Lesson 5: Treewidth

Theme: Treewidths of (undirected) graphs

In the following the term "graphs" will always denote *undirected* graphs. For a graph $G = (V, E)$, we usually write $v \in G$, or $(u, v) \in G$ to denote that a vertex v, or an edge (u, v) in G. A tree is denoted by $T = (I, F)$.

1 Treewidth

Let $G = (V, E)$ be a graph. A *tree decomposition* of G is a tree $T = (I, F)$, where:

(C1) the set I of vertices in T is a set $I = \{X_1, \ldots, X_m\}$, where each $X_i \subseteq V$;

(C2) $X_1 \cup \cdots \cup X_m = V$;

(C3) for every edge $(u, v) \in E$, there is a node X_i in T such that $u, v \in X_i$;

(C4) for every $i_1, i_2 \in I$, for every $j \in I$ in the path (in the tree T) between i_1 and i_2, the following holds:

$$X_{i_1} \cap X_{i_2} \quad \subseteq \quad X_j$$

Condition (C4) is equivalent to the following.

(C4′) For every vertex $u \in V$, the set of vertices $\{i \mid X_i \ni u\}$ is connected in T.

The *width* of a tree decomposition $T = (I, F)$ is $\max_{X_i \in I}(|X_i|) - 1$. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width among all possible tree decompositions of G.

Proposition 5.1

- *G is a forest if and only if $tw(G) = 1$.*
- *If G contains a complete graph K_m, then $tw(G) \geqslant m - 1$. In particular, $tw(K_m) = m - 1$.*

A tree decomposition $T = (I, F)$ is *nice*, if there is no edge (X_i, X_j) in T such that $X_i \subseteq X_j$. Note that every tree decomposition of a graph G can be transformed into a nice tree decomposition with the same width. So from now on, we can always assume that every tree decomposition is nice.

Proposition 5.2 *For every graph $G = (V, E)$ with a tree decomposition $T = (I, F)$ of width k, we have: $|I| \leqslant |V| - k$ and $|E| \leqslant k|V|$.*

2 Separator

Definition 5.3 Let $G = (V, E)$ be a graph.

- For a partition $V = A \cup B \cup S$, we say that S *separates A and B*, if every path from a vertex in A to a vertex in B passes at least one vertex in S.
- S is a separator for a set $W \subseteq V$, if there is partition $V = A \cup B \cup S$ such that $A \cap W \neq \emptyset$, $B \cap W \neq \emptyset$, and S separates A and B.

- S is a *good separator* for W, if S separates A and B, and $|A \cap W|, |B \cap W| \leqslant \frac{2}{3}|W|$.

Theorem 5.4 [1, 2] *Let $G = (V, E)$ be a graph.*

- *If G has treewidth k, then every subset $W \subseteq V$ of cardinality at least $2k + 3$ has a good separator S of cardinality at most $k + 1$.*
- *If every set $W \subseteq V$ of cardinality $2k + 3$ has a good separator of cardinality at most $k + 1$, then the treewidth of G is at most $4k + 3$.*

Theorem 5.4 can be used to obtain an algorithm that on input graph G of treewidth $\leqslant k$, construct a tree decomposition T of width at most $4k + 3$. See [2].

References

[1] T. D. Parsons. Pursuit-evasion in a graph. In Y. Alavi and D. R. Lick, editors, *Theory and Applications of Graphs*, pages 426–441. Springer-Verlag, 1976.

[2] B. Reed. Finding approximate separators and computing tree width quickly. In *Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC)*, pages 221–228, 1992.