low, c_high, c_sum)
```

Divide-and-Conquer Algorithm

```
MaxSubarray(A, i, j)
  if i == j // base case
    return (i, j, A[i])                                       $O(1)$
  else // recursive case
    k = floor((i + j) / 2)
    (l_low, l_high, l_sum) = MaxSubarray(A, i, k)             $T(k-i+1)$
    (r_low, r_high, r_sum) = MaxSubarray(A, k+1, j)           $T(j-k)$
    (c_low, c_high, c_sum) = MaxCrossSubarray(A, i, k, j)     $O(j-i+1)$

  if l_sum >= r_sum and l_sum >= c_sum // case 1              $O(1)$
    return (l_low, l_high, l_sum)
  else if r_sum >= l_sum and r_sum >= c_sum // case 2         $O(1)$
    return (r_low, r_high, r_sum)
  else // case 3                                              $O(1)$
    return (c_low, c_high, c_sum)
```

Algorithm Time Complexity



1. Divide

2. Conquer

3. Combine

- Divide a list of size *n* into 2 subarrays of size *n/2* $\Theta(1)$
- <u>Recursive case ($n > 1$)</u> $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$
  - find **MaxSub** for each subarrays
- <u>Base case ($n = 1$)</u> $\Theta(1)$
  - Return itself
- Find **MaxCrossSub** for the original list $\Theta(n)$
- Pick the subarray with the maximum sum among 3 subarrays

$\Theta(1)$

- $T(n)$ = time for running `MaxSubarray(A, i, j)` with $j - i + 1 = n$

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) & \text{if } n \geq 2 \end{cases}$$

Theorem 1

- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n \log n)$$

- Proof
  - There exists positive constant a, b s.t. $T(n) \leq \begin{cases} a & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b \cdot n & \text{if } n \geq 2 \end{cases}$
  - Use induction to prove $T(n) \leq 2b \cdot n \log_2 n + a \cdot n$
    - n = 1, trivial
    - n > 1, $\frac{n+1}{2} \leq \frac{n}{\sqrt{2}}$

$$\begin{aligned} & T(n) && \leq && T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + b \cdot n \\ & \text{Inductive hypothesis} && \leq && 2b \cdot (\lceil n/2 \rceil \log_2 \lceil n/2 \rceil + a \cdot \lceil n/2 \rceil) + 2b \cdot (\lfloor n/2 \rfloor \log_2 \lfloor n/2 \rfloor + a \cdot \lfloor n/2 \rfloor) + b \cdot n \\ & && \leq && 2b \cdot (\lceil n/2 \rceil \log_2 \frac{n}{\sqrt{2}} \rceil + a \cdot \lceil n/2 \rceil) + 2b \cdot (\lfloor n/2 \rfloor \log_2 \frac{n}{\sqrt{2}} + a \cdot \lfloor n/2 \rfloor) + b \cdot n \\ & && = && 2b \cdot n(\log n - \log_2 \sqrt{2}) + a \cdot n + b \cdot n = 2b \cdot n \log_2 n + a \cdot n \end{aligned}$$

Theorem 1 (Simplified)

- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases} \quad ➡ \quad T(n) = O(n \log n)$$

- Proof
  - There exists positive constant a, b s.t. $T(n) \leq \begin{cases} a & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$
  - Use induction to prove $T(n) \leq b \cdot n \log n + a \cdot n$
    - n = 1, trivial
    - n > 1,

Inductive hypothesis

$$\begin{aligned} T(n) &\leq 2T(n/2) + bn \\ &\leq 2[b \cdot \frac{n}{2} \log \frac{n}{2} + a \cdot \frac{n}{2}] + b \cdot n \\ &= b \cdot n \log n - b \cdot n + a \cdot n + b \cdot n \\ &= b \cdot n \log n + a \cdot n \end{aligned}$$

Max Subarray Problem Complexity

Upper bound $= O(n^2)$

Upper bound $= O(n \log n)$

Lower bound $= \Omega(n)$

Max Subarray Problem Complexity

Upper bound $= O(n \log n)$

Upper bound $= O(n)$

Exercise 4.1-5
page 75 of textbook

Lower bound $= \Omega(n)$

Next topic!

Solving Recurrences

Textbook Chapter 4.3 – The substitution method for solving recurrences

Textbook Chapter 4.4 – The recursion-tree method for solving recurrences

Textbook Chapter 4.5 – The master method for solving recurrences

D&C Algorithm Time Complexity

- $T(n)$: running time for input size n
- $D(n)$: time of **Divide** for input size n
- $C(n)$: time of **Combine** for input size n
- a: number of subproblems
- n/b: size of each subproblem

$$T(n) = \begin{cases} O(1) & \text{if } n \leq c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Solving Recurrences

1. **Substitution Method** (取代法)
    - Guess a bound and then prove by induction
2. **Recursion-Tree Method** (遞迴樹法)
    - Expand the recurrence into a tree and sum up the cost
3. **Master Method** (套公式大法/大師法)
    - Apply Master Theorem to a specific form of recurrences

- Useful simplification tricks
    - Ignore floors, ceilings, boundary conditions (proof in Ch. 4.6)
    - Assume base cases are constant (for small *n*)

Substitution Method

Textbook Chapter 4.3 – The substitution method for solving recurrences

Review

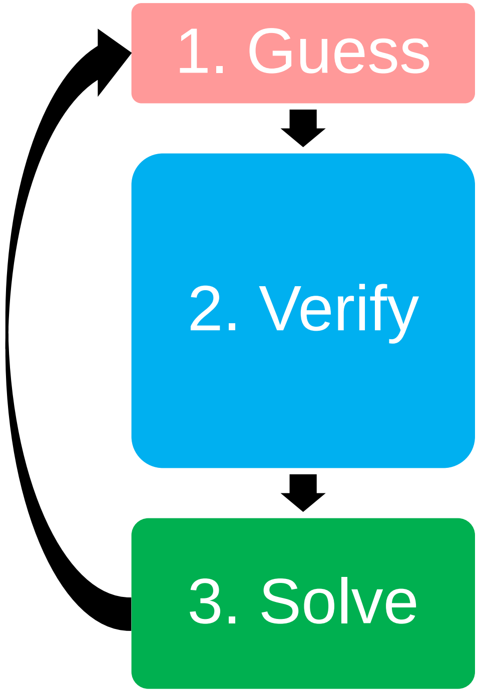
- Time Complexity for Merge Sort
- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n \log n)$$

- Proof
  - There exists positive constant *a*, *b* s.t. $T(n) \leq \begin{cases} a & \text{if } n = 1 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$
  - Use induction to prove $T(n) \leq b \cdot n \log n + a \cdot n$
    - n = 1, trivial
    - n > 1, $$\begin{aligned} T(n) &\leq 2T(n/2) + bn \\ &\leq 2[b \cdot \frac{n}{2} \log \frac{n}{2} + a \cdot \frac{n}{2}] + b \cdot n \\ &= b \cdot n \log n - b \cdot n + a \cdot n + b \cdot n \\ &= b \cdot n \log n + a \cdot n \end{aligned}$$

Substitution Method (取代法)
guess a bound and then prove by induction

Substitution Method (取代法)



- Guess the form of the solution

- Verify by mathematical induction (數學歸納法)
  - Prove it works for $n = 1$
  - Prove that if it works for $n = m$, then it works for $n = m + 1$

  → It can work for all positive integer n

- Solve constants to show that the solution works
- Prove O and Ω separately

Substitution Method Example

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 4T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

- Proof
  - $T(n) = O(n^3)$
    There exists positive constants n_0, c s.t. for all $n \geq n_0, T(n) \leq cn^3$

    Guess
  - Use induction to find the constants n_0, c
    - n = 1, trivial
    - n > 1,

$$\begin{aligned} T(n) &\leq 4T(n/2) + bn \\ &\leq 4c(n/2)^3 + bn \\ &= cn^3/2 + bn \\ &= cn^3 - (cn^3/2 - bn) \\ &\leq cn^3 \end{aligned}$$

Inductive hypothesis

Verify

$cn^3/2 - bn \geq 0$

e.g. $c \geq 2b, n \geq 1$

- $T(n) \leq cn^3$ holds when $c = 2b, n_0 = 1$

Solve

Substitution Method Example

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 4T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

Tighter upper bound?

- Proof
  - $T(n) = O(n^2)$
    There exists positive constants n_0, c s.t. for all $n \geq n_0, T(n) \leq cn^2$
  - Use induction to find the constants n_0, c
    - n = 1, trivial
    - n > 1,

$$\begin{aligned} T(n) &\leq 4T(n/2) + bn \\ &\leq 4c(n/2)^2 + bn \\ &= cn^2 + bn \end{aligned}$$

Inductive hypothesis

証不出來…
猜錯了？還是推導錯了？

沒猜錯 推導也沒錯
這是取代法的小盲點

Substitution Method Example

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 4T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

Strengthen the inductive hypothesis
by subtracting a low-order term

- Proof
  - $T(n) = O(n^2)$
    There exists positive constants n_0, c_1, c_2 s.t. for all $n \geq n_0, T(n) \leq c_1 n^2 - c_2 n$

Guess

  - Use induction to find the constants n_0, c_1, c_2
    - n = 1, $T(1) \leq c_1 - c_2$ holds for $c_1 \geq c_2 + 1$
    - n > 1,

Verify

$$\begin{aligned} T(n) &\leq 4T(n/2) + bn \\ &\leq 4[c_1(n/2)^2 - c_2(n/2)] + bn \\ &= c_1 n^2 - 2c_2 n + bn \\ &= c_1 n^2 - c_2 n - (c_2 n - bn) \\ &\leq c_1 n^2 - c_2 n \end{aligned}$$

Inductive hypothesis

$c_2 n - bn \geq 0$
e.g. $c_2 \geq b, n \geq 0$

- $T(n) \leq c_1 n^2 - c_2 n$ holds when $c_1 = b + 1, c_2 = b, n_0 = 0$

Solve

Useful Tricks

- Guess based on seen recurrences
- Use the recursion-tree method
- From loose bound to tight bound
- Strengthen the inductive hypothesis by subtracting a low-order term
- Change variables
  - E.g., $T(n) = 2T(\sqrt{n}) + \log n$

1. Change variable: $k = \log n, n = 2^k \to T(2^k) = 2T(2^{k/2}) + k$
2. Change variable again: $S(k) = T(2^k) \to S(k) = 2S(k/2) + k$
3. Solve recurrence $S(k) = \Theta(k \log k) \to T(2^k) = \Theta(k \log k) \to T(n) = \Theta(\log n \log \log n)$

Recursion-Tree Method

Textbook Chapter 4.4 – The recursion-tree method for solving recurrences

Review

- Time Complexity for Merge Sort
- Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n \log n)$$

- Proof

Recursion-Tree Method (遞迴樹法)
Expand the recurrence into a tree and sum up the cost

$$\begin{aligned} T(n) &\leq 2T(\frac{n}{2}) + cn \\ &\leq 2[2T(\frac{n}{4}) + c\frac{n}{2}] + cn = 4T(\frac{n}{4}) + 2cn \\ &\leq 4[2T(\frac{n}{8}) + c\frac{n}{4}] + 2cn = 8T(\frac{n}{8}) + 3cn \\ &\vdots \\ &\leq 2^kT(\frac{n}{2^k}) + kcn \end{aligned}$$

1st expansion

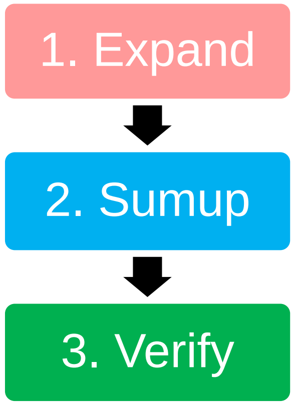
2nd expansion

kth expansion

The expansion stops when $2^k = n$

$$\begin{aligned} T(n) &\leq nT(1) + cn\log_2 n \\ &= O(n) + O(n \log n) \\ &= O(n \log n) \end{aligned}$$

Recursion-Tree Method (遞迴樹法)



- Expand a recurrence into a tree
- Sum up the cost of all nodes as a good guess
- Verify the guess as in the substitution method

- Advantages
  - Promote intuition
  - Generate good guesses for the substitution method

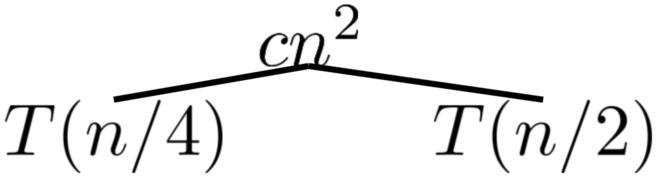
Recursion-Tree Example

$$T(n) = T(n/4) + T(n/2) + cn^2$$

$$T(n)$$

Recursion-Tree Example

$$T(n) = T(n/4) + T(n/2) + cn^2$$

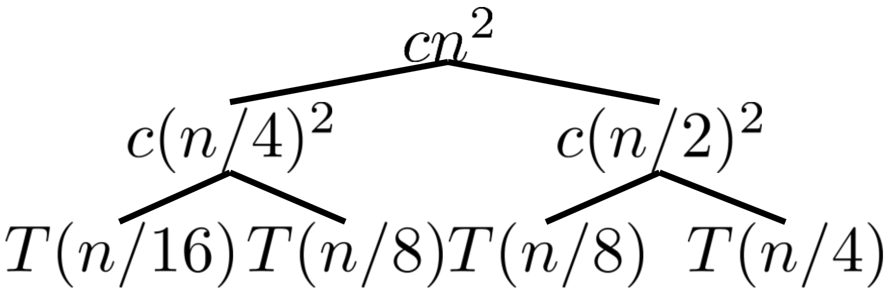


cn^2

$T(n/4)$ $T(n/2)$

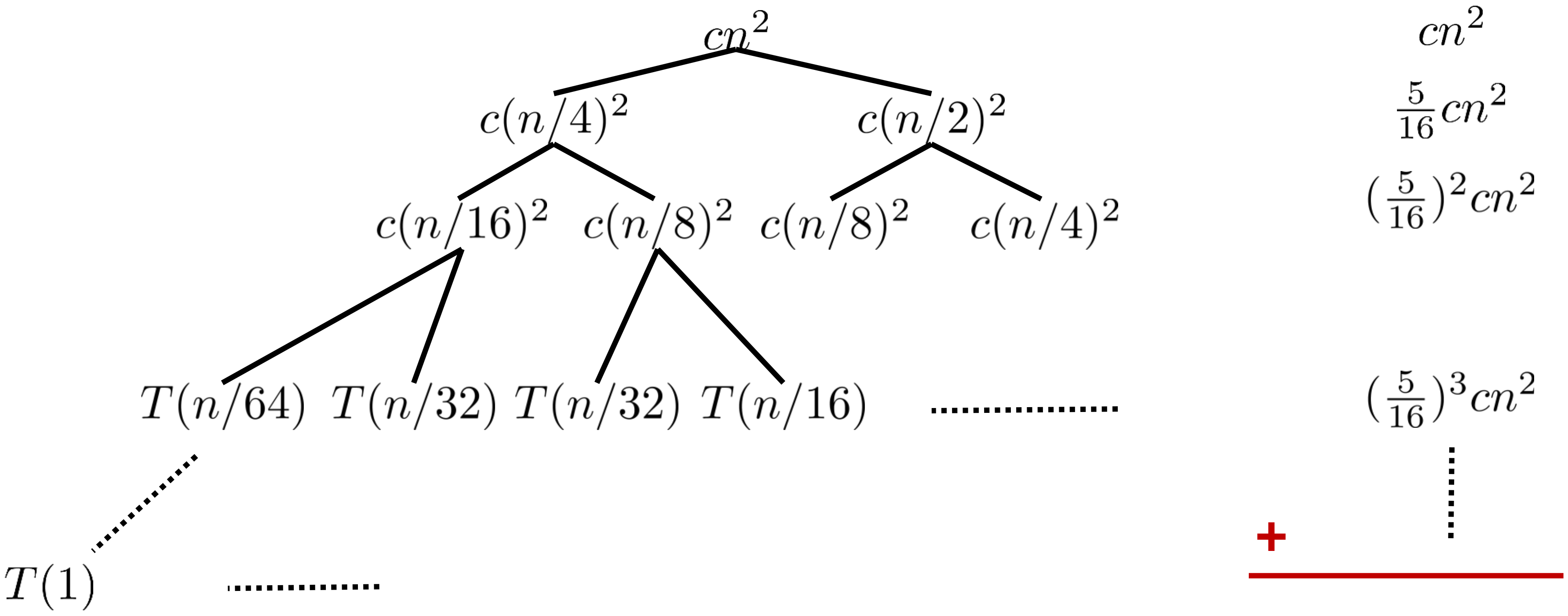
Recursion-Tree Example

$$T(n) = T(n/4) + T(n/2) + cn^2$$



Recursion-Tree Example

$$T(n) = T(n/4) + T(n/2) + cn^2$$



cn^2

$c(n/4)^2$ $c(n/2)^2$

$c(n/16)^2$ $c(n/8)^2$ $c(n/8)^2$ $c(n/4)^2$

$T(n/64)$ $T(n/32)$ $T(n/32)$ $T(n/16)$

$T(1)$

cn^2

$\frac{5}{16}cn^2$

$(\frac{5}{16})^2cn^2$

$(\frac{5}{16})^3cn^2$

+

$$T(n) \leq (1 + \frac{5}{16} + (\frac{5}{16})^2 + (\frac{5}{16})^3 + \cdots)cn^2 = \frac{1}{1-\frac{5}{16}}cn^2 = \frac{16}{11}cn^2 = O(n^2)$$

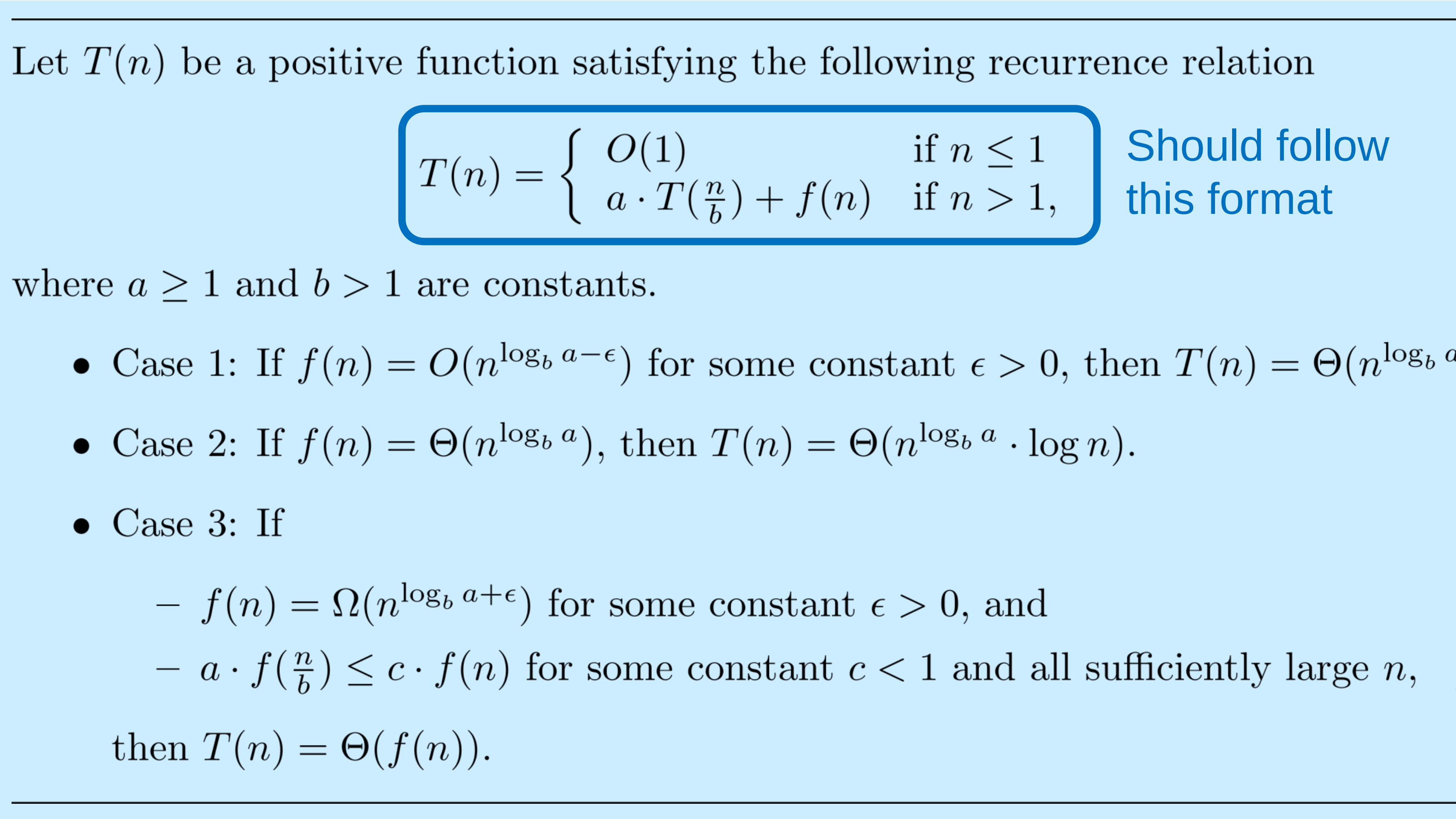
Master Theorem

Textbook Chapter 4.4 – The recursion-tree method for solving recurrences

Master Theorem

The proof is in Ch. 4.6

divide a problem of size n into a subproblems, each of size $\frac{n}{b}$ is solved in time $T\left(\frac{n}{b}\right)$ recursively



Let $T(n)$ be a positive function satisfying the following recurrence relation

$$\boxed{T(n) = \begin{cases} O(1) & \text{if } n \le 1 \\ a \cdot T(\frac{n}{b}) + f(n) & \text{if } n > 1, \end{cases}}$$

Should follow this format

where $a \ge 1$ and $b > 1$ are constants.

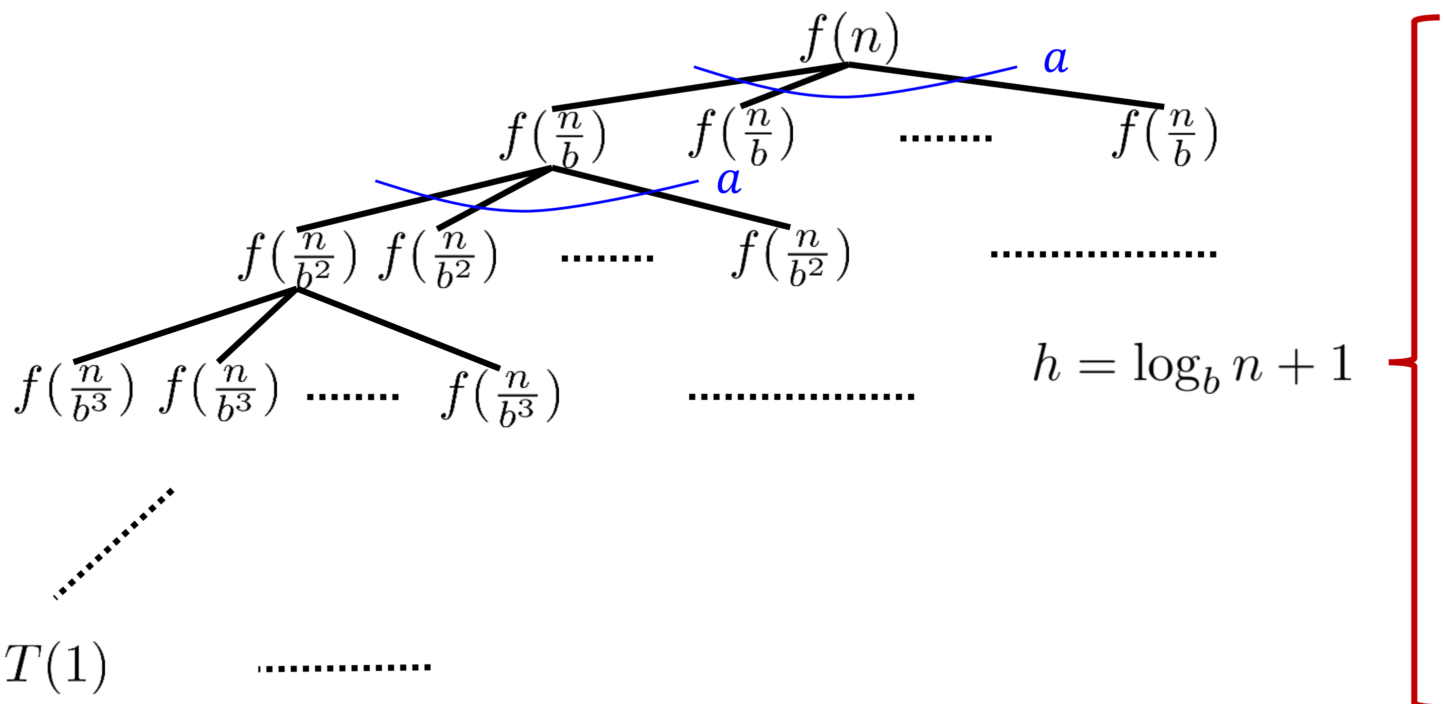
- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and
  - $a \cdot f(\frac{n}{b}) \le c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

compare $f(n)$ with $n^{\log_b a}$

Recursion-Tree for Master Theorem

$$T(n) = aT(\frac{n}{b}) + f(n)$$



$f(n)$
a
$f(\frac{n}{b})$ $f(\frac{n}{b})$ .... $f(\frac{n}{b})$
a
$f(\frac{n}{b^2})$ $f(\frac{n}{b^2})$ .... $f(\frac{n}{b^2})$
$f(\frac{n}{b^3})$ $f(\frac{n}{b^3})$ .... $f(\frac{n}{b^3})$
$T(1)$
$h = \log_b n + 1$

$f(n)$

$af(\frac{n}{b})$

$a^2f(\frac{n}{b^2})$

$a^3f(\frac{n}{b^3})$

$\vdots$

$+\ a^{\log_b n}T(1)$

$$T(n) = f(n) + af(\frac{n}{b}) + a^2f(\frac{n}{b^2}) + a^3f(\frac{n}{b^3}) + \cdots + a^{\log_b n}T(1)$$

$$a^{\log_b n}T(1) = n^{\log_b a}T(1)$$

Three Cases

- $T(n) = aT(\frac{n}{b}) + f(n)$
  - $a \geq 1$, the number of subproblems
  - $b > 1$, the factor by which the subproblem size decreases
  - $f(n)$ = work to divide/combine subproblems

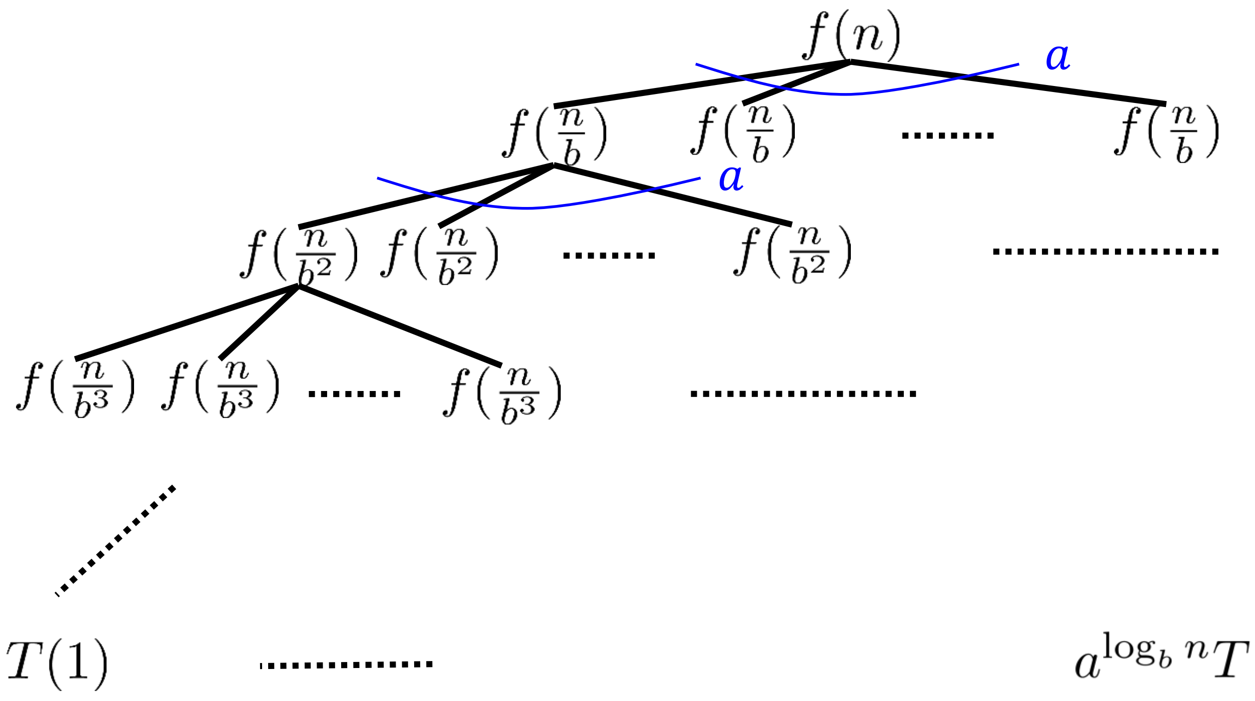
$$T(n) = f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + a^3 f(\frac{n}{b^3}) + \cdots + n^{\log_b a} T(1)$$

- Compare $f(n)$ with $n^{\log_b a}$

1. Case 1: $f(n)$ grows polynomially slower than $n^{\log_b a}$
2. Case 2: $f(n)$ and $n^{\log_b a}$ grow at similar rates
3. Case 3: $f(n)$ grows polynomially faster than $n^{\log_b a}$

Case 1:
Total cost dominated by the leaves

$$T(n) = 9T(\tfrac{n}{3}) + n, T(1) = 1$$


$f(n)$
a
$f(\frac{n}{b})$ $f(\frac{n}{b})$ .... $f(\frac{n}{b})$
a
$f(\frac{n}{b^2})$ $f(\frac{n}{b^2})$ .... $f(\frac{n}{b^2})$
$f(\frac{n}{b^3})$ $f(\frac{n}{b^3})$ .... $f(\frac{n}{b^3})$
$T(1)$

$$f(n) = n$$

$$af(\tfrac{n}{b}) = \tfrac{9}{3}n$$

$$a^2 f(\tfrac{n}{b^2}) = (\tfrac{9}{3})^2 n$$

$$a^3 f(\tfrac{n}{b^3}) = (\tfrac{9}{3})^3 n$$

$$a^{\log_b n} T(1) = 9^{\log_3 n} = (\tfrac{9}{3})^{\log_3 n} n$$

$f(n)$ grows polynomially slower than $n^{\log_b a}$

Case 1:
Total cost dominated by the leaves

$$T(n) = 9T(\tfrac{n}{3}) + n, T(1) = 1$$

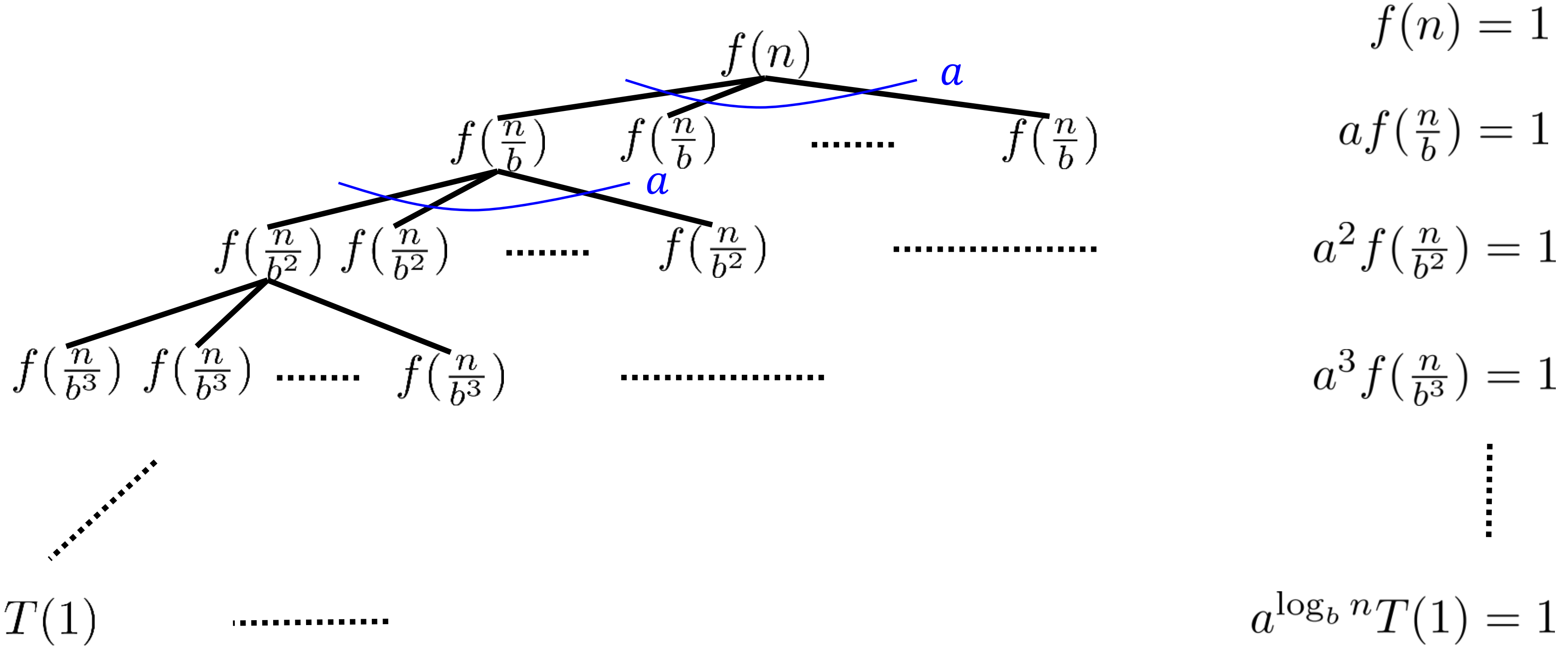
$$\begin{aligned}
T(n) &= (1 + \frac{9}{3} + (\frac{9}{3})^2 + \cdots + (\frac{9}{3})^{\log_3 n})n \\
&= \frac{(\frac{9}{3})^{1+\log_3 n} - 1}{3-1} n \\
&= \frac{3n}{2} \cdot \frac{9^{\log_3 n}}{3^{\log_3 n}} - \frac{1}{2} n \\
&= \frac{3n}{2} \cdot \frac{n^{\log_3 9}}{n} - \frac{1}{2} n \\
&= \Theta(n^{\log_3 9}) = \Theta(n^2)
\end{aligned}$$

- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

Case 2:

Total cost evenly distributed among levels

$$T(n) = T(\tfrac{2n}{3}) + 1, T(1) = 1$$


$f(n)$
a
$f(\frac{n}{b})$ $f(\frac{n}{b})$ .... $f(\frac{n}{b})$
a
$f(\frac{n}{b^2})$ $f(\frac{n}{b^2})$ .... $f(\frac{n}{b^2})$ ....
$f(\frac{n}{b^3})$ $f(\frac{n}{b^3})$ .... $f(\frac{n}{b^3})$ ....
$T(1)$ ....

$f(n) = 1$

$af(\frac{n}{b}) = 1$

$a^2 f(\frac{n}{b^2}) = 1$

$a^3 f(\frac{n}{b^3}) = 1$

$a^{\log_b n} T(1) = 1$

$f(n)$ and $n^{\log_b a}$ grow at similar rates

Case 2:
Total cost evenly distributed among levels

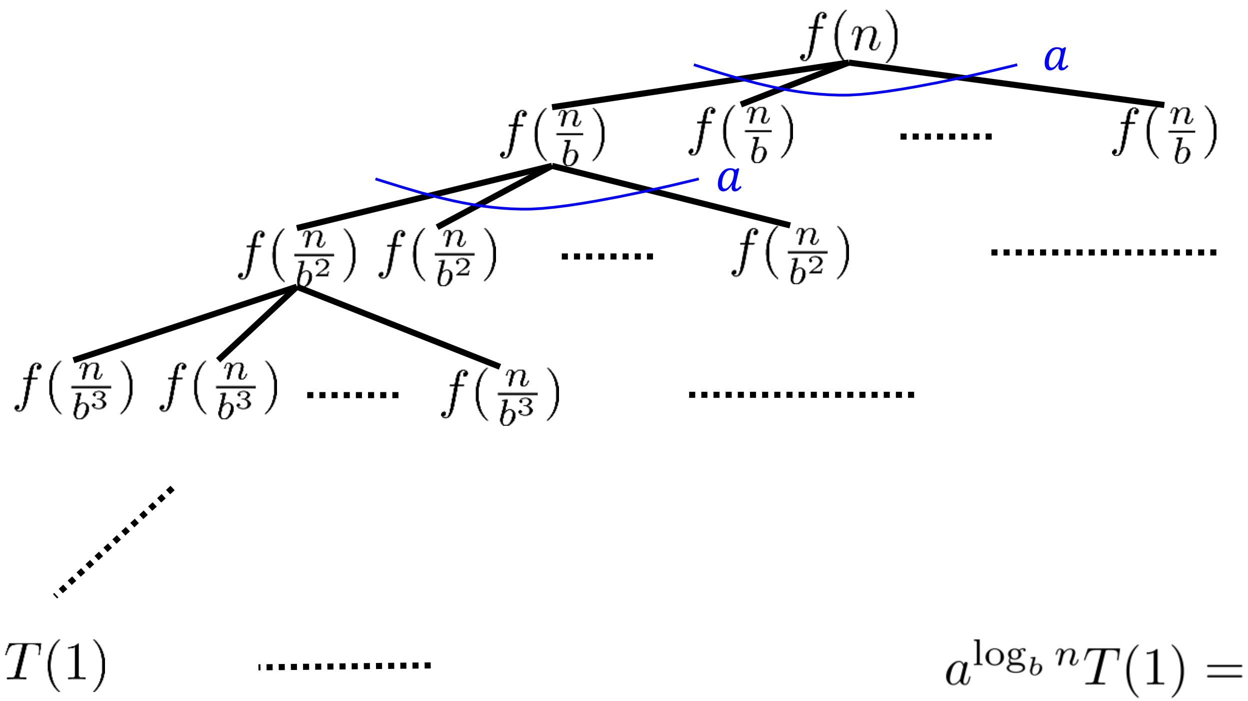
$$T(n) = T(\tfrac{2n}{3}) + 1, T(1) = 1$$

$$\begin{aligned} T(n) &= 1 + 1 + 1 + \cdots + 1 \\ &= \log_{\frac{3}{2}} n + 1 \\ &= \Theta(\log n) \end{aligned}$$

- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.

Case 3:
Total cost dominated by root cost

$$T(n) = 3T(\frac{n}{4}) + n^5, T(1) = 1$$


$f(n)$
a
$f(\frac{n}{b})$ $f(\frac{n}{b})$ .... $f(\frac{n}{b})$
a
$f(\frac{n}{b^2})$ $f(\frac{n}{b^2})$ .... $f(\frac{n}{b^2})$ ....
$f(\frac{n}{b^3})$ $f(\frac{n}{b^3})$ .... $f(\frac{n}{b^3})$ ....
$T(1)$ ....

$$f(n) = n^5$$

$$af(\frac{n}{b}) = 3(\frac{n}{4})^5$$

$$a^2 f(\frac{n}{b^2}) = 3^2(\frac{n}{4^2})^5$$

$$a^3 f(\frac{n}{b^3}) = 3^3(\frac{n}{4^3})^5$$

$$\vdots$$

$$a^{\log_b n} T(1) = 3^{\log_4 n} = 3^{\log_4 n}(\frac{n}{4^{\log_4 n}})^5$$

$f(n)$ grows polynomially faster than $n^{\log_b a}$

Case 3:
Total cost dominated by root cost

$$T(n) = 3T(\tfrac{n}{4}) + n^5, T(1) = 1$$

$$T(n) = (1 + \tfrac{3}{4^5} + (\tfrac{3}{4^5})^2 + \cdots + (\tfrac{3}{4^5})^{\log_4 n})n^5$$

$$T(n) > n^5$$

$$T(n) \leq \frac{1}{1 - \frac{3}{4^5}} n^5$$

$$T(n) = \Theta(n^5)$$

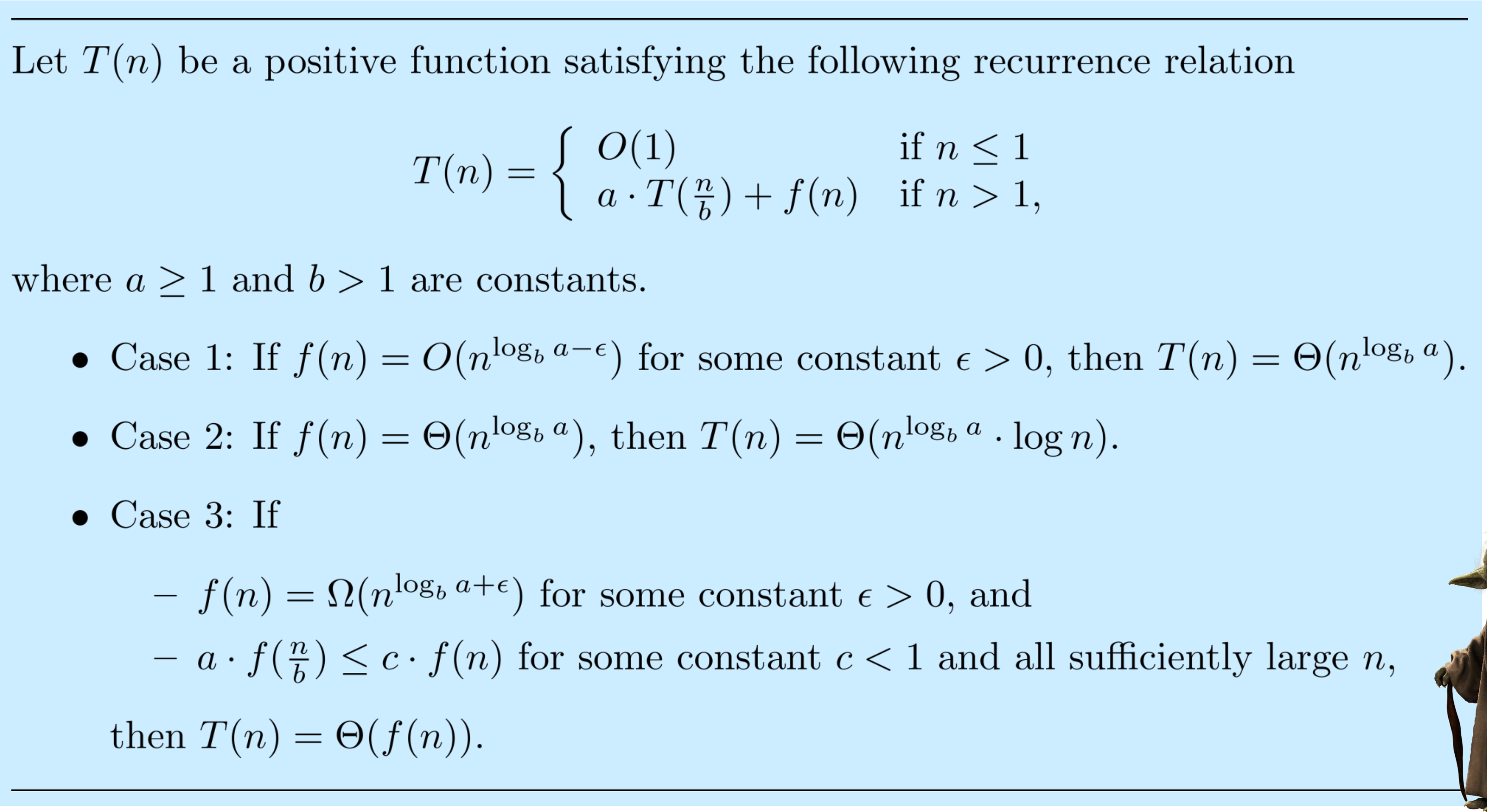
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and
  - $\boxed{a \cdot f(\frac{n}{b}) \leq c \cdot f(n)}$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

Master Theorem

The proof is in Ch. 4.6

divide a problem of size n into a subproblems, each of size $\frac{n}{b}$ is solved in time $T\left(\frac{n}{b}\right)$ recursively



Let $T(n)$ be a positive function satisfying the following recurrence relation

$$T(n) = \begin{cases} O(1) & \text{if } n \le 1 \\ a \cdot T(\frac{n}{b}) + f(n) & \text{if } n > 1, \end{cases}$$

where $a \geq 1$ and $b > 1$ are constants.

- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
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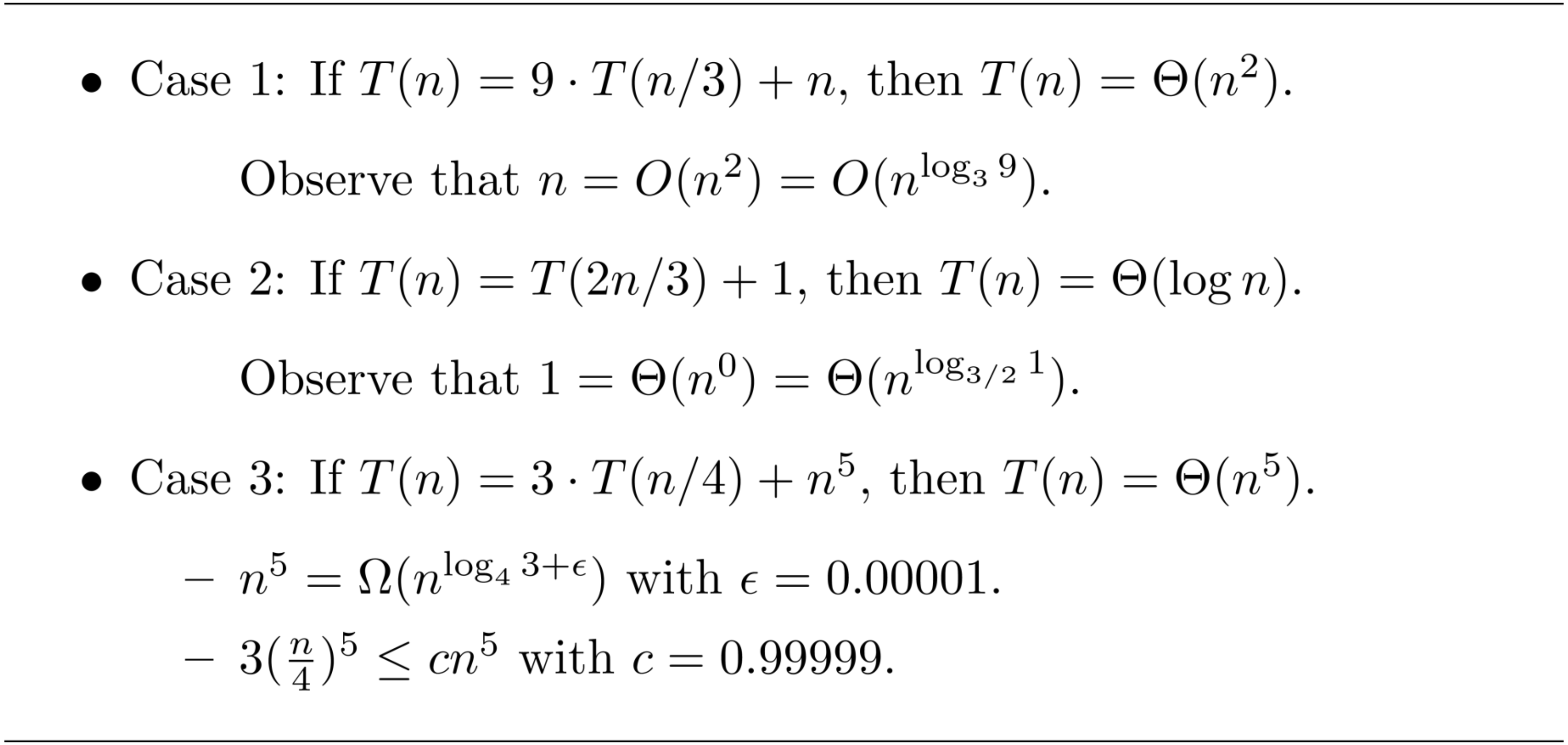
  then $T(n) = \Theta(f(n))$.

compare $f(n)$ with $n^{\log_b a}$

Examples

compare $f(n)$ with $n^{\log_b a}$



- Case 1: If $T(n) = 9 \cdot T(n/3) + n$, then $T(n) = \Theta(n^2)$.

  Observe that $n = O(n^2) = O(n^{\log_3 9})$.

- Case 2: If $T(n) = T(2n/3) + 1$, then $T(n) = \Theta(\log n)$.

  Observe that $1 = \Theta(n^0) = \Theta(n^{\log_{3/2} 1})$.

- Case 3: If $T(n) = 3 \cdot T(n/4) + n^5$, then $T(n) = \Theta(n^5)$.
  - $n^5 = \Omega(n^{\log_4 3+\epsilon})$ with $\epsilon = 0.00001$.
  - $3(\frac{n}{4})^5 \le cn^5$ with $c = 0.99999$.

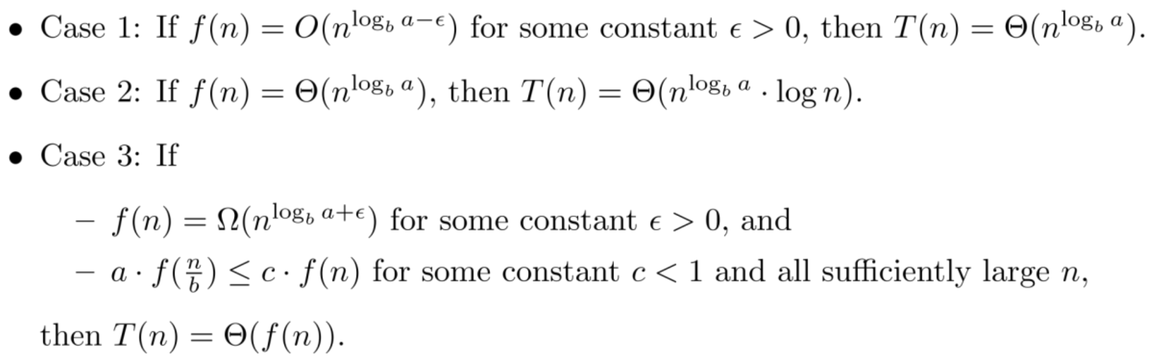
Floors and Ceilings

- Master theorem can be extended to recurrences with floors and ceilings
- The proof is in the Ch. 4.6

$$T(n) = aT(\lceil \tfrac{n}{b} \rceil) + f(n)$$

$$T(n) = aT(\lfloor \tfrac{n}{b} \rfloor) + f(n)$$

Theorem 1



- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and
  - $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

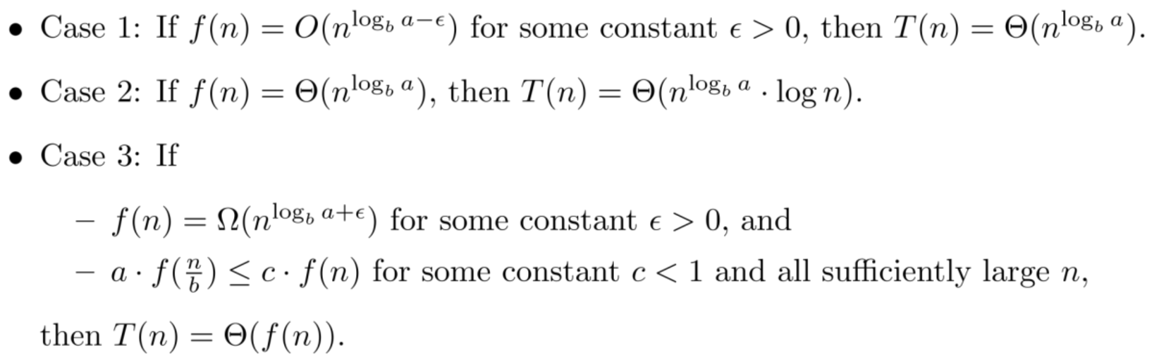
$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases} \quad \Rightarrow \quad T(n) = O(n \log n)$$

- Case 2

$$f(n) = \Theta(n) = \Theta(n^1) = \Theta(n^{\log_2 2}) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(f(n) \log n) = O(n \log n)$$

Theorem 2



- Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and
  - $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

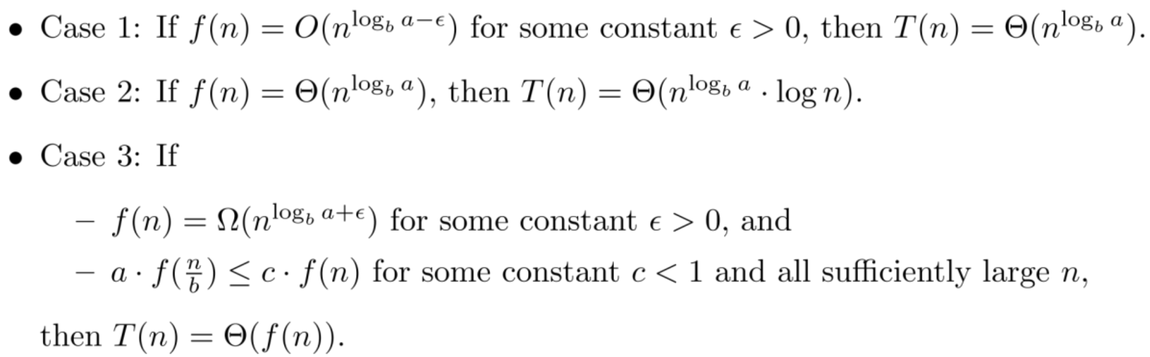
$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(1) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(n)$$

- Case 1

$$f(n) = O(1) = O(n) = O(n^{\log_2 2}) = O(n^{\log_b a})$$

$$T(n) = \Theta(n^{\log_2 2}) = \Theta(n)$$

Theorem 3



- Case 1: If $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
- Case 2: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log n)$.
- Case 3: If
  - $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant $\epsilon > 0$, and
  - $a \cdot f(\frac{n}{b}) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n,

  then $T(n) = \Theta(f(n))$.

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(n/2) + O(1) & \text{if } n \geq 2 \end{cases} \Rightarrow T(n) = O(\log n)$$

- Case 2

$$f(n) = \Theta(1) = \Theta(n^0) = \Theta(n^{\log_2 1}) = \Theta(n^{\log_b a})$$

$$T(n) = \Theta(f(n) \log n) = O(\log n)$$

To Be Continue…

Question?

Important announcement will be sent to @ntu.edu.tw mailbox & post to the course website

Course Website: http://ada.miulab.tw
Email: ada-ta@csie.ntu.edu.tw