P/NP, NP Complete, NP Hard concept introduction

NP Completeness (2)

Dec 13th, 2018

Algorithm Design and Analysis

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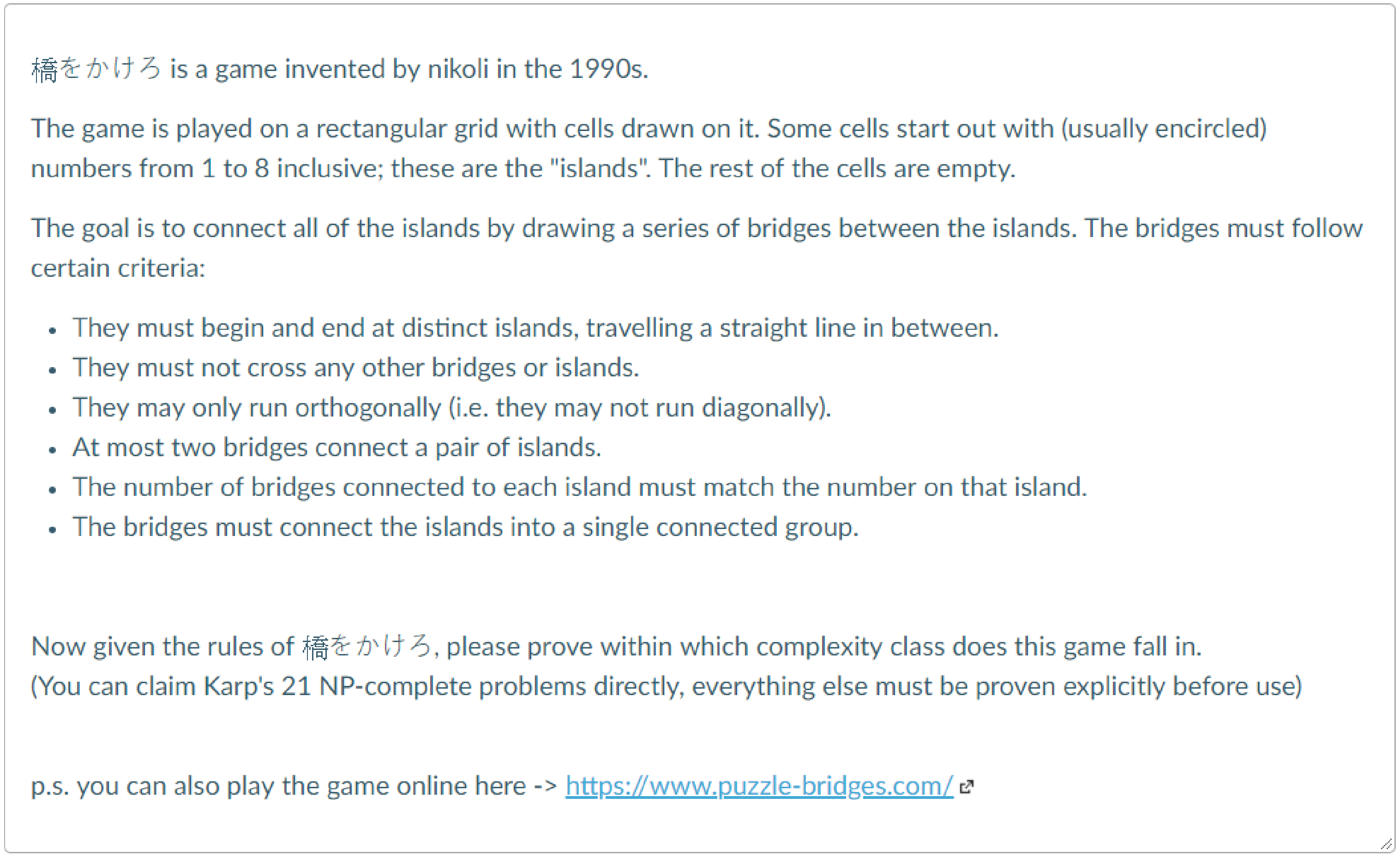
Slides credited from Hsueh-I Lu & Hsu-Chun Hsiao

Announcement

- Homework 4 released
  - Due on 1/3 (Thur) 14:20 (three weeks later)
- Mini-HW 10 released
  - Due on 12/20 (Thur) 14:20
- Next week
  - Break
  - Watch online videos
- Class 12/27
  - A small test for Christmas (optional)

Mini-HW 10



橋をかけろ is a game invented by nikoli in the 1990s.

The game is played on a rectangular grid with cells drawn on it. Some cells start out with (usually encircled) numbers from 1 to 8 inclusive; these are the "islands". The rest of the cells are empty.

The goal is to connect all of the islands by drawing a series of bridges between the islands. The bridges must follow certain criteria:

- They must begin and end at distinct islands, travelling a straight line in between.
- They must not cross any other bridges or islands.
- They may only run orthogonally (i.e. they may not run diagonally).
- At most two bridges connect a pair of islands.
- The number of bridges connected to each island must match the number on that island.
- The bridges must connect the islands into a single connected group.

Now given the rules of 橋をかけろ, please prove within which complexity class does this game fall in.
(You can claim Karp's 21 NP-complete problems directly, everything else must be proven explicitly before use)

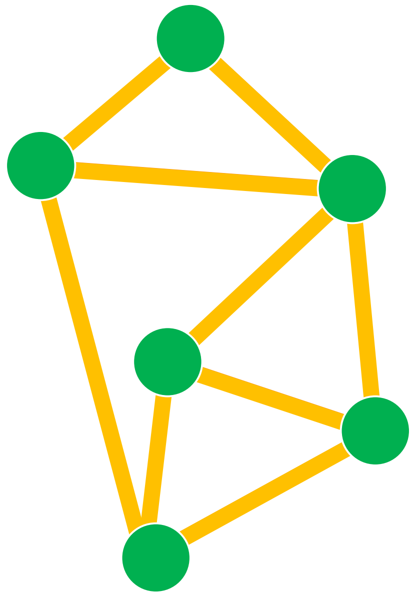
p.s. you can also play the game online here -> https://www.puzzle-bridges.com/

Outline

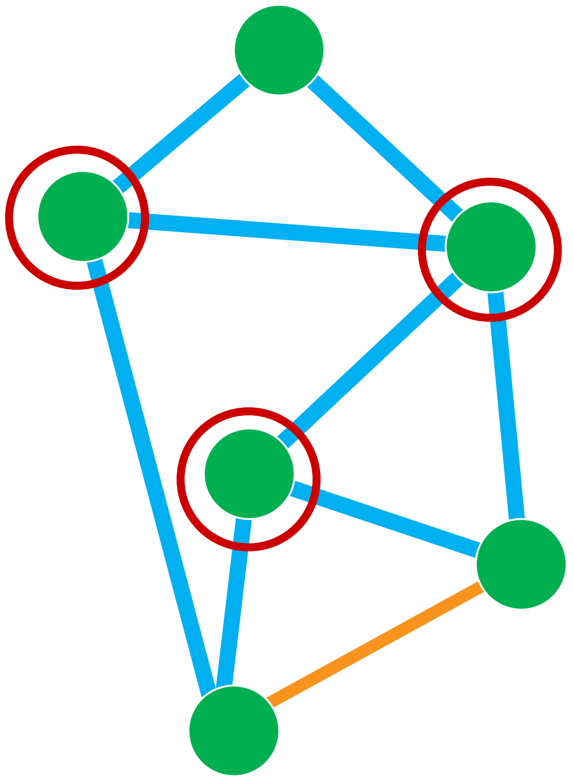
- Complexity Classes
  - P v.s. NP
  - NP, NP-Complete, NP-Hard
- Polynomial-Time Reduction

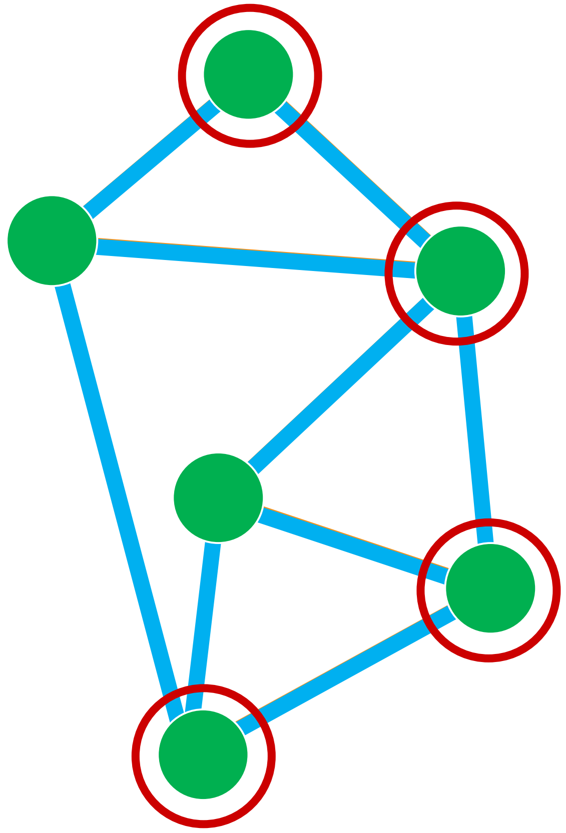
Vertex Cover Problem (路燈問題)

- Input: a graph *G*
- Output: a smallest vertex subset of *G* that covers all edges of *G*.
- Known to be NP-complete



Illustration





Vertex Cover (Decision Version)

- Input: a Graph *G* and an integer *k*.
- Output: Does *G* contain a vertex cover of size no more than *k*?

- Original problem → optimization problem
  - 原先的路燈問題是要算出放路燈的方法
- Yes/No → decision problem
  - 問*k*盞路燈夠不夠照亮整個公園

Non-Deterministic Algorithm

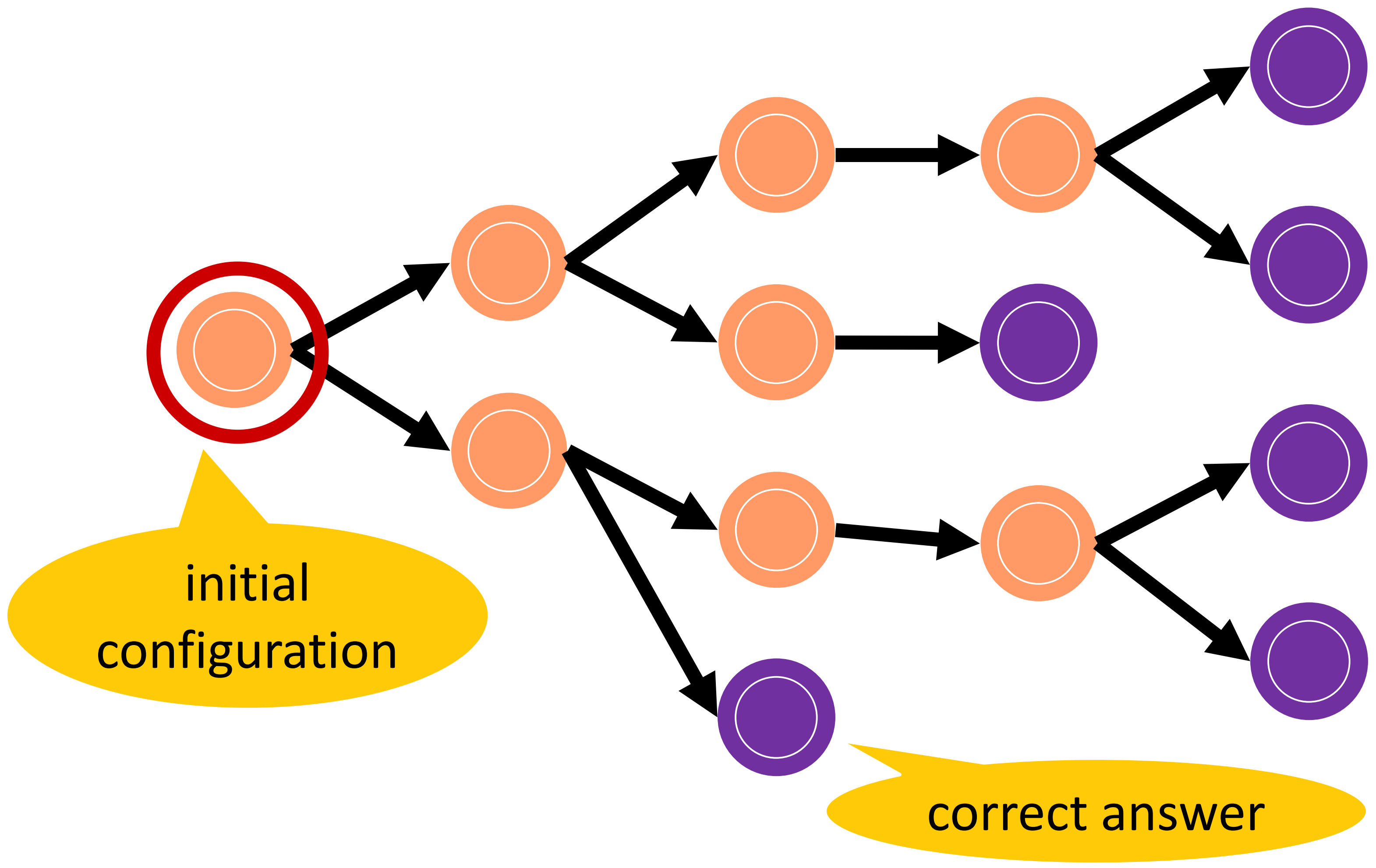
```
Non-Deterministic-Vertex-Cover(G, k)
  set S = {}
  for each vertex x of G
    non-deterministically insert x to S
  if |S| > k
    output no
  if S is not a vertex cover
    output no
  output yes
```

Algorithm Correctness

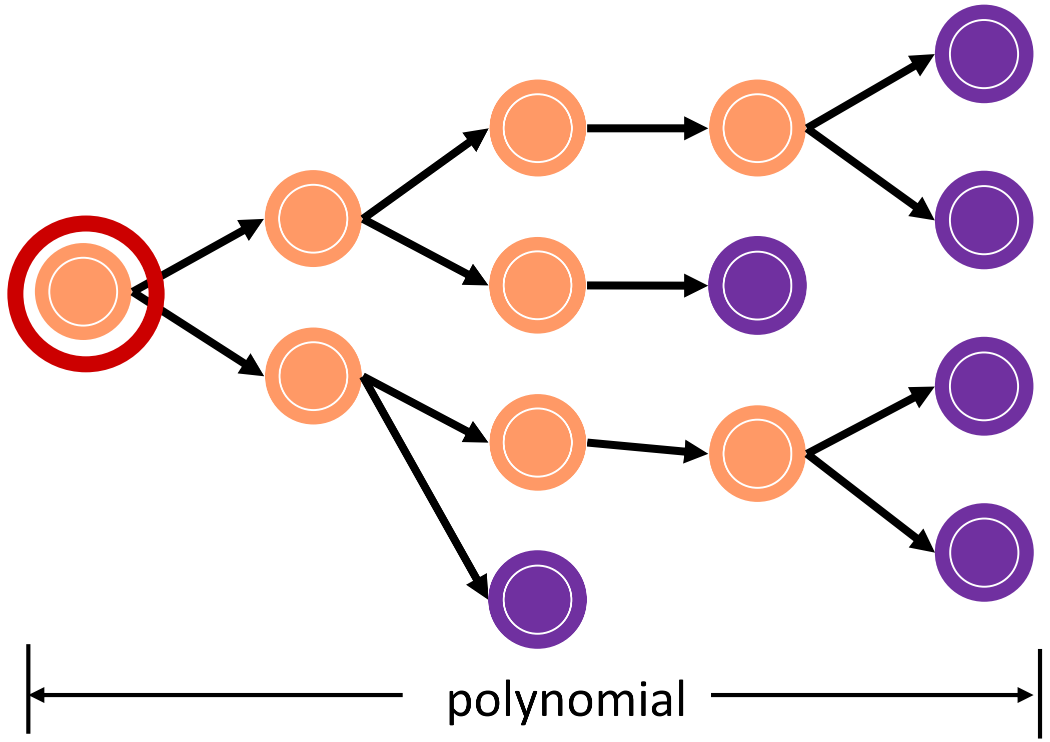
```
Non-Deterministic-Vertex-Cover(G, k)
  set S = {}
  for each vertex x of G
    non-deterministically insert x to S
  if |S| > k
    output no
  if S is not a vertex cover
    output no
  output yes
```

- If the correct answer is *yes*, then there is a computation path of the algorithm that leads to *yes*.
  - 至少有一條路是對的
- If the correct answer is *no*, then all computation paths of the algorithm lead to *no*.
  - 每一條路都是對的

Non-Deterministic Problem Solving



Non-Deterministic Polynomial



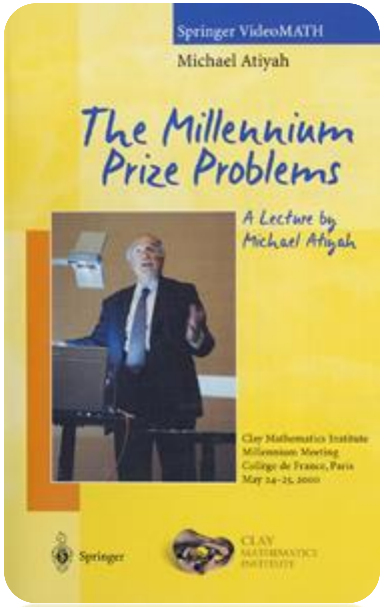
"solved" in non-deterministic polynomial time
= "verified" in polynomial time

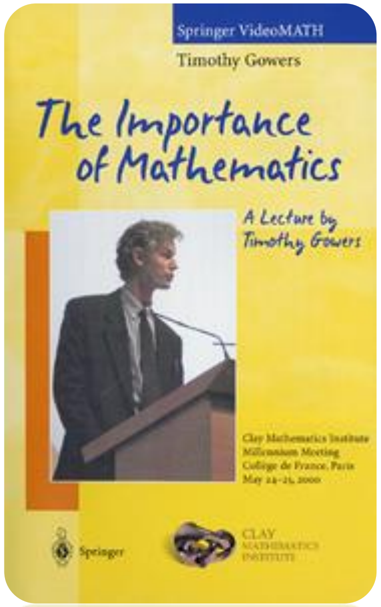
P ⊆ NP or NP ⊆ P?

- P ⊆ NP
  - A problem solvable in polynomial time is verifiable in polynomial time as well
- Any NP problem can be solved in (deterministically) exponential time?
  - Yes
- Any NP problem can be solved in (deterministically) polynomial time?
  - Open problem

US$1,000,000 Per Problem

- http://www.claymath.org/millennium-problems


Springer VideoMATH
Michael Atiyah
The Millennium Prize Problems
A Lecture by Michael Atiyah
Clay Mathematics Institute Millennium Meeting Collège de France, Paris May 24–25, 2000
Springer
CLAY MATHEMATICS INSTITUTE


Springer VideoMATH
Timothy Gowers
The Importance of Mathematics
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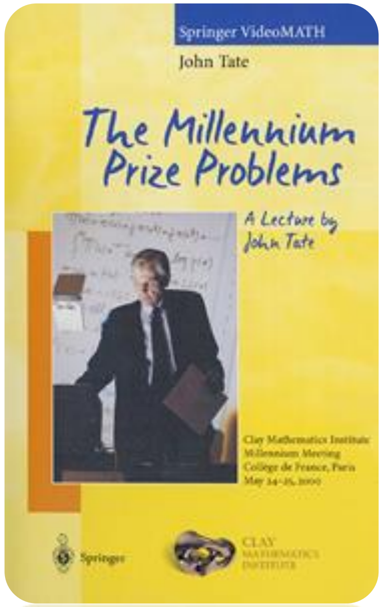
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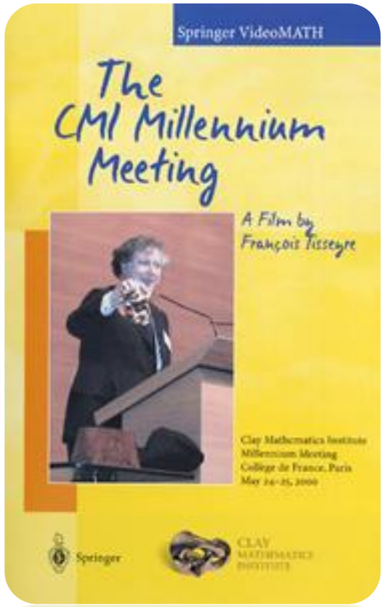
Springer VideoMATH
The CMI Millennium Meeting
A Film by François Tisseyre
Clay Mathematics Institute Millennium Meeting Collège de France, Paris May 24–25, 2000
Springer
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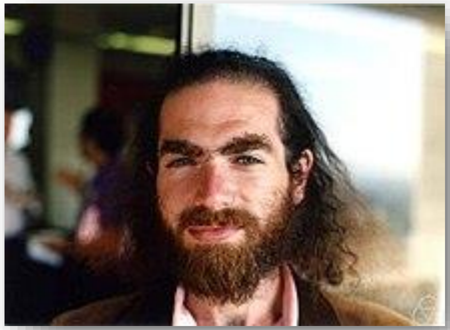
Millennium Problems

- Yang–Mills and Mass Gap
- Riemann Hypothesis
- P vs NP Problem
- Navier–Stokes Equation
- Hodge Conjecture
- Poincaré Conjecture (solved by Grigori Perelman)
- Birch and Swinnerton-Dyer Conjecture



Grigori Perelman
Fields Medal (2006), declined
Millennium Prize (2010), declined

Vinay Deolalikar

- Aug 2010 claimed a proof of P is not equal to NP.

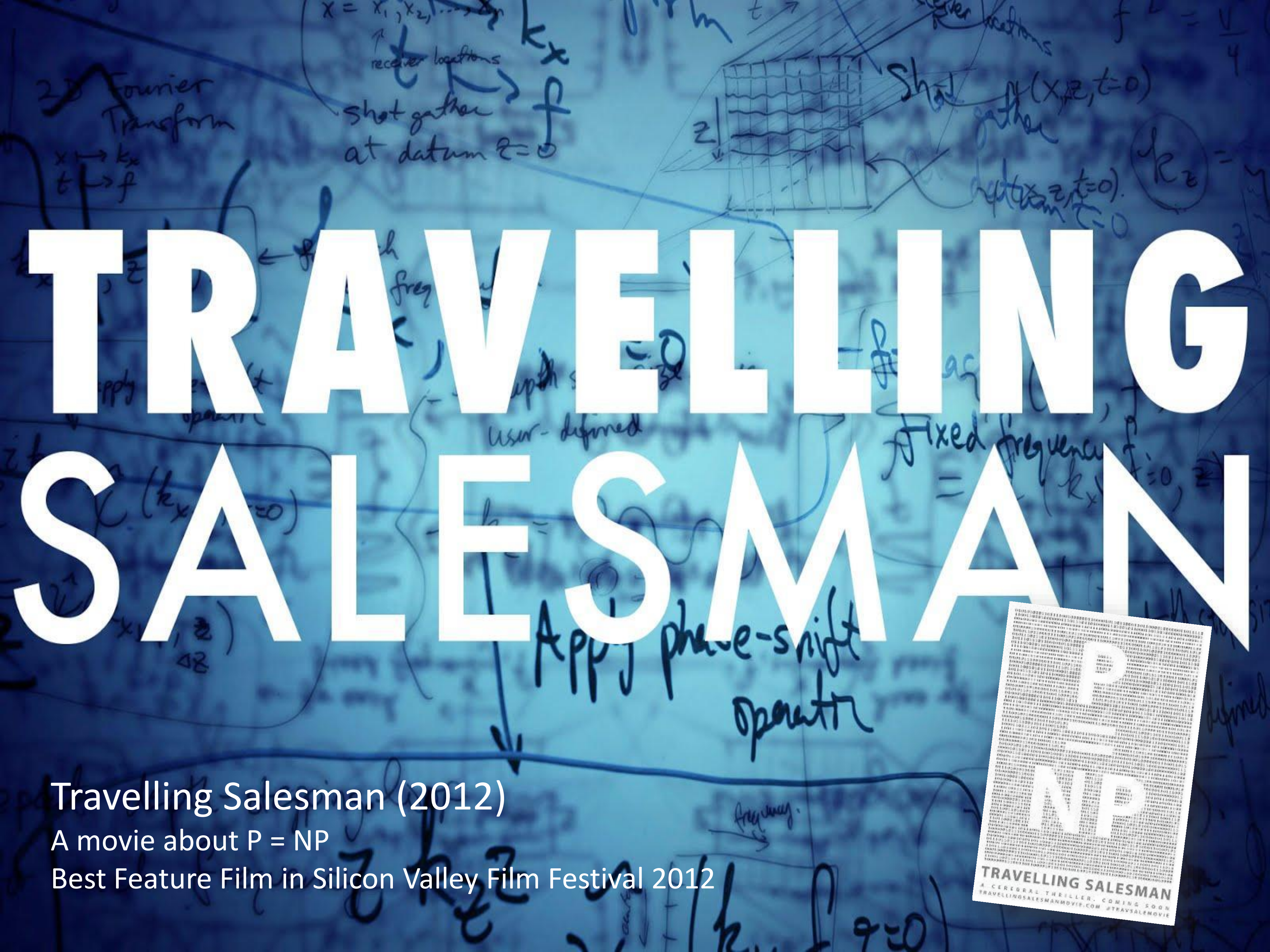
If P = NP

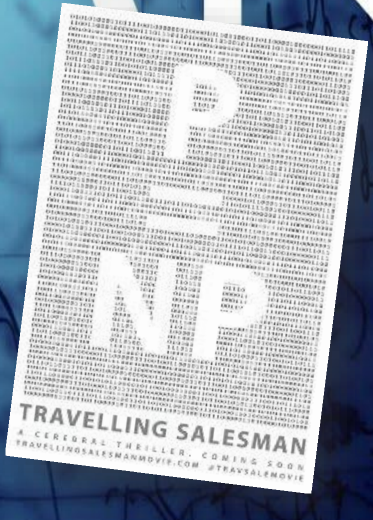
- problems that are verifiable → solvable

- public-key cryptography will be broken

"If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss..." – Scott Aaronson, MIT

Widespread belief in P ≠ NP




P
=
NP
TRAVELLING SALESMAN
A CEREBRAL THRILLER. COMING SOON
TRAVELLINGSALESMANMOVIE.COM @TRAVSALEMOVIE

Travelling Salesman (2012)

A movie about P = NP

Best Feature Film in Silicon Valley Film Festival 2012

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NP, NP-Complete, NP-Hard

NP-Hardness

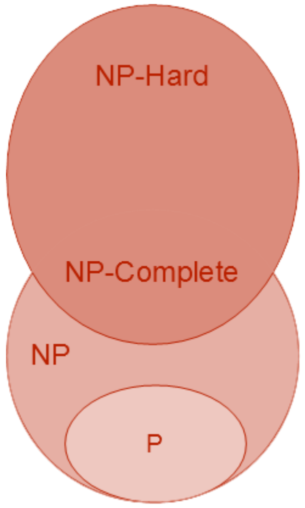
- A problem is NP-hard if it is as least as hard as all NP problems.
- In other words, a problem *X* is NP-hard if the following condition holds:
    - If *X* can be solved in (deterministic) polynomial time, then all NP problems can be solved in (deterministic) polynomial time.

NP-Completeness (NPC)

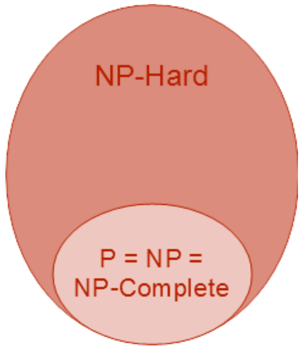
- A problem is NP-complete if
  - it is NP-hard and
  - it is in NP.
- In other words, an NP-complete problem is one of the “hardest” problems in the class NP.
- In other words, an NP-complete problem is a hardness representative problem of the class NP.
- Hardest in NP → solving one NPC can solve all NP problems (“complete”)
- It is wildly believed that NPC problems have no polynomial-time solution → good reference point to judge whether a problem is in P
  - We can decide whether a problem is “too hard to solve” by showing it is as hard as an NPC problem
  - We then focus on designing approximate algorithms or solving special cases

Complexity Classes

- Class P: class of problems that can be solved in $O(n^k)$
- Class NP: class of problems that can be verified in $O(n^k)$
- Class NP-hard: class of problems that are “at least as hard as all NP problems”
- Class NP-complete: class of problems in both NP and NP-hard

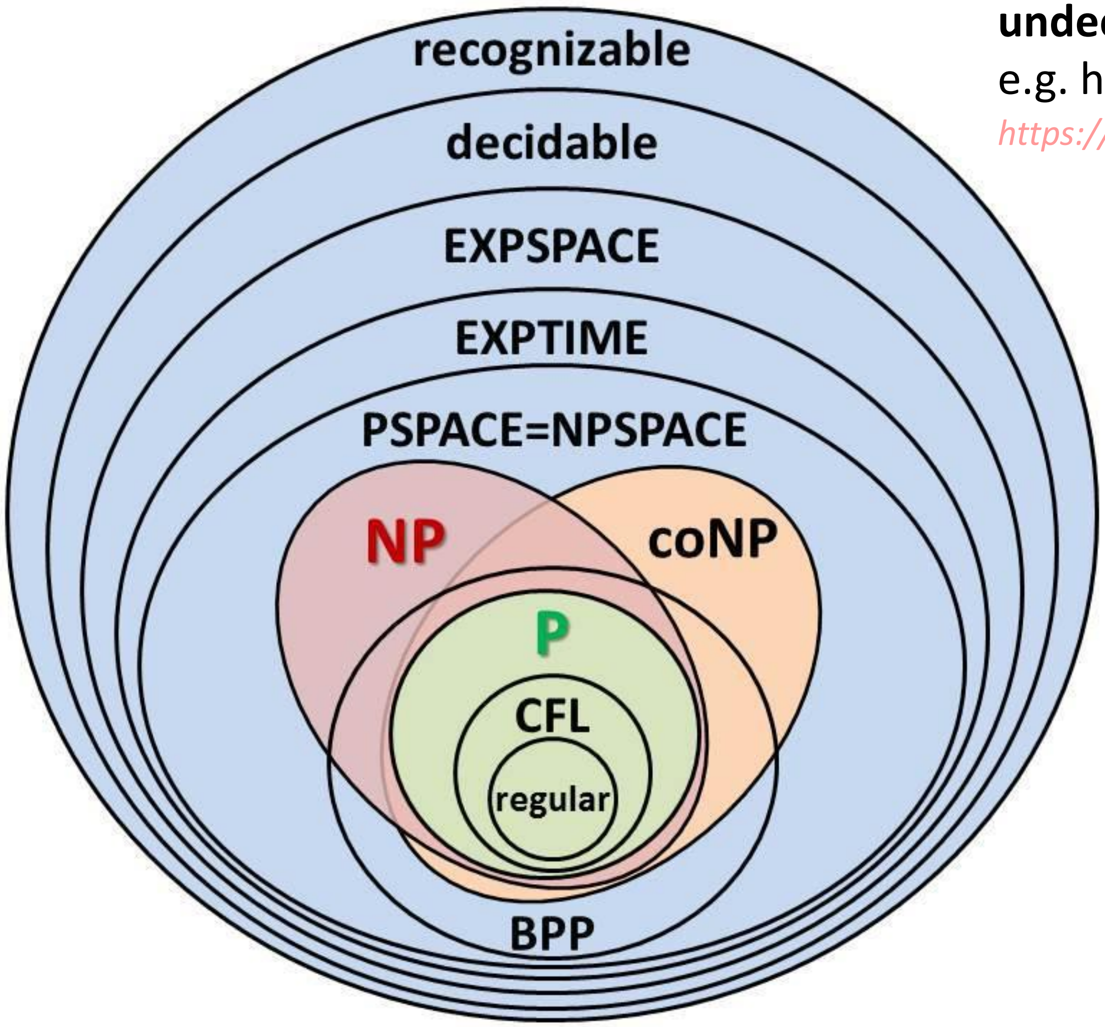


P ≠ NP



P = NP

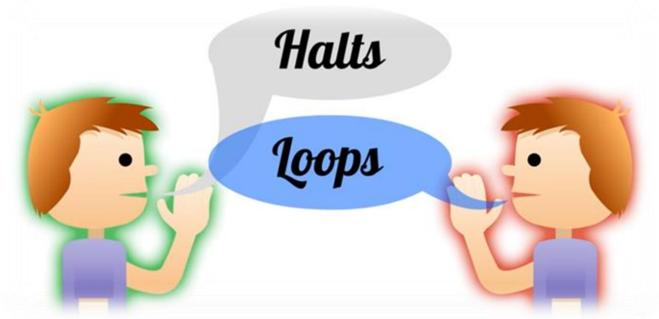
More Complexity Classes



undecidable: no algorithm;
e.g. halting problem
https://www.youtube.com/watch?v=wGLQiHXHWNk

An Undecidable Problem – Halting Problem

- Halting problem is to determine whether a program p halts on input x
- Proof for undecidable via a counterexample
  - Suppose h can determine whether a program p halts on input x
  - $h(p, x)$ = return (p halts on input x)
  - Define g(p) = if h(p,p) is 0 then return 0 else HANG
  - → g(g) = if h(g,g) is 0 then return 0 else HANG
- Both cases contradict the assumption:
  1. g halts on g: then h(g,g)=1, which would make g(g) hang
  2. g does not halt on g: then h(g,g)=0, which would make g(g) halt



Complexity Classes

- Which one is in P?

Shortest Simple Path

Longest Simple Path

Euler Tour

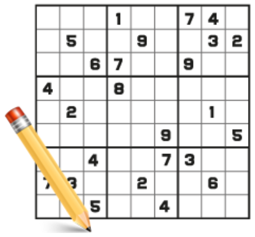
Hamitonian Cycle

LCS with 2 Input Sequences

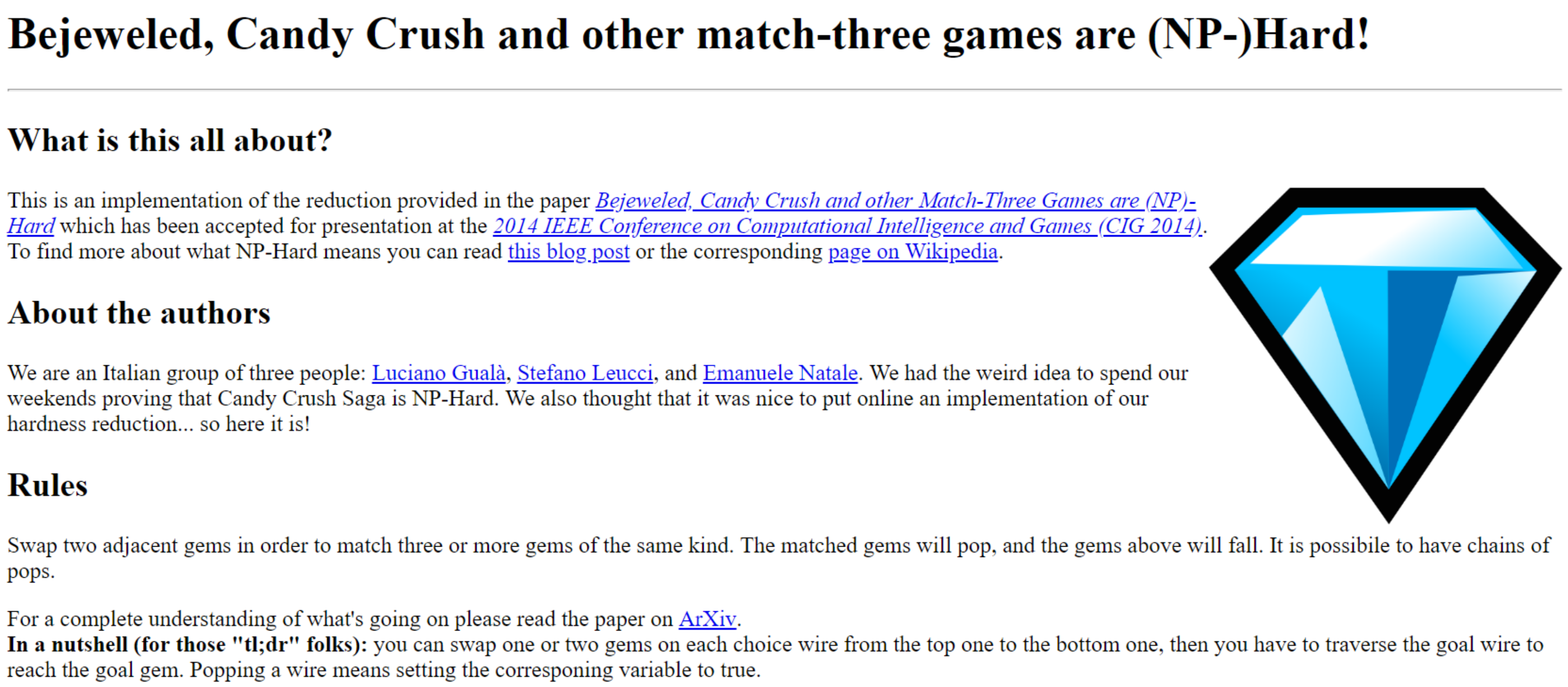
LCS with Arbitrary Input Sequences

Degree-Constrained Spanning Tree

Minimal Spanning Tree



Candy Crush is NP-Hard



Bejeweled, Candy Crush and other match-three games are (NP-)Hard!

What is this all about?

This is an implementation of the reduction provided in the paper *Bejeweled, Candy Crush and other Match-Three Games are (NP)-Hard* which has been accepted for presentation at the *2014 IEEE Conference on Computational Intelligence and Games (CIG 2014)*. To find more about what NP-Hard means you can read this blog post or the corresponding page on Wikipedia.

About the authors

We are an Italian group of three people: Luciano Gualà, Stefano Leucci, and Emanuele Natale. We had the weird idea to spend our weekends proving that Candy Crush Saga is NP-Hard. We also thought that it was nice to put online an implementation of our hardness reduction... so here it is!

Rules

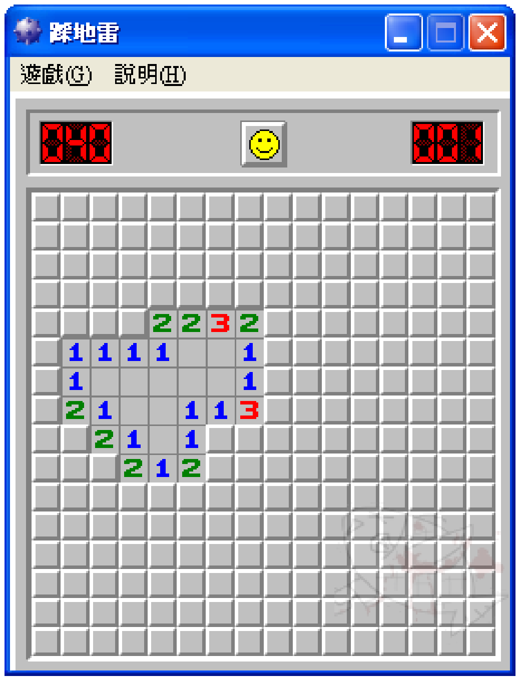
Swap two adjacent gems in order to match three or more gems of the same kind. The matched gems will pop, and the gems above will fall. It is possibile to have chains of pops.

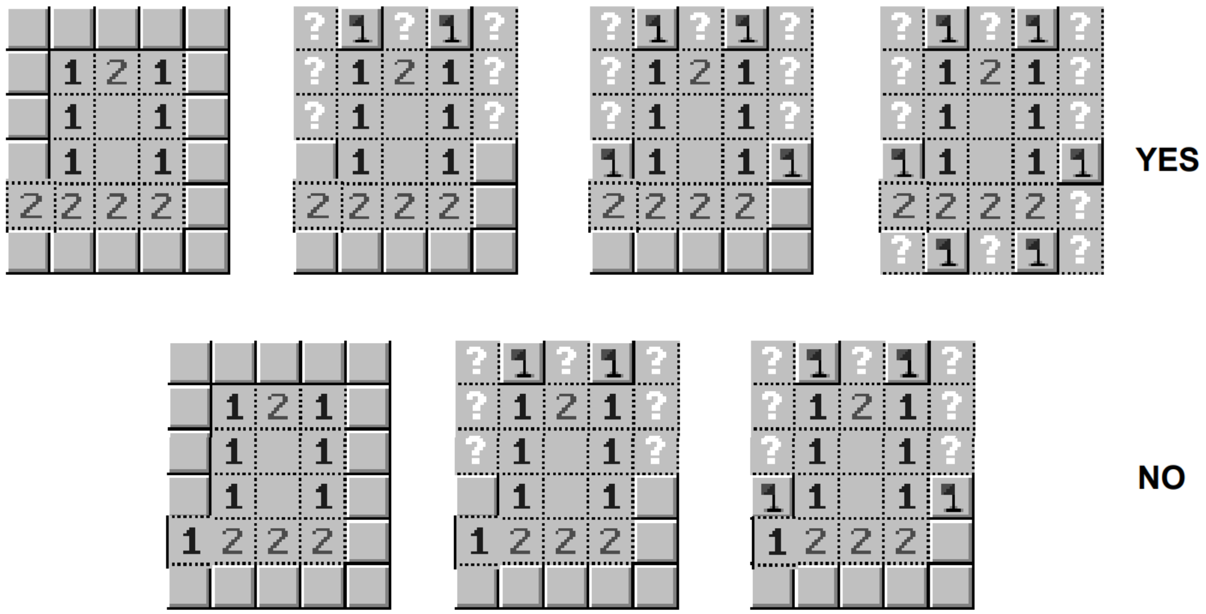
For a complete understanding of what's going on please read the paper on ArXiv.
In a nutshell (for those "tl;dr" folks): you can swap one or two gems on each choice wire from the top one to the bottom one, then you have to traverse the goal wire to reach the goal gem. Popping a wire means setting the corresponing variable to true.

Sudoku is NPC

Minesweeper Consistency is NPC

- Minesweeper Consistency: Given a state of what purports to be a Minesweeper games, is it logically consistent?





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Polynomial-Time Reduction

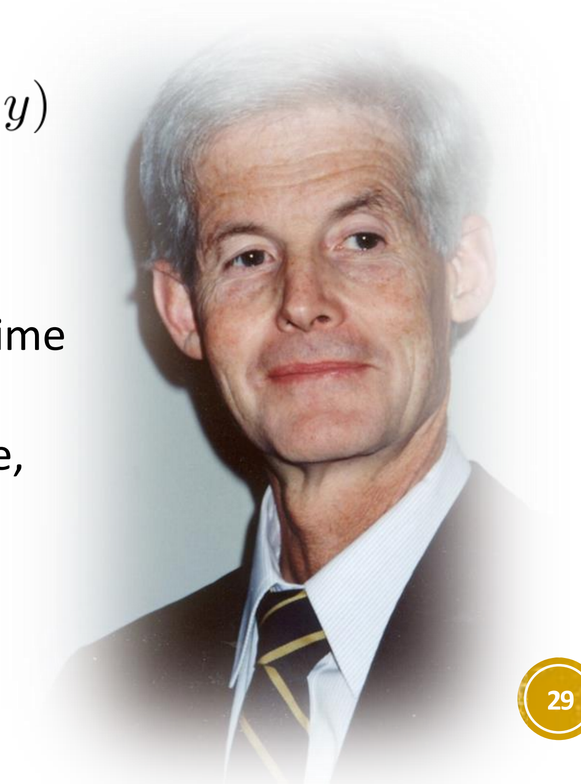
Textbook Chapter 34.3 – NP-completeness and reducibility

First NP-Complete Problem – SAT (Satisfiability)

- Input: a Boolean formula with variables
- Output: whether there is a truth assignment for the variables that satisfies the input Boolean formula

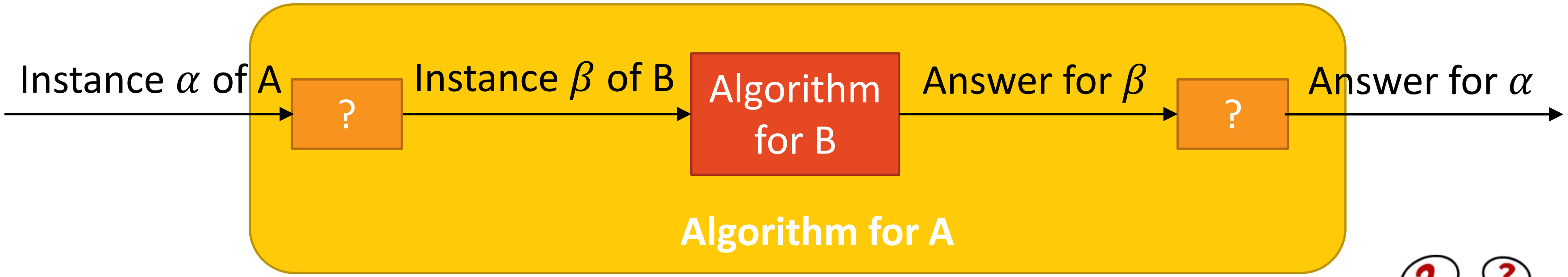
$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee y)$$

- Stephan A. Cook [FOCS 1971] proved that
  - SAT can be solved in non-deterministic polynomial time → SAT ∈ NP
  - If SAT can be solved in deterministic polynomial time, then so can any NP problems → SAT ∈ NP-hard



Reduction

- Problem A can be reduced (in polynomial time) to Problem B
  = Problem B can be reduced (in polynomial time) from Problem A
  - We can find an algorithm that solves Problem B to help solve Problem A

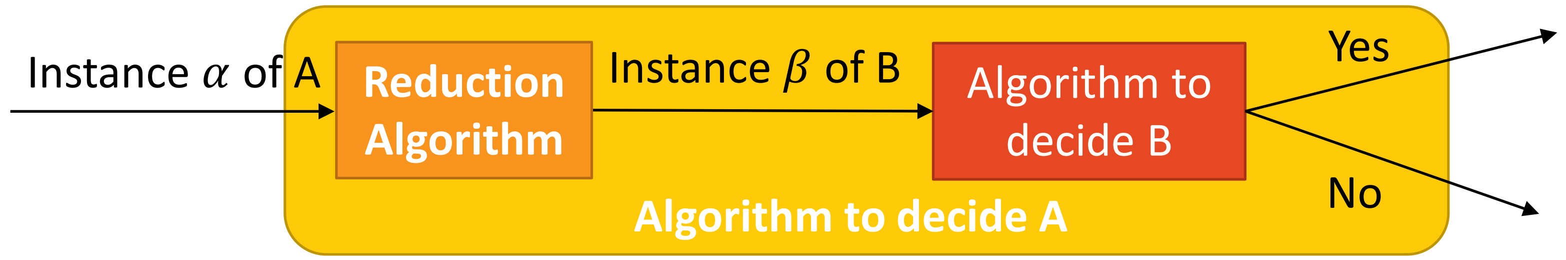


What is the complexity of Algorithm for A?

- If problem B has a polynomial-time algorithm, then so does problem A
- Practice: design a MULTIPLY() function by ADD(), DIVIDE(), and SQUARE()

Reduction

- A reduction is an algorithm for **transforming a problem instance into another**



- Definition
  - Reduction from A to B implies A is not harder than B
  - $A \leq_p B$ if A can be reduced to B in polynomial time
- Applications
  - Designing algorithms: given algorithm for B, we can also solve A
  - Classifying problems: establish relative difficulty between A and B
  - **Proving limits: if A is hard, then so is B**

This is why we need it for proving NP-completeness!

Questions

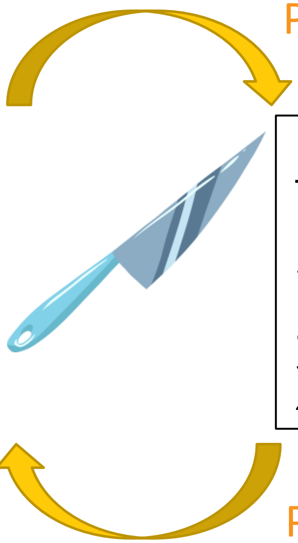
- If *A* is an NP-hard problem and *B* can be reduced from *A*, then *B* is an NP-hard problem?
- If *A* is an NP-complete problem and *B* can be reduced from *A*, then *B* is an NP-complete problem?
- If *A* is an NP-complete problem and *B* can be reduced from *A*, then *B* is an NP-hard problem?

Problem Difficulty

- Q: Which one is harder?

KNAPSACK: Given a set $\{a_1, \dots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$.

Polynomial-time reducible?



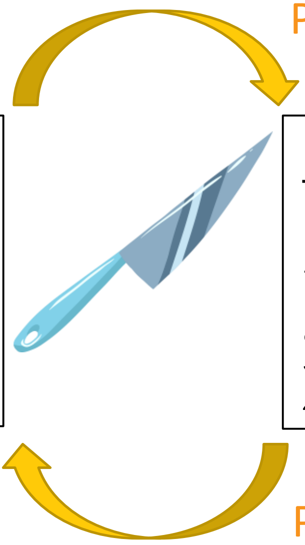
PARTITION: Given a set of n non-negative integers $\{a_1, \dots, a_n\}$, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = \sum_{i \notin P} a_i$.

Polynomial-time reducible?

- A: They have equal difficulty.
- Proof:
  - PARTITION $\leq_p$ KNAPSACK
  - KNAPSACK $\leq_p$ PARTITION

Polynomial Time Reduction

KNAPSACK: Given a set $\{a_1, \dots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$.



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- PARTITION $\leq_p$ KNAPSACK
  - If we can solve KNAPSACK, how can we use that to solve PARTITION?
- KNAPSACK $\leq_p$ PARTITION
  - If we can solve PARTITION, how can we use that to solve KNAPSACK?

PARTITION $\leq_p$ KNAPSACK

KNAPSACK: Given a set $\{a_1, \dots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$.

PARTITION: Given a set of n non-negative integers $\{a_1, \dots, a_n\}$, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = \sum_{i \notin P} a_i$.

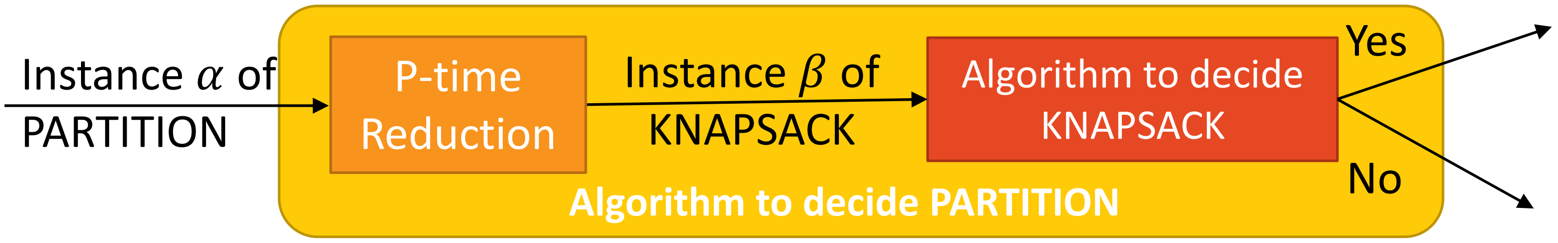
- If we can solve KNAPSACK, how can we use that to solve PARTITION?
- Polynomial-time reduction
  - Set $K = \frac{1}{2}\sum_{i=1}^{n} a_i$

$$\boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8} \xrightarrow{\text{p-time reduction}} \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}$$

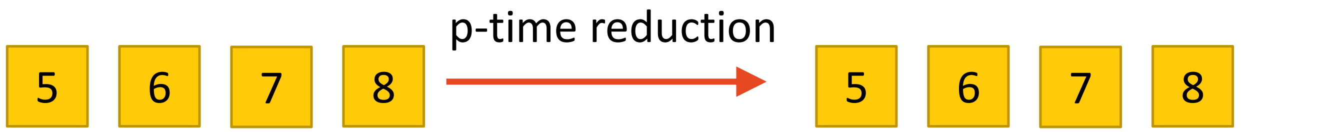
PARTITION instance

KNAPSACK instance with $K = \frac{1}{2} \times 26 = 13$

PARTITION $\leq_p$ KNAPSACK


Instance α of PARTITION
P-time Reduction
Instance β of KNAPSACK
Algorithm to decide KNAPSACK
Yes
No
Algorithm to decide PARTITION

- If we can solve KNAPSACK, how can we use that to solve PARTITION?
- Polynomial-time reduction
  - Set $K = \frac{1}{2}\sum_{i=1}^{n} a_i$



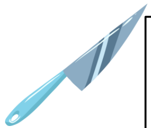
PARTITION instance

KNAPSACK instance with $K = \frac{1}{2} \times 26 = 13$

- Correctness proof: KNAPSACK returns yes if and only if an equal-size partition exists

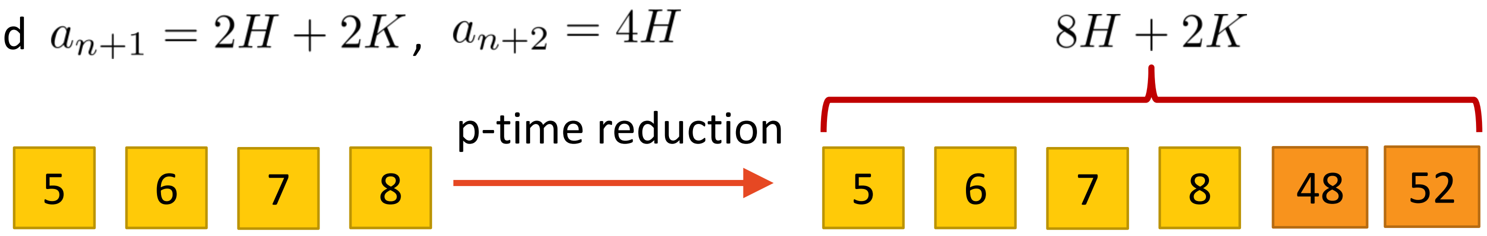
KNAPSACK $\leq_p$ PARTITION

KNAPSACK: Given a set $\{a_1, \dots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$.



PARTITION: Given a set of n non-negative integers $\{a_1, \dots, a_n\}$, decide if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = \sum_{i \notin P} a_i$.

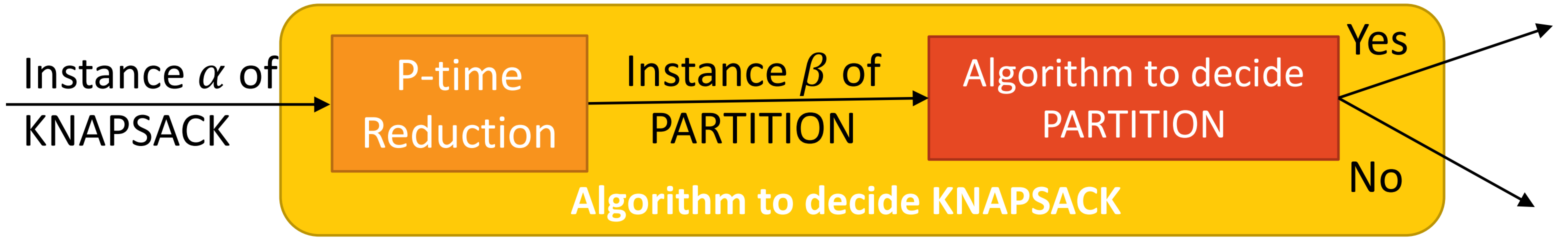
- If we can solve PARTITION, how can we use that to solve KNAPSACK?
- Polynomial-time reduction
  - Set $H = \frac{1}{2} \sum_{i=1}^{n} a_i$
  - Add $a_{n+1} = 2H + 2K$, $a_{n+2} = 4H$



KNAPSACK instance with $K = 11$

PARTITION instance

KNAPSACK $\leq_p$ PARTITION


Instance α of KNAPSACK
P-time Reduction
Instance β of PARTITION
Algorithm to decide PARTITION
Yes
No
Algorithm to decide KNAPSACK

- If we can solve PARTITION, how can we use that to solve KNAPSACK?
- Polynomial-time reduction
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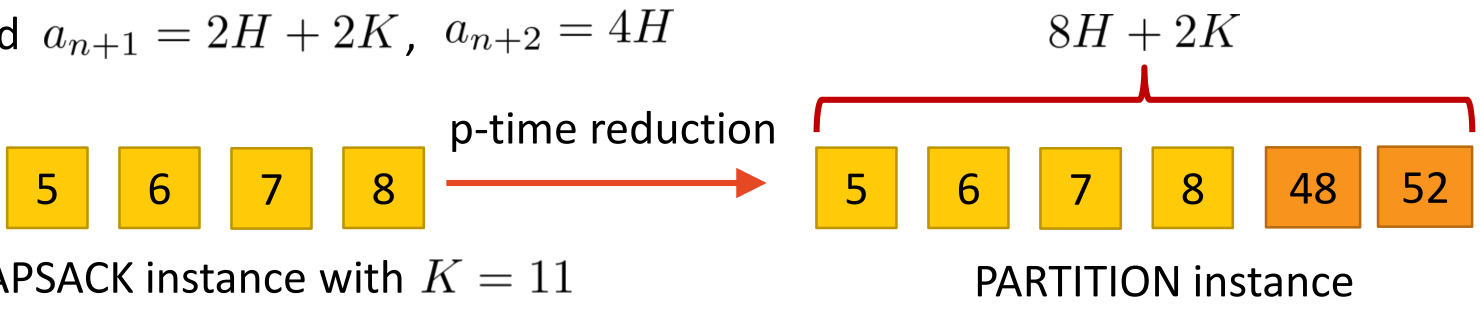
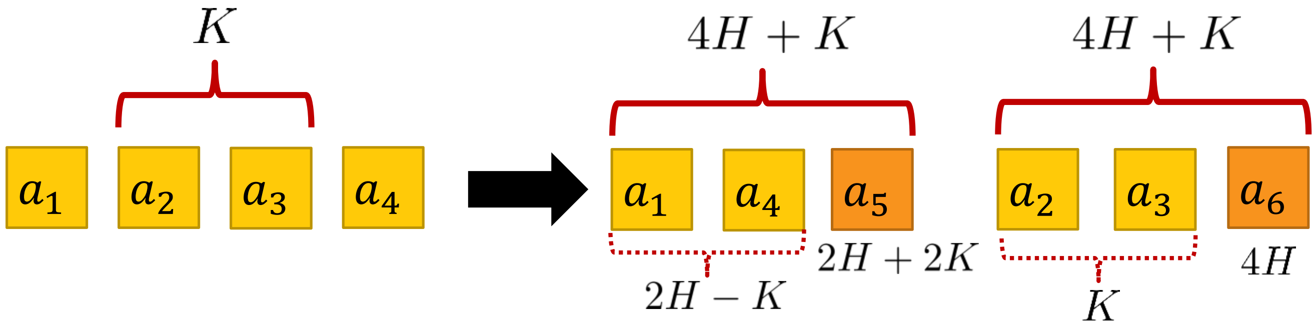


- Correctness proof: PARTITION returns yes if and only if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$

KNAPSACK $\leq_p$ PARTITION

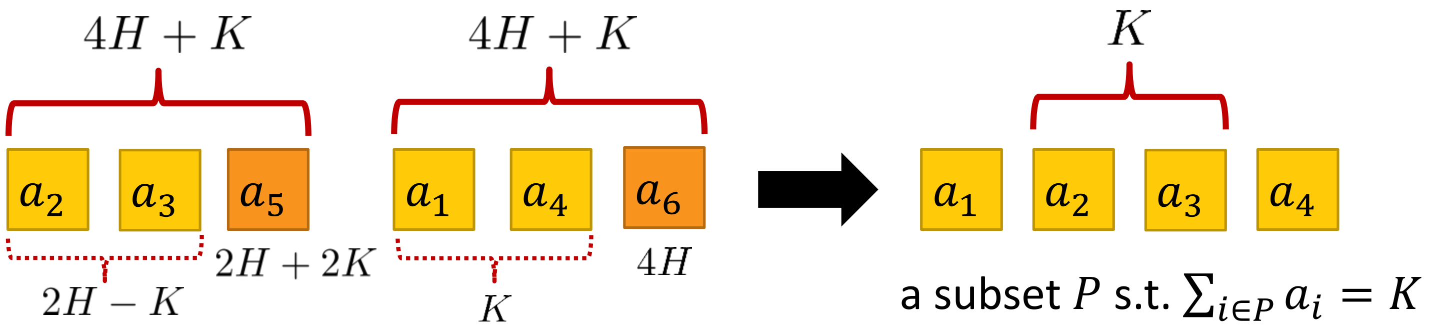
- Polynomial-time reduction
  - Set $H = \frac{1}{2}\sum_{i=1}^{n} a_i$
  - Add $a_{n+1} = 2H + 2K$, $a_{n+2} = 4H$
- Correctness proof: PARTITION returns yes if and only if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$
  - "if" direction



PARTITION returns yes!

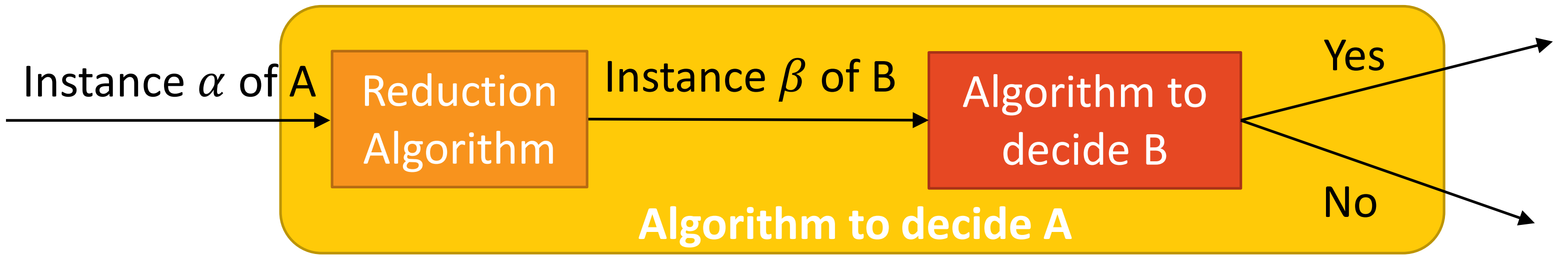
KNAPSACK $\leq_p$ PARTITION

- Polynomial-time reduction
  - Set $H = \frac{1}{2}\sum_{i=1}^{n} a_i$
  - Add $a_{n+1} = 2H + 2K$, $a_{n+2} = 4H$
- Correctness proof: PARTITION returns yes if and only if there is a subset $P \subseteq [1, n]$ such that $\sum_{i \in P} a_i = K$
  - "only if" direction
    - Because $\sum_{i=1}^{n+2} a_i = 8H + 2K$, if PARTITION returns yes, each set has $4H + K$
    - $\{a_1, \dots, a_n\}$ must be divided into $2H - K$ and K



a subset P s.t. $\sum_{i \in P} a_i = K$

Reduction for Proving Limits



- Definition
  - Reduction from A to B implies A is not harder than B
  - $A \leq_p B$ if A can be reduced to B in polynomial time
- NP-completeness proofs
  - Goal: prove that B is NP-hard
  - Known: A is NP-complete/NP-hard
  - Approach: construct a polynomial-time reduction algorithm to convert α to β
  - Correctness: if we can solve B, then A can be solved → $A \leq_p B$
  - B is no easier than A → A is NP-hard, so B is NP-hard

If the reduction is not p-time, does this argument hold?

Proving NP-Completeness

Formal Language Framework

- Focus on decision problems
- A ***language*** *L* over ∑ is any set of strings made up of symbols from ∑
- Every language *L* over ∑ is a subset of $\sum^*$

$$\sum^* = \{\epsilon; 0; 1; 10; 11; 101; 111; \cdots\}$$

The formal-language framework allows us to express concisely the relation between decision problems and algorithms that solve them.

- An algorithm A ***accepts*** a string $x \in \{0,1\}^*$ if $A(x) = 1$
- The language ***accepted*** by an algorithm A is the set of strings

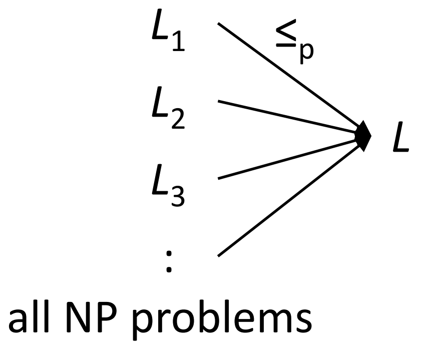
$$L = \{x \in \{0,1\}^* : A(x) = 1\}$$

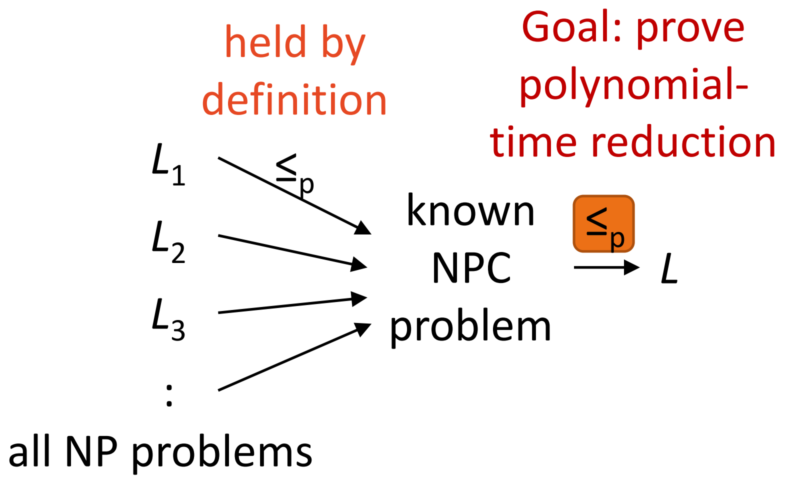
- An algorithm A ***rejects*** a string *x* if $A(x) = 0$

Proving NP-Completeness

- NP-Complete (NPC): class of decision problems in both NP and NP-hard
- In other words, a decision problem L is NP-complete if
  1. $L \in \text{NP}$
  2. $L \in \text{NP-hard}$ (that is, $L' \leq_p L$ for every $L' \in \text{NP}$)

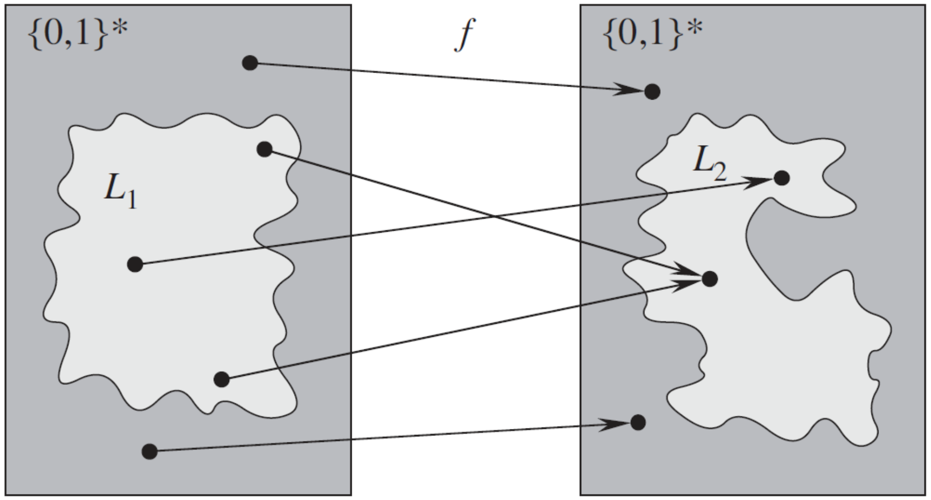
How to prove *L* is NP-hard ?

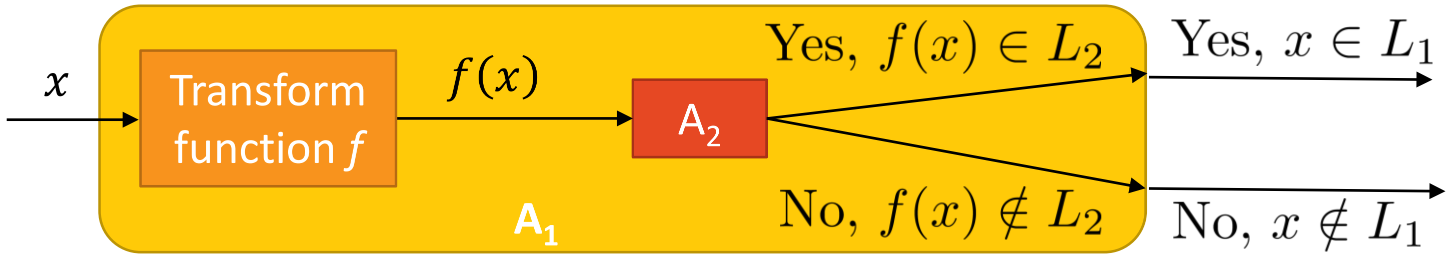




Polynomial-Time Reducible

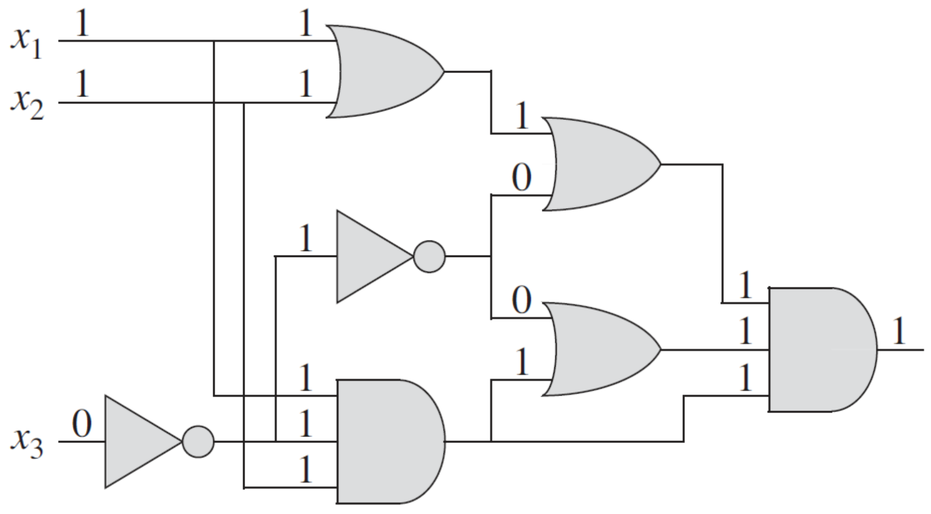
- If $L_1, L_2 \subset \{0,1\}^*$ are languages s.t. $L_1 \le_p L_2$, then $L_2 \in$ P implies $L_1 \in$ P.



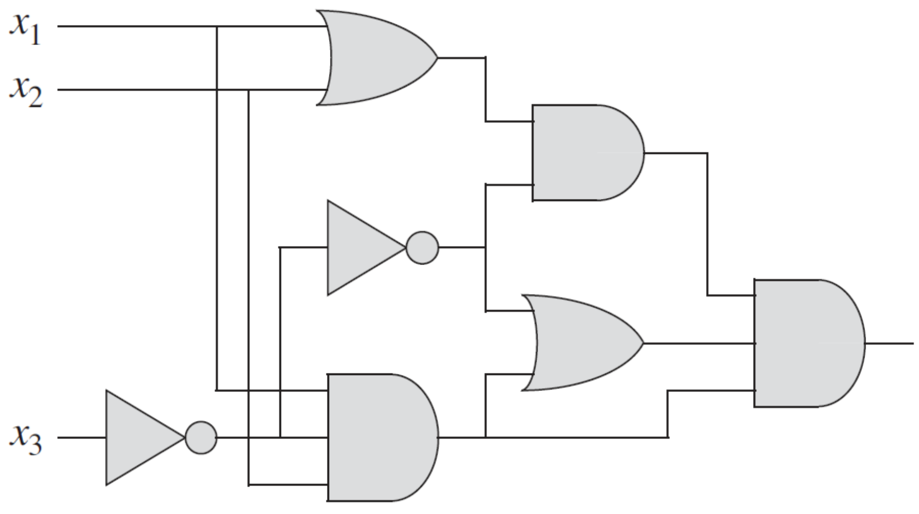


Circuit Satisfiability Problem

- Given a Boolean combinational circuit composed of AND, OR, and NOT gates, is it satisfiable?
  - Satisfiable: there exists an assignment s.t. outputs = 1



Satisfiable



Unsatisfiable

CIRCUIT-SAT

CIRCUIT-SAT = {<C>: C is a satisfiable Boolean combinational circuit}

- CIRCUIT-SAT can be solved in non-deterministic polynomial time

  $\rightarrow \in$ NP

- If CIRCUIT-SAT can be solved in deterministic polynomial time, then so can any NP problems

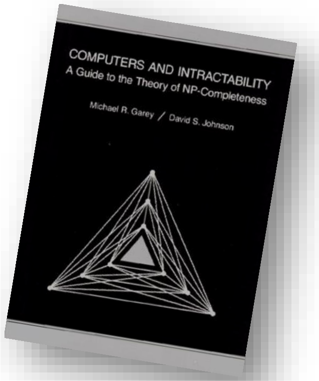
  $\rightarrow \in$ NP-hard

- (proof in textbook 34.3)
- CIRCUIT-SAT is NP-complete

P v.s. NP

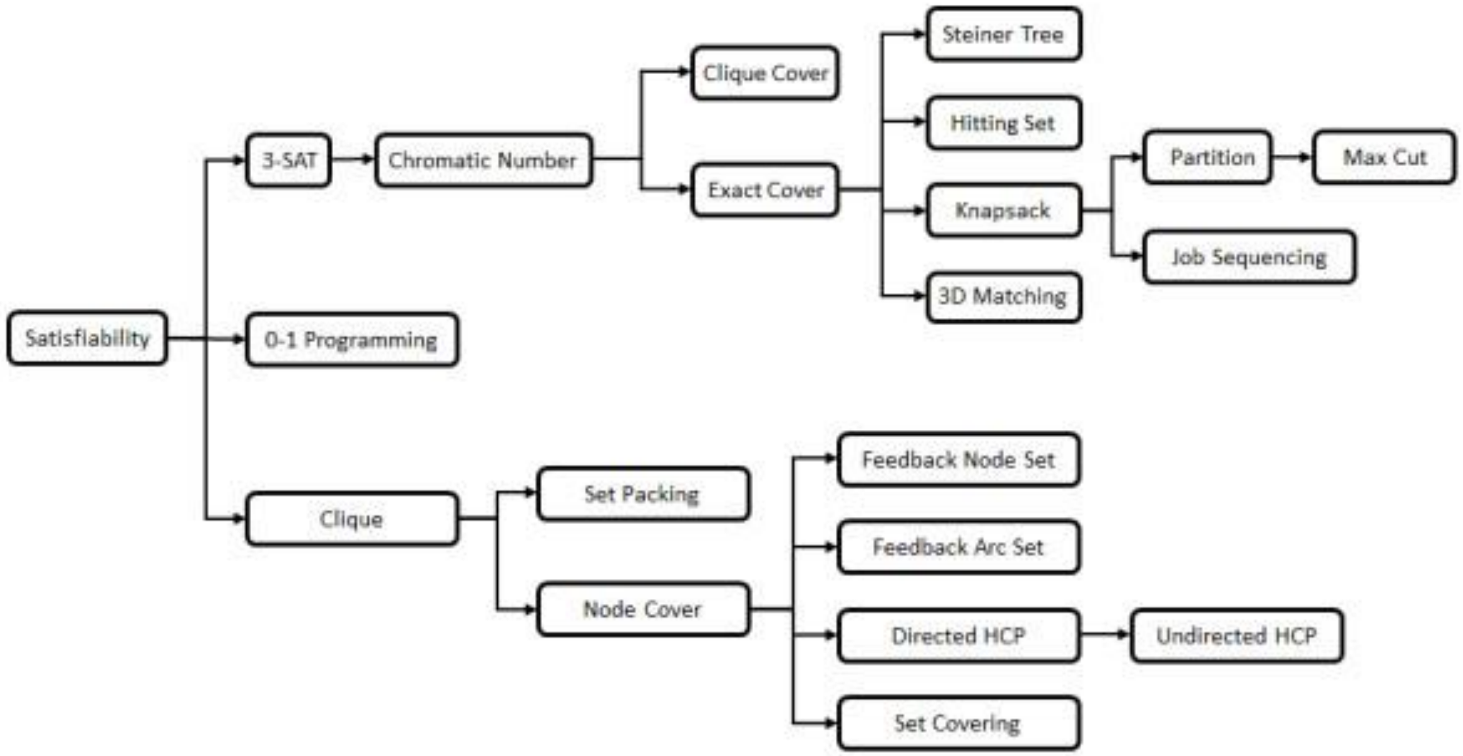
- If one proves that SAT can be solved by a polynomial-time algorithm, then NP = P.
- If somebody proves that SAT cannot be solved by any polynomial-time algorithm, then NP ≠ P.

Karp’s NP-Complete Problems


COMPUTERS AND INTRACTABILITY
A Guide to the Theory of NP-Completeness
Michael R. Garey / David S. Johnson

1. CNF-SAT
2. 0-1 INTEGER PROGRAMMING
3. CLIQUE
4. SET PACKING
5. VERTEX COVER
6. SET COVERING
7. FEEDBACK ARC SET
8. FEEDBACK NODE SET
9. DIRECTED HAMILTONIAN CIRCUIT
10. UNDIRECTED HAMILTONIAN CIRCUIT
11. 3-SAT
12. CHROMATIC NUMBER
13. CLIQUE COVER
14. EXACT COVER
15. 3-dimensional MATCHING
16. STEINER TREE
17. HITTING SET
18. KNAPSACK
19. JOB SEQUENCING
20. PARTITION
21. MAX-CUT

Karp’s NP-Complete Problems



Formula Satisfiability Problem (SAT)

- Given a Boolean formula Φ with variables, is there a variable assignment satisfying Φ

$$\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$

  - ∧ (AND), ∨ (OR), ¬ (NOT), → (implication), ↔ (if and only if)
  - Satisfiable: Φ is evaluated to 1

$$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$$

$$\begin{aligned}\phi &= ((0 \to 0) \vee \neg((\neg 0 \leftrightarrow 1) \vee 1)) \wedge \neg 0\\ &= (1 \vee \neg((1 \leftrightarrow 1) \vee 1)) \wedge 1\\ &= (1 \vee \neg(1 \vee 1)) \wedge 1\\ &= (1 \vee 0) \wedge 1\\ &= 1 \wedge 1\\ &= 1\end{aligned}$$

SAT

SAT = {Φ | Φ is a Boolean formula with a satisfying assignment }

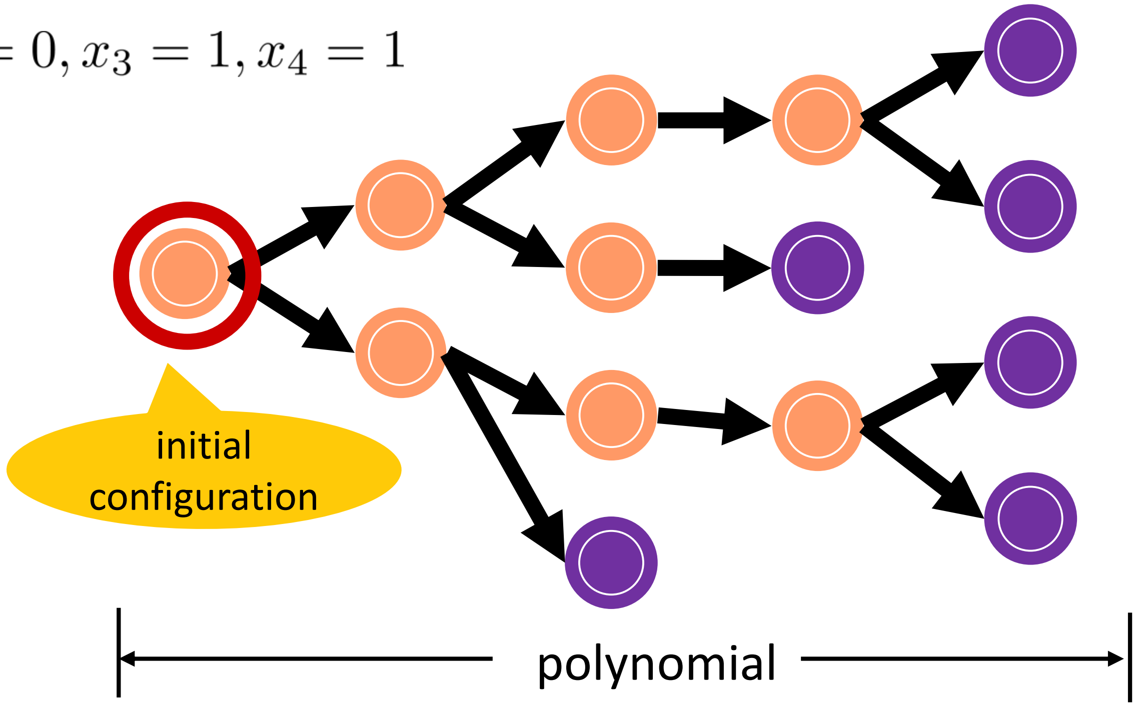
- Is SAT ∈ NP-Complete?
- To prove that SAT is NP-Complete, we show that
  - SAT ∈ NP
  - SAT ∈ NP-hard (CIRCUIT-SAT $\leq_p$ SAT)
    1) CIRCUIT-SAT is a known NPC problem
    2) Construct a reduction *f* transforming every CIRCUIT-SAT instance to an SAT instance
    3) Prove that x ∈ CIRCUIT-SAT iff *f*(x) ∈ SAT
    4) Prove that *f* is a polynomial time transformation

SAT ∈ NP

- **Polynomial-time verification**: replaces each variable in the formula with the corresponding value in the certificate and then evaluates the expression

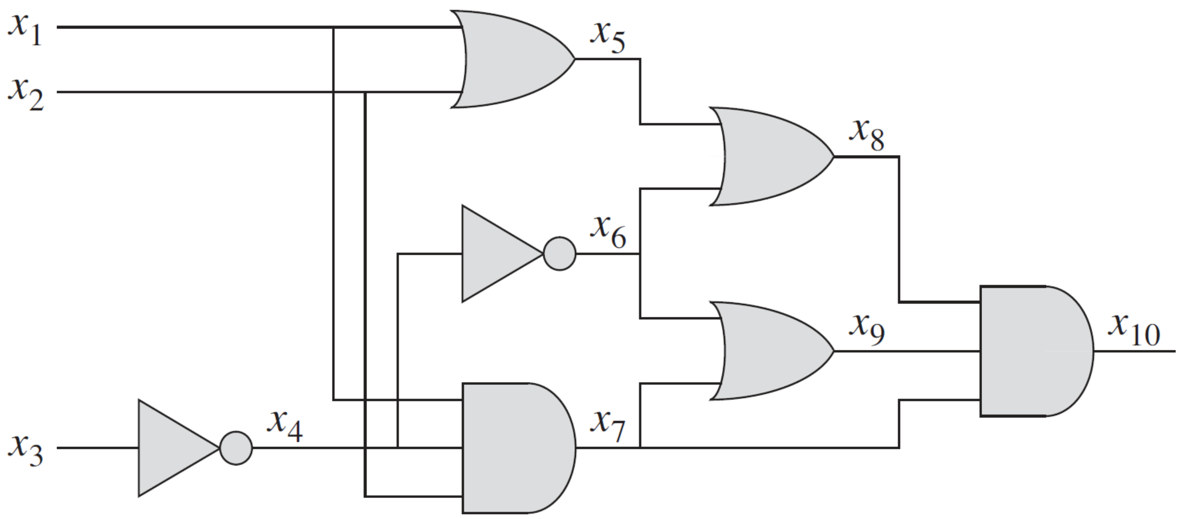
$$\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$

$$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$$

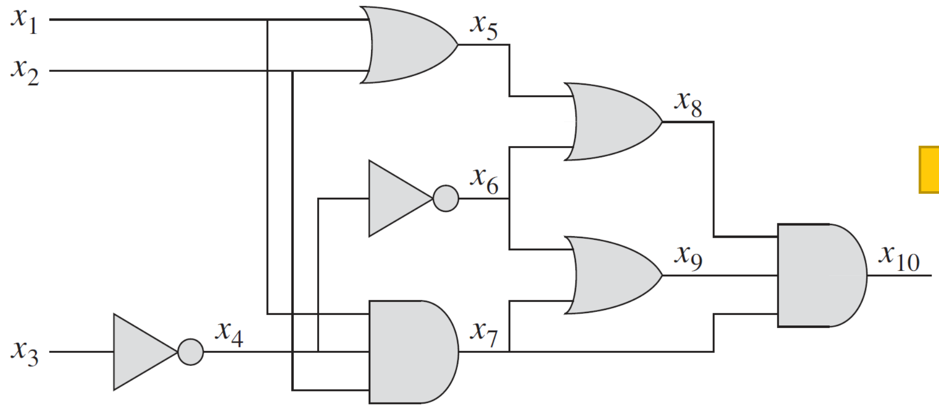


SAT ∈ NP-Hard

1) CIRCUIT-SAT is a known NPC problem
2) Construct a reduction *f* transforming every CIRCUIT-SAT instance to an SAT instance
   - Assign a variable to each **wire** in circuit C
   - Represent the operation of each gate using a formula, e.g. $x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)$
   - Φ = AND the **output variable** and the **operations of all gates**



SAT ∈ NP-Hard



$$
\begin{aligned}
\phi = x_{10} &\wedge (x_4 \leftrightarrow \neg x_3)\\
&\wedge (x_5 \leftrightarrow (x_1 \vee x_2))\\
&\wedge (x_6 \leftrightarrow \neg x_4)\\
&\wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4))\\
&\wedge (x_8 \leftrightarrow (x_5 \vee x_6))\\
&\wedge (x_9 \leftrightarrow (x_6 \vee x_7))\\
&\wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9))
\end{aligned}
$$

3. Prove that *x* ∈ CIRCUIT-SAT ↔ *f*(*x*) ∈ SAT
   - *x* ∈ CIRCUIT-SAT → *f*(*x*) ∈ SAT
   - *f*(*x*) ∈ SAT → *x* ∈ CIRCUIT-SAT
4. *f* is a polynomial time transformation

CIRCUIT-SAT $\leq_p$ SAT → SAT ∈ NP-hard

3-CNF-SAT Problem

- 3-CNF-SAT: Satisfiability of Boolean formulas in 3-*conjunctive normal form* (3-CNF)

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

  - 3-CNF = AND of clauses, each of which is the OR of exactly 3 distinct literals
  - A literal is an occurrence of a variable or its negation, e.g., x_1 or $\neg x_1$

$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$ → satisfiable

3-CNF-SAT

3-CNF-SAT = {Φ | Φ is a Boolean formula in 3-conjunctive normal form (3-CNF) with a satisfying assignment }

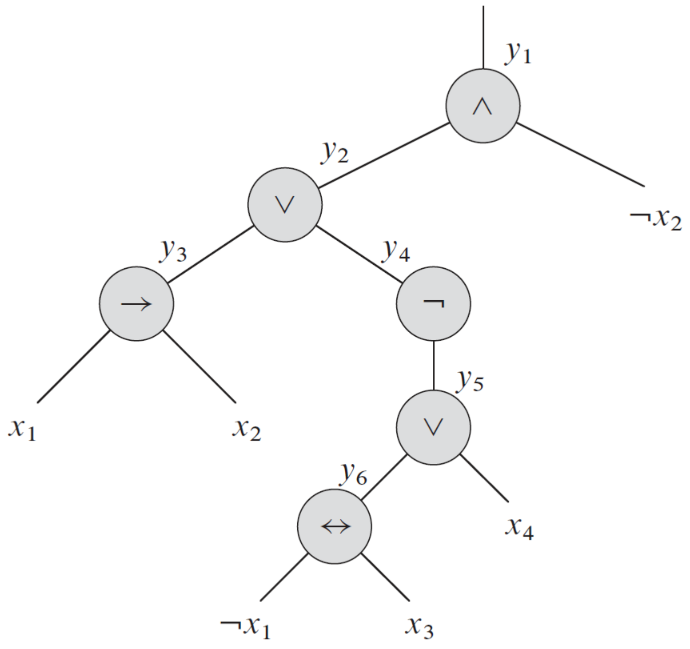
- Is 3-CNF-SAT ∈ NP-Complete?
- To prove that SAT is NP-Complete, we show that
  - 3-CNF-SAT ∈ NP
  - 3-CNF-SAT ∈ NP-hard (SAT $\leq_p$ 3-CNF-SAT)
    1) SAT is a known NPC problem
    2) Construct a reduction *f* transforming every SAT instance to an 3-CNF-SAT instance
    3) Prove that x ∈ SAT iff *f*(x) ∈ 3-CNF-SAT
    4) Prove that *f* is a polynomial time transformation

We focus on the reduction construction from now on, but remember that a full proof requires showing that all other conditions are true as well

SAT $\leq_p$ 3-CNF-SAT

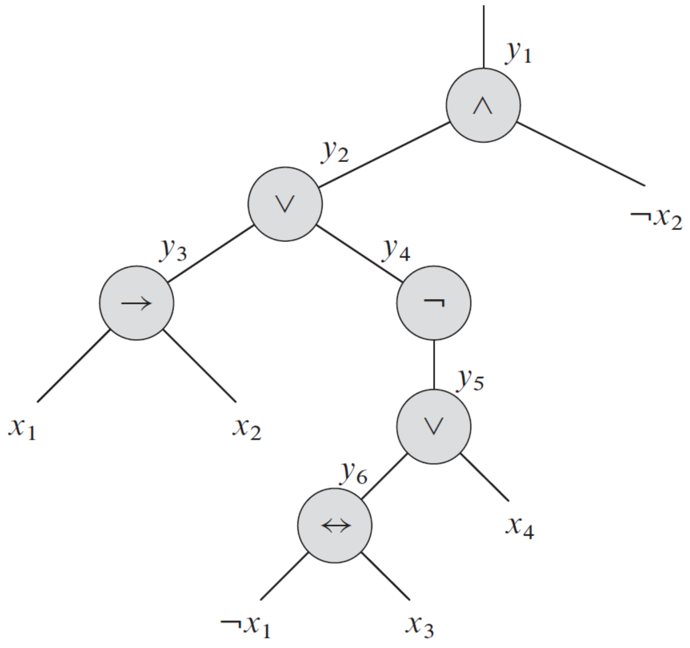
a) Construct a binary parser tree for an input formula Φ and introduce a variable y_i for the output of each internal node

$$\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$



SAT $\leq_p$ 3-CNF-SAT

b) Rewrite Φ as the AND of the root variable and clauses describing the operation of each node



$$\begin{aligned}\phi' = y_1 &\wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2))\\ &\wedge (y_2 \leftrightarrow (y_3 \vee y_4))\\ &\wedge (y_3 \leftrightarrow (x_1 \wedge x_2))\\ &\wedge (y_4 \leftrightarrow \neg y_5)\\ &\wedge (y_5 \leftrightarrow (y_6 \vee x_4))\\ &\wedge (y_6 \leftrightarrow (\neg x_1 \wedge x_3))\end{aligned}$$

SAT $\leq_p$ 3-CNF-SAT

$$\phi' = y_1 \wedge \boxed{(y_1 \leftrightarrow (y_2 \wedge \neg x_2))}$$
$$\wedge (y_2 \leftrightarrow (y_3 \vee y_4))$$
$$\wedge (y_3 \leftrightarrow (x_1 \wedge x_2))$$
$$\wedge (y_4 \leftrightarrow \neg y_5)$$
$$\wedge (y_5 \leftrightarrow (y_6 \vee x_4))$$
$$\wedge (y_6 \leftrightarrow (\neg x_1 \wedge x_3))$$

c) Convert each clause Φi' to CNF
- Construct a truth table for each clause Φi'
- Construct the disjunctive normal form for ¬Φi'
- Apply DeMorgan's Law to get the CNF formula Φi''

y_1	y_2	y_2	Φ_1'	$\neg\Phi_1'$
1	1	1	0	**1**
1	1	0	1	**0**
1	0	1	0	**1**
1	0	0	0	**1**
0	1	1	1	**0**
0	1	0	0	**1**
0	0	1	1	**0**
0	0	0	1	**0**

$$\neg\phi_1' = (y_1 \wedge y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge x_2)$$
$$\vee (y_1 \wedge \neg y_2 \wedge \neg x_2) \vee (\neg y_1 \wedge y_2 \wedge \neg x_2)$$

$$\Downarrow$$

$$\phi_1' = (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2)$$
$$\wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)$$

$$\boxed{\begin{array}{l} \neg(a \wedge b) = \neg a \vee \neg b \\ \neg(a \vee b) = \neg a \wedge \neg b \end{array}}$$

SAT $\leq_p$ 3-CNF-SAT

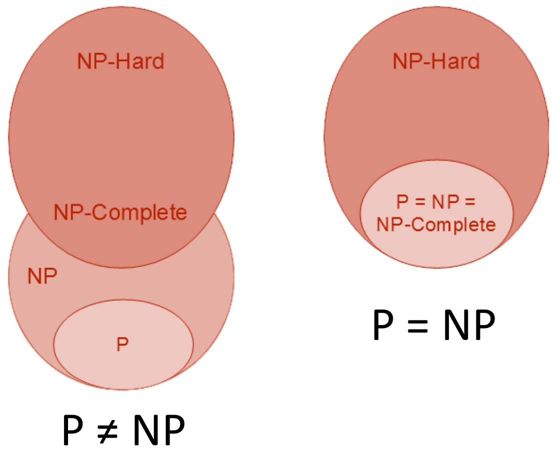
d) Construct Φ''' in which each clause C_i exactly 3 distinct literals
   - 3 distinct literals: $C_i = l_1 \vee l_2 \vee l_3$
   - 2 distinct literals: $C_i = l_1 \vee l_2$

     $C_i = l_1 \vee l_2 = (l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$
   - 1 literal only: $C_i = l$

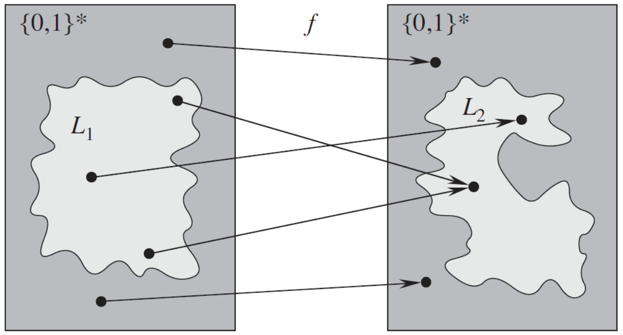
     $C_i = l = (l \vee p \vee q) \wedge (l \vee \neg p \vee q) \wedge (l \vee p \vee \neg q) \wedge (l \vee \neg p \vee \neg q)$

- Φ''' is satisfiable iff Φ is satisfiable
- All transformation can be done in polynomial time
- → 3-CNF-SAT is NP-Complete

Concluding Remarks



- Proving NP-Completeness: L ∈ NPC iff L ∈ NP and L ∈ NP-hard
- Step-by-step approach for proving *L* in NPC:
  - Prove L ∈ NP
  - Prove L ∈ NP-hard
    1) Select a known NPC problem C
    2) Construct a reduction f transforming every instance of C to an instance of L
    3) Prove that $x \in C \iff f(x) \in C, \forall x \in \{0,1\}^*$
    4) Prove that *f* is a polynomial time transformation *L* ∈ NP



Question?

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& post to the course website

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