Lesson 9. Non-deterministic Turing machines

CSIE 3110 – Formal Languages and Automata Theory

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Intuitively, non-deterministic Turing machines are algorithms with capability to "guess correctly."

They are an important model of computation for defining complexity classes such as the class NP-complete.

The definition of non-deterministic Turing machine

(Def.) A *non-deterministic Turing machine* (NTM) is:

$$\mathcal{M} = \langle \Sigma, \Gamma, Q, q_0, q_{\mathrm{acc}}, q_{\mathrm{rej}}, \delta \rangle$$

All the components are defined as in the standard Turing machine.

The difference is that δ is now a relation, where there is one or two transitions applicable on every pair $(p, a) \in Q \times \Gamma$.

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The difference is that δ is now a relation, where there is one or two transitions applicable on every pair $(p, a) \in Q \times \Gamma$.

More precisely, for every pair $(p, a) \in Q \times \Gamma$:

Either there is exactly *one* (q, b, α) such that:

$$(p, a) \to (q, b, \alpha) \ \in \ \delta$$

or there are exactly *two* (q_1, b_1, α_1) *and* (q_2, b_2, α_2) such that:

$$(p, a) \to (q_1, b_1, \alpha_1) \ \in \ \delta \qquad \text{and} \qquad (p, a) \to (q_2, b_2, \alpha_2) \ \in \ \delta$$

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It works "deterministically":

For every (p, a), there is only one (q, b, α) such that $(p, a) \to (q, b, \alpha) \in \delta$.

It is usually called deterministic Turing machine (DTM).

NTM vs. DTM

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The notions of configuration, initial configuration, accepting/rejecting configuration and run for NTM are all defined exactly as in DTM.

- For every input word w, there is exactly one run of a DTM on w.
- For every input word w, there are many runs of an NTM on w.

Illustration

Let $\mathcal{M}$ be an NTM and w be the input word.

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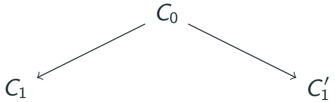

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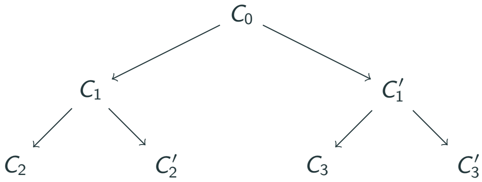

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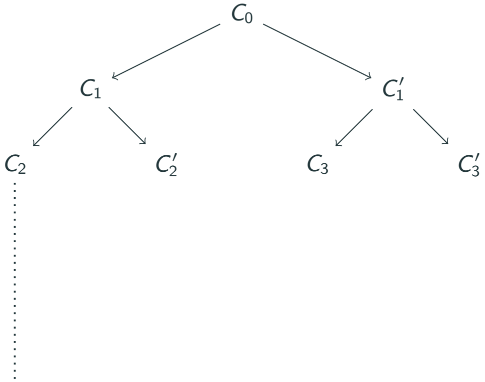

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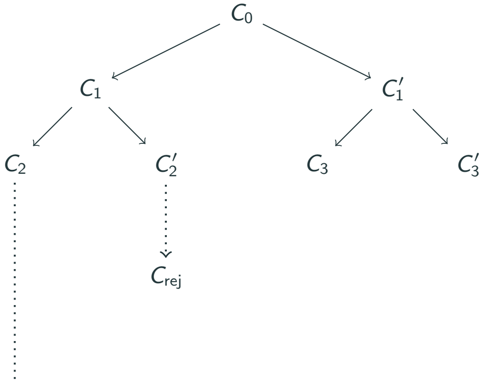

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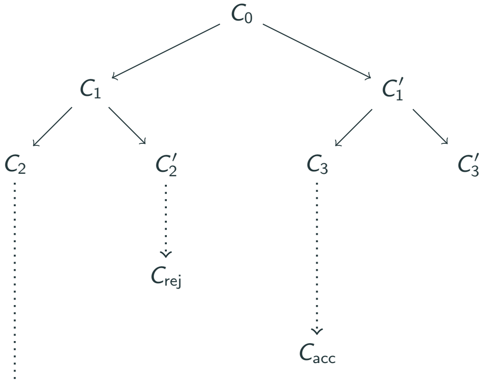

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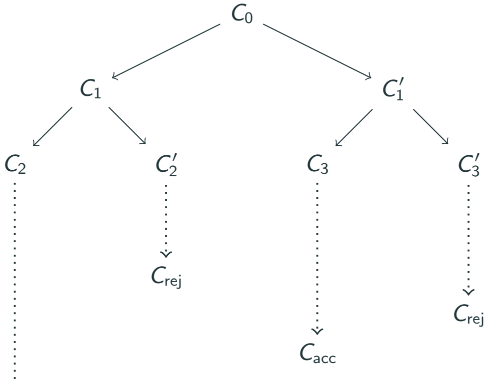

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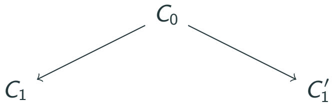

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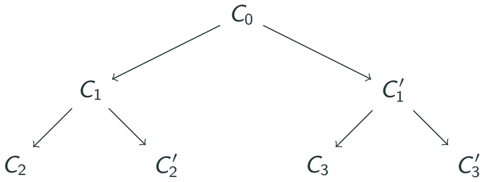

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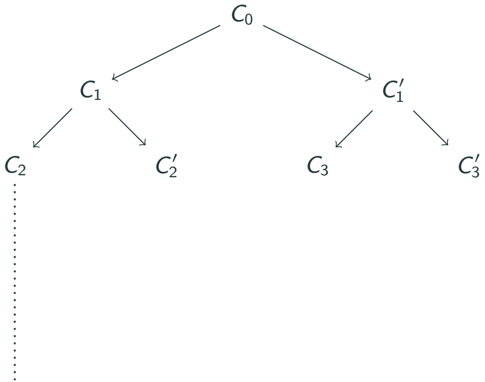

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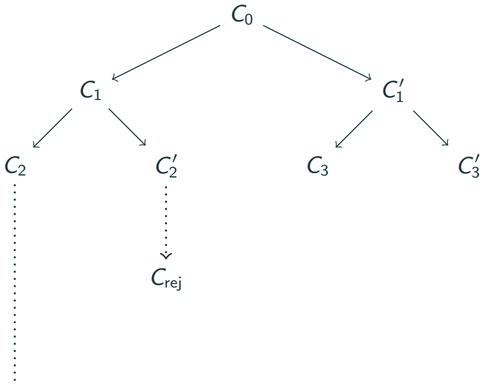

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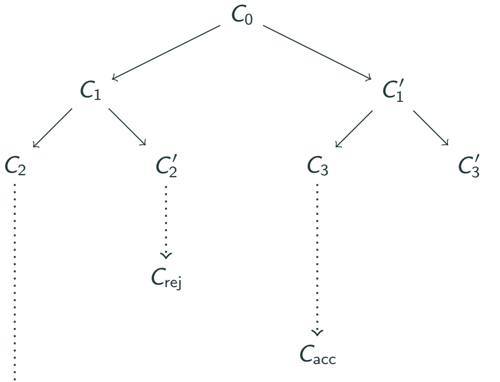

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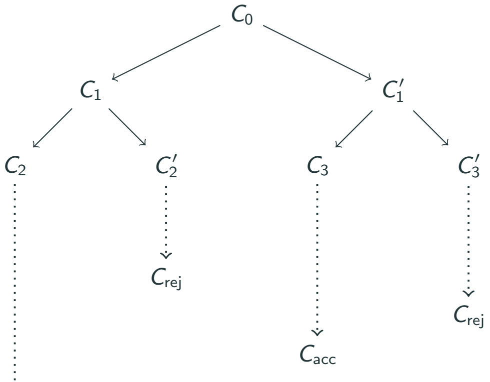

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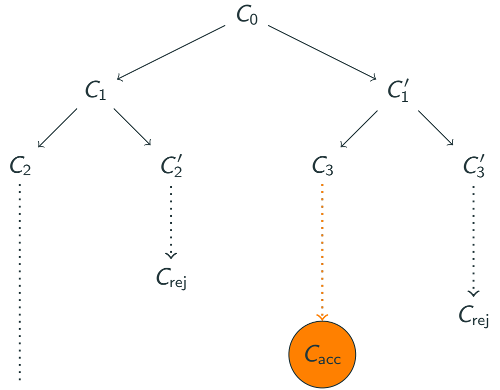

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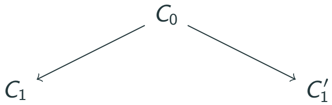

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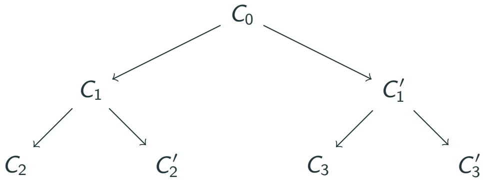

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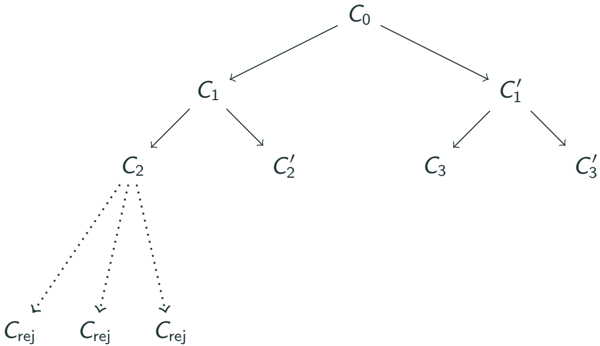

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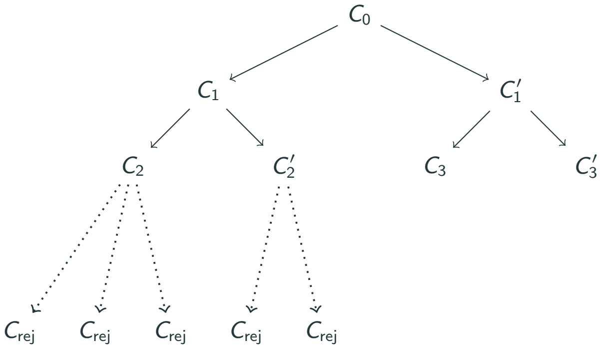

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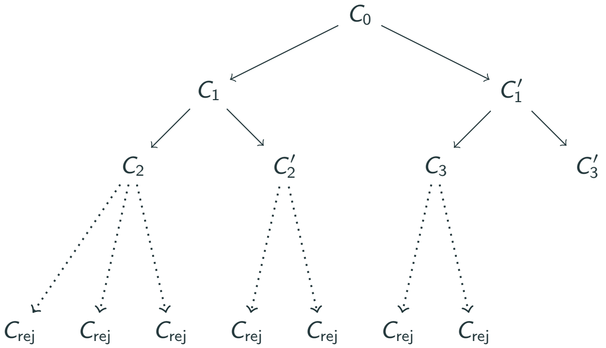

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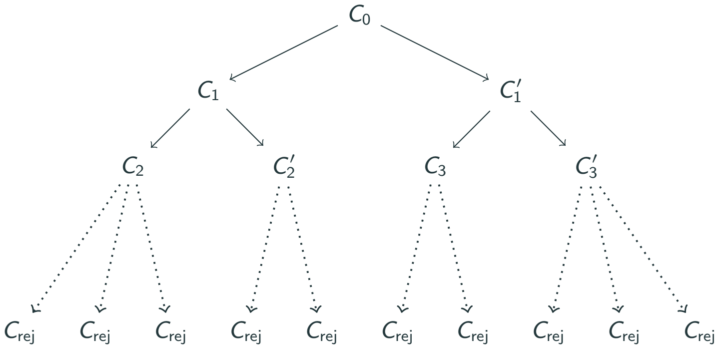

When NTM does not halt

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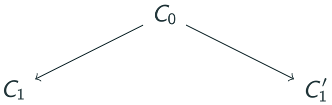

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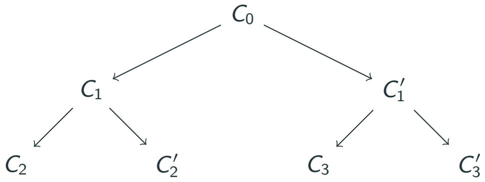

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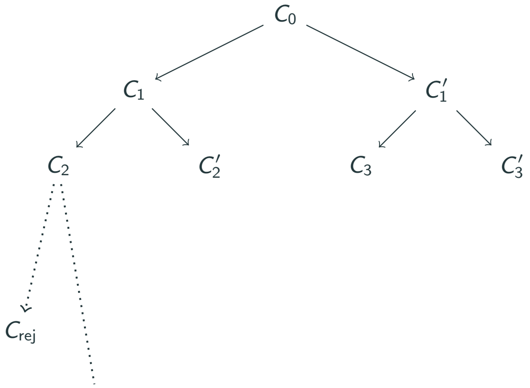

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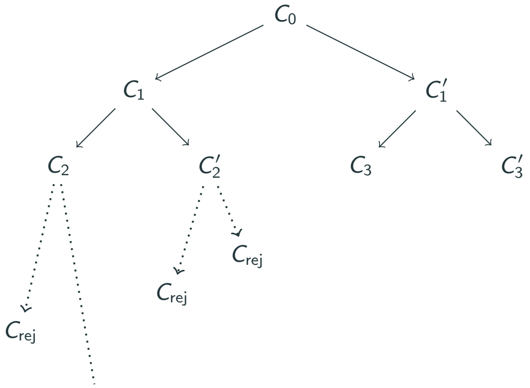

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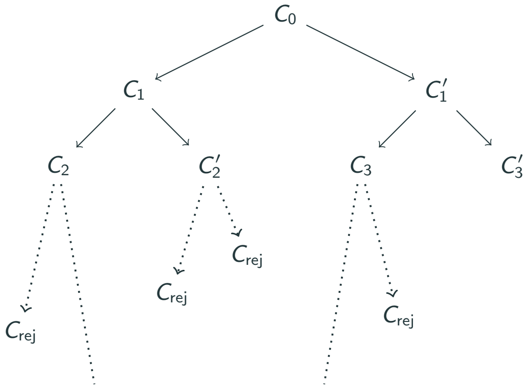

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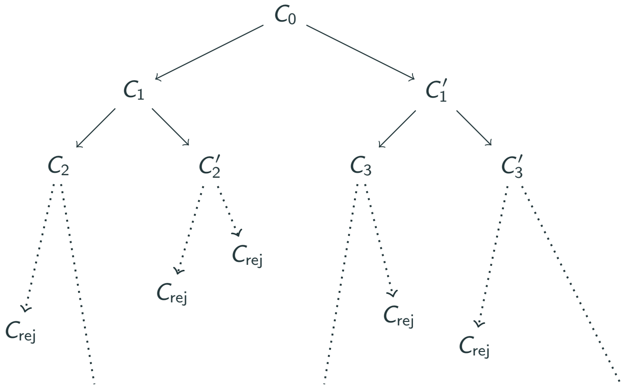

Decidable and recognizable languages by NTM

(Def.) An NTM $\mathcal{M}$ decides a language L, if:

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(Def.) An NTM $\mathcal{M}$ recognizes a language L, if:

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NTM is equivalent to DTM

Theorem 9.1

For every language NTM $\mathcal{M}$, there is DTM $\mathcal{M}'$ such that for every input word w, the following holds.

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- *If $\mathcal{M}$ does not halt on w, then $\mathcal{M}'$ does not halt on w.*

In other words, $\mathcal{M}$ and $\mathcal{M}'$ are equivalent.

Proof of Theorem 9.1

Let $\mathcal{M}$ be an NTM.

The DTM $\mathcal{M}'$ works by simulating $\mathcal{M}$ on the input word.

On input word w, do the following.

- Let C_0 be the initial configuration of $\mathcal{M}$ on w.
- Let $S = \{C_0\}$, i.e., a set that contains only one element C_0.
- while ($S \neq \emptyset$) or (S contains an accepting configuration):
  - Delete all the rejecting configurations from S.
  - Compute the next configuration of each element in S.
    Store them all in S.
- If S contains an accepting configuration, ACCEPT.
  If $S = \emptyset$, REJECT.

Proof of Theorem 9.1 – Illustration

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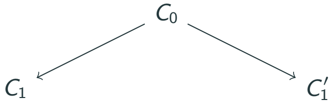

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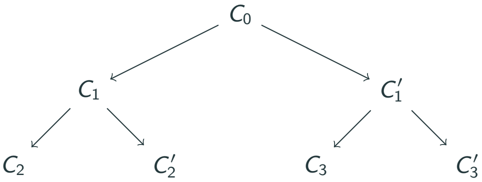

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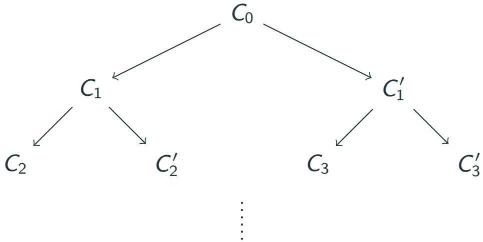

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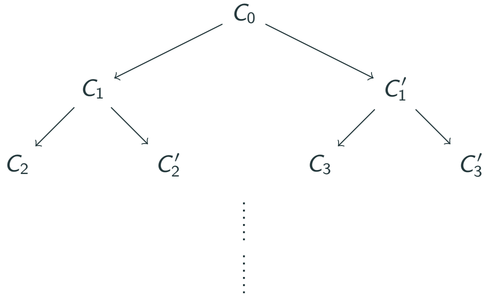

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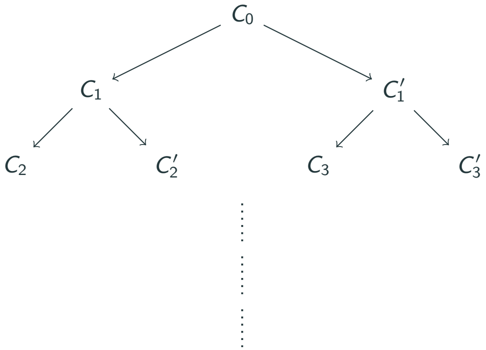

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NTM can be generalized to multiple tape and Theorem 9.1 still holds.

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Theorem 9.2

Recognizable languages are closed under concatenation and Kleene star.

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(Proof) Let L_1 and L_2 be recognizable languages and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be DTM that recognize them. We assume that $\Sigma = \{0, 1\}$.

Closure property of recognizable languages

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(Proof) Let L_1 and L_2 be recognizable languages and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be DTM that recognize them. We assume that $\Sigma = \{0, 1\}$.

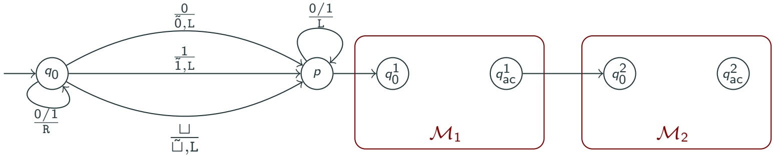
(Closure under concatenation) We present a 2-tape NTM $\mathcal{M}$ that recognizes $L_1 L_2$. On input word w:

- "Guess" a partition $v_1 v_2$ of w.
- Copy v_1 onto the second tape.
- Run $\mathcal{M}_1$ on v_1 (on the second tape).
- If $\mathcal{M}_1$ accepts, erase the second tape and copy v_2 onto the second tape
- Run $\mathcal{M}_2$ on v_2 (on the second tape).
- If $\mathcal{M}_2$ accepts, ACCEPT.

“Guess” a partition of w into $w = v_1 v_2$ – Illustration

We have new symbols $\tilde{0}, \tilde{1}, \tilde{\sqcup}$.

The NTM looks like this:



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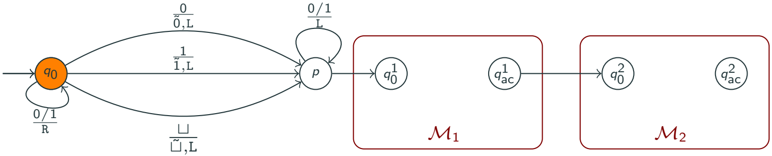

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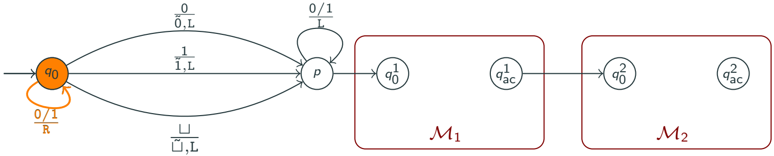

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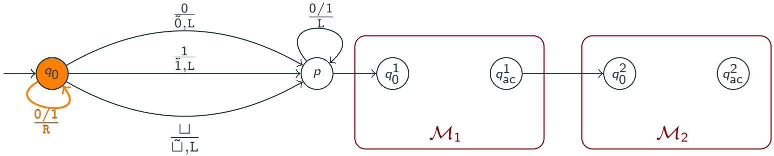

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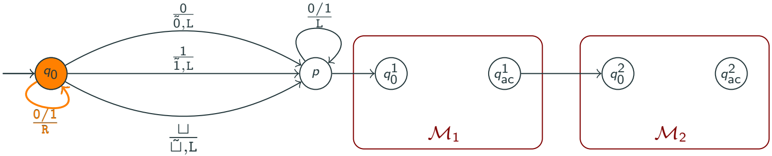

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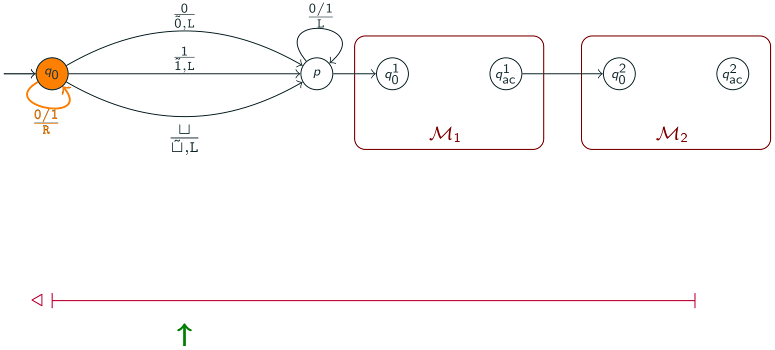

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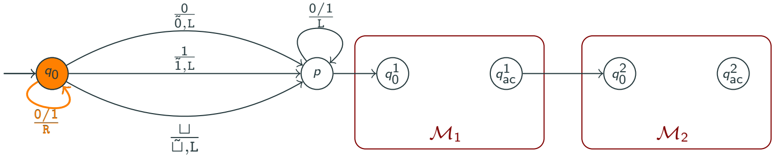

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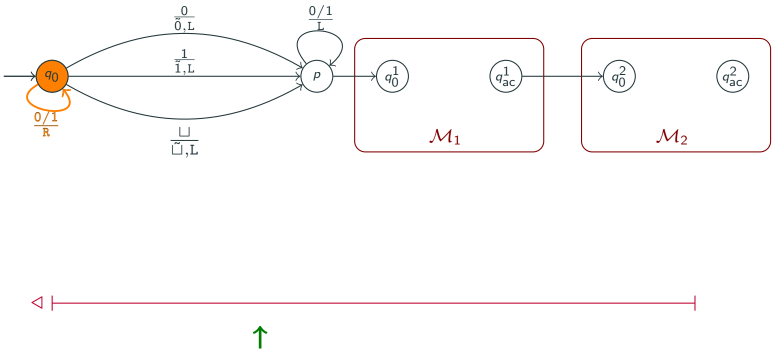

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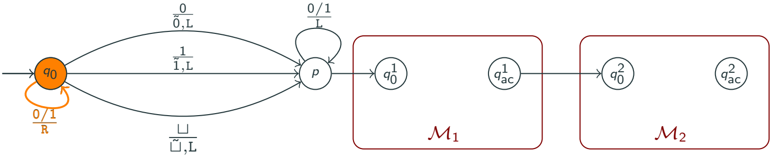

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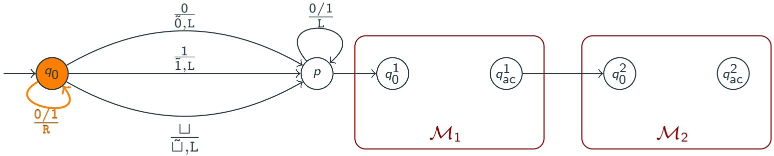

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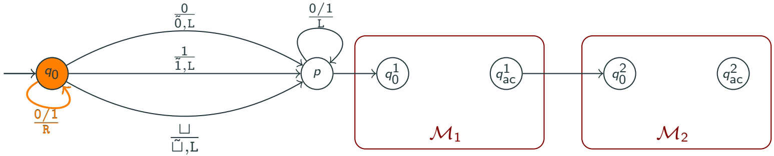

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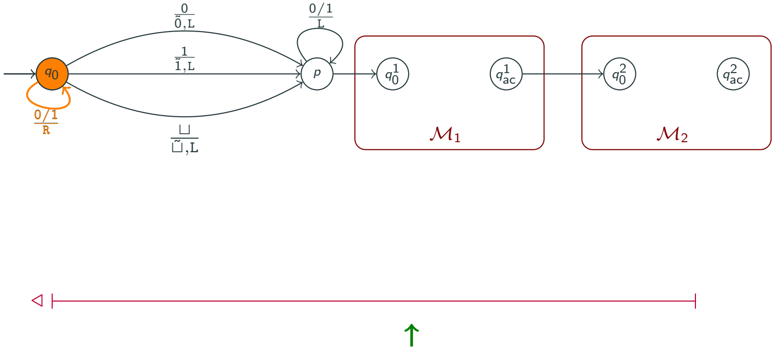

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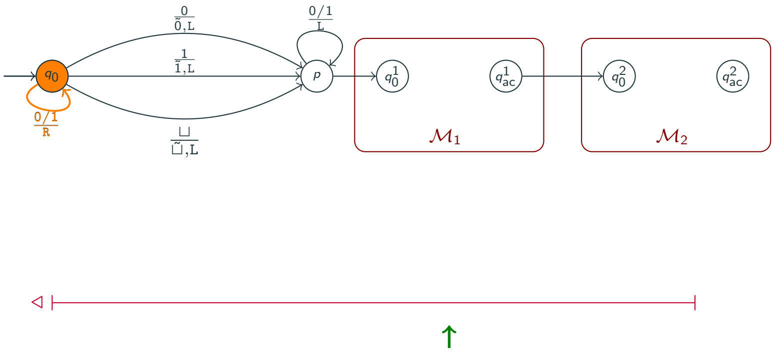

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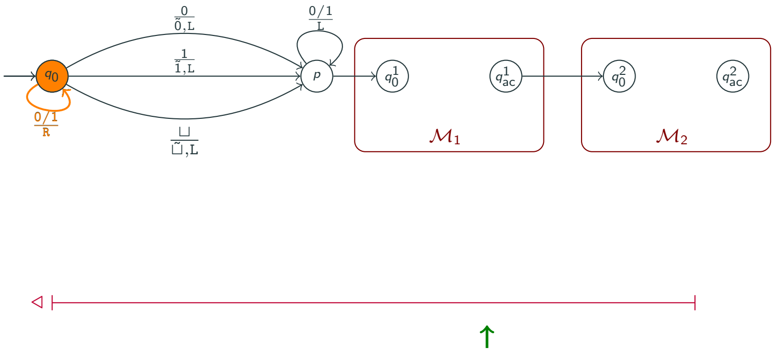

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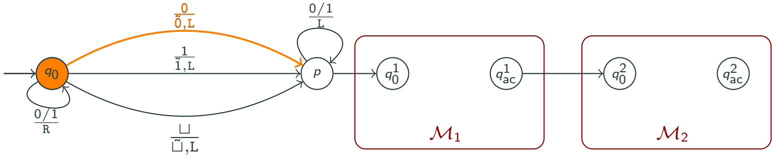

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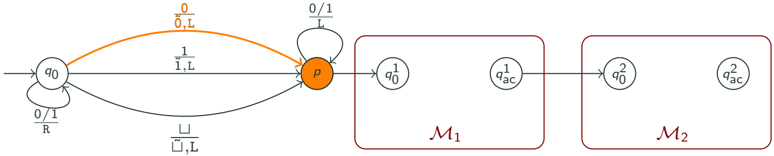

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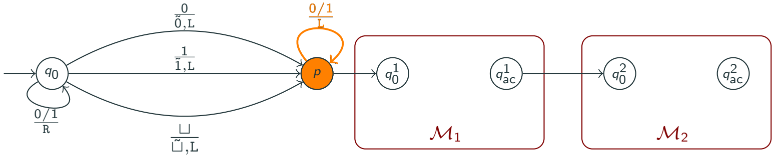

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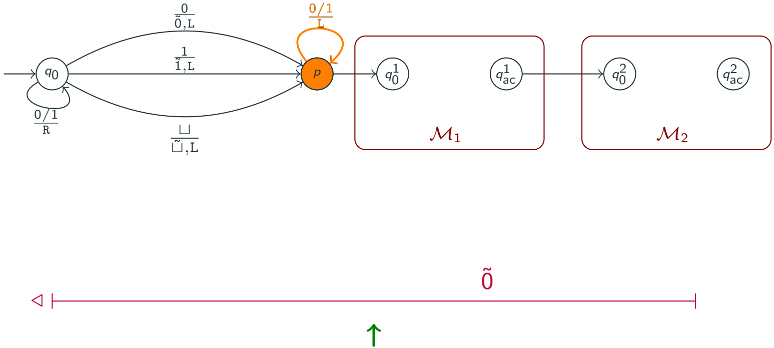

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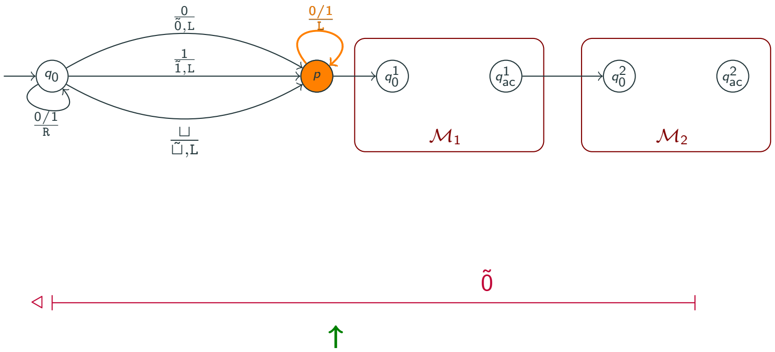

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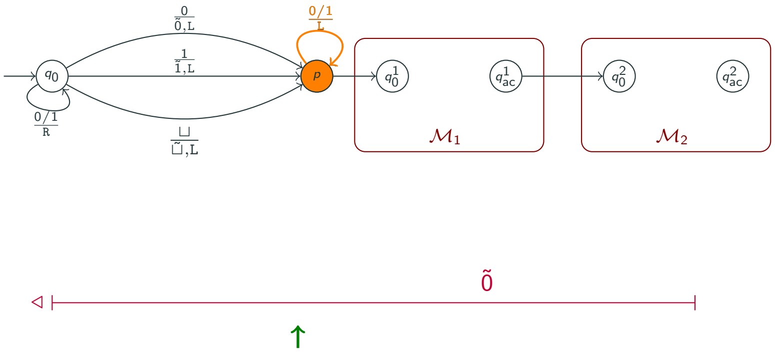

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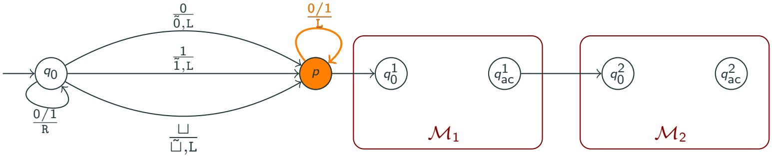

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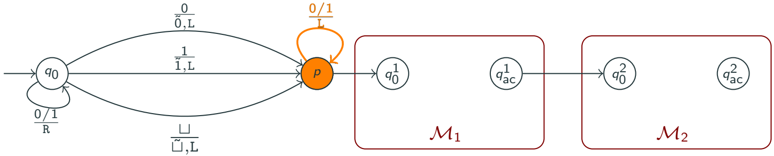

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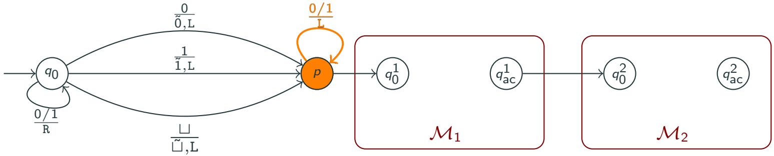

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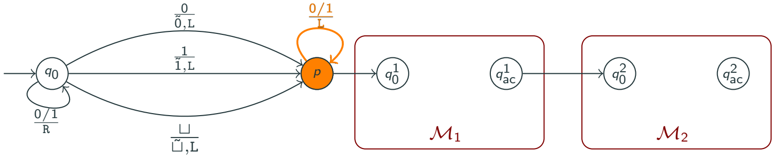

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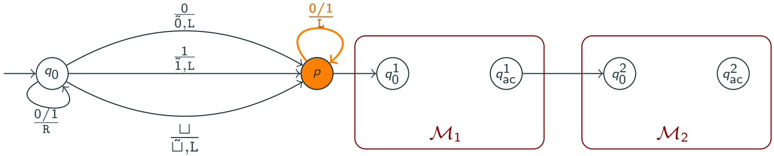

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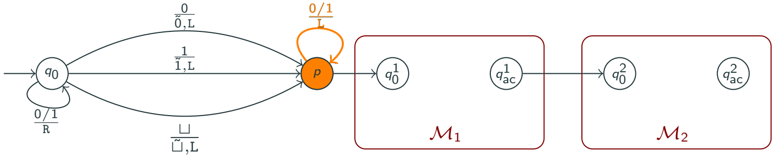

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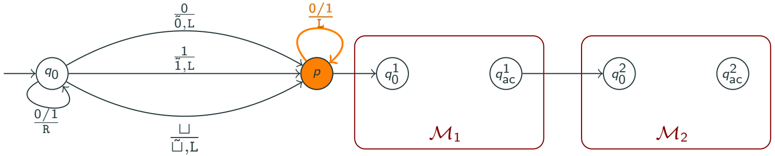

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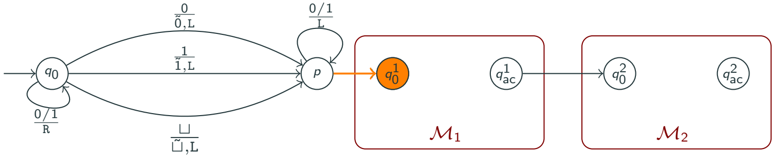

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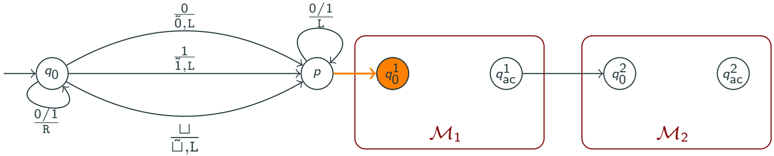

Proof of the closure under Kleene star

(Closure under Kleene star) We present a 2-tape NTM $\mathcal{M}$ that recognizes L_1^*.
On input word w:

- "Guess" a partition $v_1 \cdots v_k$ of w, for some $k \geqslant 1$.
- For each $i = 1, \ldots, k$:
  - Copy v_i onto the second tape.
  - Run $\mathcal{M}_1$ on v_i (on the second tape).
  - If $\mathcal{M}_1$ accepts, erase the second tape.
- ACCEPT.

Table of contents

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(Def.) A non-deterministic algorithm A "accepts" an input word w, if on every instruction:

$$z \quad := \quad 0 \parallel 1;$$

variable z can be assigned with 0 or 1 such that A will "return true."

Example: The problem SAT

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Input: A propositional formula φ.
Task: Output `True`, if φ has (at least one) satisfying assignment.
Otherwise, output `False`.

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Otherwise, output `False`.

(Algo.) On input formula φ:

- Let $x_1, \ldots, x_n$ be the variables in φ.
- For each $i = 1, \ldots, n$ do:
  - $z \; := \; 0 \parallel 1;$
  - If $z == 1$, assign x_i with `True`.
  - If $z == 0$, assign x_i with `False`.
- Check if the formula φ evaluates to true under the assignment.
- If it evaluates to `True`, then ACCEPT.
  If it evaluates to `False`, then REJECT.

Example: The problem Independent-Set

Independent-Set

Input: An undirected graph $G = (V, E)$ and an integer $k \geqslant 1$ (written in binary).
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(Algo.) On input graph $G = (V, E)$ and an integer $k \geqslant 1$:

- $S := \emptyset$.
- For each vertex $v \in V$ do:
  - $z \ := \ 0 \parallel 1$;
  - If $z == 1$, insert v into S.
- Check if the set S is an independent set and $|S| \geqslant k$.
- If it is, ACCEPT.
  If it is not, REJECT.

End of Lesson 9