Lesson 8: Hypergraphs

Theme: The notion of acyclicity in hypergraphs.

1 Hypergraphs

A *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$, where:

- V is a set of vertices;
- $\mathcal{E} = \{e_1, \dots, e_m\}$ is a set of hyperedges, where each hyperedge e_i is a subset of V.

When it is clear from the context, hyperedges are simply called edges.

The *reduction* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is the hypergraph obtained by deleting every edge $e \in \mathcal{E}$ that is a subset of another edge in $\mathcal{E}$. A hypergraph is called *reduced*, if it is the same as its reduction.

It is usually assumed that $V = \bigcup \mathcal{E}$, i.e., every vertex in V occurs at least in one edge. So, a hypergraph $\mathcal{H}$ can simply be defined as a collection of finite sets $\{e_1, \dots, e_m\}$.

2 Berge acyclicity

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. A *Berge cycle* in $\mathcal{H}$ is a sequence

$$a_1 \; E_1 \; a_2 \; E_2 \; \cdots \; a_n \; E_n \; a_{n+1}$$

such that

- each $a_i \in V$ and $E_i \in \mathcal{E}$,
- $a_1 = a_{n+1}$,
- $a_i, a_{i+1} \in E_i$, for every $i = 1, \dots, n$.

A hypergraph is called *Berge cyclic*, if it contains a Berge cycle. Otherwise, it is called *Berge acyclic*.

Although seems natural at the first glance, Berge acyclicity is actually a bit "weird." Consider the following hypergraph:

$$\mathcal{H} \quad := \quad \Big\{ \; \{1,2,3\}, \; \{2,3,4,5\}, \; \{4,5,6\} \Big\} \tag{1}$$

Naturally, we would like to think of $\mathcal{H}$ as "acyclic." However, it contains a Berge cycle:

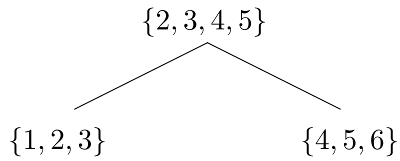
$$1 \quad \{1,2,3\} \quad 2 \quad \{2,3,4,5\} \quad 4 \quad \{4,5,6\} \quad 5 \quad \{2,3,4,5\} \quad 3 \quad \{1,2,3\} \quad 1$$

3 α-acyclicity

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *α-acyclic*, or *acyclic*, in short, if there is a tree $T = (I, F)$, where

- the set nodes in T is $I = \mathcal{E}$,
- for every vertex $v \in V$, the set $\{e \mid e \text{ contains } v\}$ is connected in T.

The tree T is called *generalized hypertree decomposition* (GHD) of $\mathcal{H}$. The hypergraph in (1) above is acyclic with GHD:



To test whether a hypergraph $\mathcal{H} = \{e_1, \ldots, e_m\}$ is acyclic, we can perform the following procedure, called GYO algorithm.

- Repeat the following until it is not possible.
  (1) If there is a vertex u that appears in *exactly* one edge e_i, delete u from e_i.
  (2) If e_i is empty or $e_i \subseteq e_j$ for some $j \neq i$, delete e_i.
- If at the end the hypergraph becomes empty, the original $\mathcal{H}$ is acyclic. Otherwise, it is cyclic.

Appendix

There is an alternative definition of α-acyclicity in [1], which is a generalisation of the definition of acyclicity in standard graphs. In the same paper, Fagin also discussed two more notions of acyclicity, called β-acyclicity and γ-acyclicity, and their relations with Berge-acyclicity and α-acyclicity. Nowadays, the term "acyclic" means α-acyclic.

GYO stands for Graham, Yu, and Özsoyoğlu, after [2, 3].

References

[1] R. Fagin. Degrees of acyclicity for hypergraphs and relational database schemes. *Journal of the ACM*, 30(3):514–550, 1983.

[2] M. Graham. On the universal relation. Technical report, University of Toronto, September 1979.

[3] C. T. Yu and M. Özsoyoğlu. Algorithm for tree-query membership of a distributed query. In *Proceedings of the IEEE Computer Society's International Computer Software and Applications Conference*, pages 306–312. IEEE, 1979.