Lesson 3. Context-free languages

CSIE 3110 – Formal Languages and Automata Theory

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Table of contents

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Context-free grammar (CFG)

(Def.) A *context-free grammar* (CFG) is a system $\mathcal{G} = \langle \Sigma, V, R, S \rangle$, where each component is as follows.

- Σ is a finite set of symbols, called *terminals*.
- V is a finite set of *variables*, and $V \cap \Sigma = \emptyset$.
- R is a finite set of *rules*, where each rule is of the form $A \to w$, where $A \in V$ and $w \in (V \cup \Sigma)^*$.
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$$A \to w_1, \quad A \to w_2, \ldots\ldots, \quad A \to w_m \qquad\qquad \text{in } R$$

usually abbreviated as:

$$A \quad \to \quad w_1 \mid w_2 \mid \cdots \mid w_m$$

Note also that we may have a rule of the form $A \to \varepsilon$.

Some examples of CFG

(Example 1) $\mathcal{G}_1 = \langle \Sigma, V, R, S \rangle$ where:

- $\Sigma = \{a, b\}$.
- $V = \{S\}$.
- R contains the rules: $S \to aSb \mid \varepsilon$.
- S is the start variable.

Some examples of CFG

(Example 2) $\mathcal{G}_2 = \langle \Sigma, V, R, S \rangle$ where:

- $\Sigma = \{a, b\}$.
- $V = \{S, T\}$.
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(Example 3) $\mathcal{G}_3 = \langle \Sigma, V, R, S \rangle$ where:

- $\Sigma = \{a, b\}$.
- $V = \{S\}$.
- R contains the rules: $S \to S$.
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Some examples of CFG

(Example 4) $\mathcal{G}_4 = \langle \Sigma, V, R, S \rangle$ where:

- $\Sigma = \{a, b\}$.
- $V = \{S, X, A, B, C\}$.
- R contains the rules:

$$\begin{array}{rcl} S & \rightarrow & XAXBXC \mid AA \mid BBA \mid CCA \\ X & \rightarrow & \varepsilon \mid aX \mid bX \\ A & \rightarrow & aaX \mid bbA \\ B & \rightarrow & baX \mid bbB \\ C & \rightarrow & abX \mid bbC \end{array}$$

- S is the start variable.

Notations and terminology

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We say that *uAv yields uwv*, denoted by $uAv \Rightarrow uwv$, if there is a rule $A \to w$ in R.

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Intuitively, the rule $A \rightarrow w$ means variable A can be replaced with w.

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(Def.) For $x, y \in (\Sigma \cup V)^*$, we say that *$x$ derives y*, denoted by $x \Rightarrow^* y$, if either $x = y$, or $x \Rightarrow z_1 \Rightarrow z_2 \Rightarrow \cdots \Rightarrow y$ (finitely many).

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Intuitively, the rule $A \to w$ means variable A can be replaced with w.

(Def.) For $x, y \in (\Sigma \cup V)^*$, we say that x *derives* y, denoted by $x \Rightarrow^* y$, if either $x = y$, or $x \Rightarrow z_1 \Rightarrow z_2 \Rightarrow \cdots \Rightarrow y$ (finitely many).

We will also say "y is derived from x" or "from x we can derive y."

Notations and terminology — continued

(Def.) For a variable A, $L(\mathcal{G}, A)$ denotes the language of all words over Σ that can be derived from variable A. Formally,

$$L(\mathcal{G}, A) \quad = \quad \{w \in \Sigma^* \;\mid\; A \Rightarrow^* w\}$$

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(Def.) $L(\mathcal{G})$ denotes the language $L(\mathcal{G}, S)$, i.e., the language of all words over Σ that can be derived from the start variable S. Formally,

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$L(\mathcal{G})$ is called the language generated/defined/derived from/by $\mathcal{G}$.

(Def.) A language L is called a *context-free language* (CFL), if there is a CFG $\mathcal{G}$ such that $L(\mathcal{G}) = L$.

Example 1: $\mathcal{G}_1 = \langle \Sigma, V, R, S \rangle$

$\Sigma = \{a, b\}$, $V = \{S\}$, S is the start variable and R contains the rule:

$$S \rightarrow aSb \mid \varepsilon$$

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- $S \Rightarrow \varepsilon$.

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- $S \Rightarrow \varepsilon$. So, $\varepsilon \in L(\mathcal{G}_1)$.
- $S \Rightarrow aSb \Rightarrow ab$. So, $ab \in L(\mathcal{G}_1)$.
- $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$. So, $aabb \in L(\mathcal{G}_1)$.

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- In general, for every integer $n \geqslant 0$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \cdots \Rightarrow \underbrace{a \cdots a}_{n}\underbrace{b \cdots b}_{n}$$

That is, $S \Rightarrow^* a^n b^n$, i.e., $a^n b^n \in L(\mathcal{G}_1)$, for every integer $n \geqslant 0$.

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In fact,

$$L(\mathcal{G}_1) \quad = \quad \{a^n b^n \mid n \geqslant 0\}$$

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In fact,

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Thus,

$$L(\mathcal{G}_3) \quad = \quad \emptyset$$

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Rules such as:

$$
\begin{aligned}
A_{\text{seq}} &\to A_{\text{1-ins}} \mid A_{\text{1-ins}} A_{\text{seq}} \\
A_{\text{if}} &\to \texttt{if } A_{\text{bool-cond}} \{ A_{\text{seq}} \} \mid \texttt{if } A_{\text{bool-cond}} \{ A_{\text{seq}} \} \texttt{ else } \{ A_{\text{seq}} \} \\
A_{\text{while}} &\to \texttt{while } A_{\text{bool-cond}} \{ A_{\text{seq}} \} \\
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A_{\text{while}} &\rightarrow \texttt{while } A_{\text{bool-cond}} \; \{ \; A_{\text{seq}} \; \} \\
&\vdots
\end{aligned}$$

In fact, any programming language is defined by a CFG.

Closure under union, concatenation and Kleene star

Theorem 3.2
Context-free languages are closed under union, concatenation and Kleene star.

(Proof) Let $\mathcal{G}_1 = \langle \Sigma, V_1, R_1, S_1 \rangle$ and $\mathcal{G}_2 = \langle \Sigma, V_2, R_2, S_2 \rangle$. First, we rename the variables in V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$.

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(Closure under union)

Consider the CFG $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ defined as follows.

- $V = V_1 \cup V_2 \cup \{S\}$, where S is a "new" variable, i.e., $S \notin V_1 \cup V_2$.
- $R = R_1 \cup R_2 \cup \{S \to S_1 | S_2\}$.
- S is the start variable.

It can be verified that $L(\mathcal{G}) = L(\mathcal{G}_1) \cup L(\mathcal{G}_2)$.

Closure under union, concatenation and Kleene star

(Proof — continued)

(Closure under concatenation)

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- S is the start variable.

It can be verified that $L(\mathcal{G}) = L(\mathcal{G}_1)L(\mathcal{G}_2)$.

Closure under union, concatenation and Kleene star

(Proof — continued)

(Closure under Kleene star)

Consider the CFG $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ defined as follows.

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- $R = R_1 \cup \{S \to S_1 S | \varepsilon\}$.
- S is the start variable.

It can be verified that $L(\mathcal{G}) = L(\mathcal{G}_1)^*$. See Note 3 for more details.

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Later we will see that context-free languages are <u>not</u> closed under intersection and complement.

Regular languages are CFL

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Every regular language is a context-free language.

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(Note:) There is a context-free language that is not regular!

Table of contents

Derivation trees as an alternative condition for CFL membership

(Def.) A *derivation tree*, or a *parse tree*, of a CFG $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ is a tree T in which:

- every vertex has a *label*, which is a symbol from $V \cup \Sigma \cup \{\varepsilon\}$;
- the label of an interior vertex is a variable from V;
- the label of a leaf vertex is either ε or a terminal from Σ;
- if an interior vertex has a label $A \in V$ and it has k children $n_1, \ldots, n_k$ (in the order from left to right) with labels $X_1, \ldots, X_k$, respectively, then $A \to X_1 \cdots X_k$ must be a rule in R.

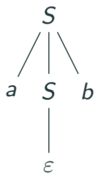
An example

- Every vertex has a *label*, which is a symbol from $V \cup \Sigma \cup \{\varepsilon\}$.
- The label of an interior vertex is a variable from V.
- The label of a leaf vertex is either ε or a terminal from Σ.
- If an interior vertex has a label $A \in V$ and it has k children $n_1, \ldots, n_k$ (in the order from left to right) with labels $X_1, \ldots, X_k$, respectively, then $A \to X_1 \cdots X_k$ must be a rule in R.

(Example) Consider a CFG with the following rules:

$$T \to SS \qquad \text{and} \qquad S \to aSb \mid \varepsilon$$

S, T are variables and the alphabet is $\{a, b\}$.



Definition of derivation trees — continued

(Def.) If the label of the root is a variable A, and the leaf vertices of T are $n_1, \ldots, n_m$ (in the order from left to right) with labels $u_1, \ldots, u_m$, we say that T is the *derivation tree of $\mathcal{G}$ from variable A on word $u_1 \cdots u_m$*.

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Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ be a CFG. For every variable $A \in V$, for every word $w \in \Sigma^*$, the following holds.

$$A \Rightarrow^* w \quad \text{if and only if} \quad \text{there is a derivation tree of } \mathcal{G} \text{ from } A \text{ on } w.$$

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The proof is straightforward, but the idea is best illustrated by examples.

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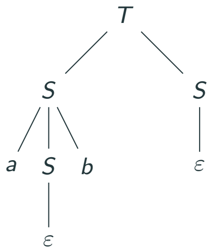

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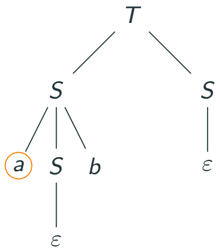

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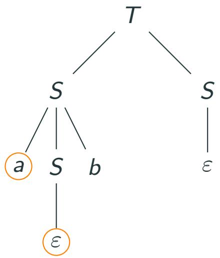

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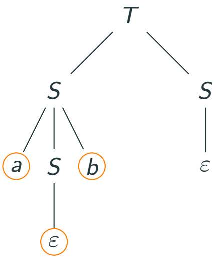

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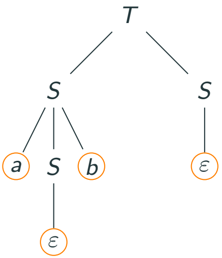

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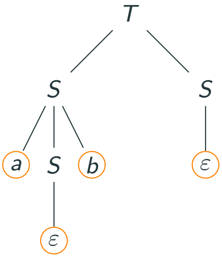

This is a derivation tree of the CFG on ab.

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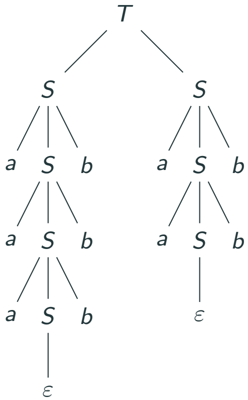

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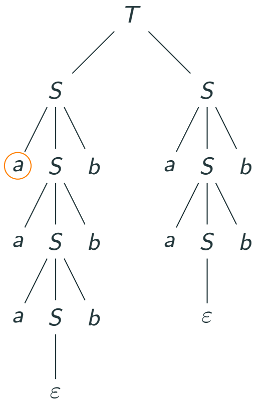

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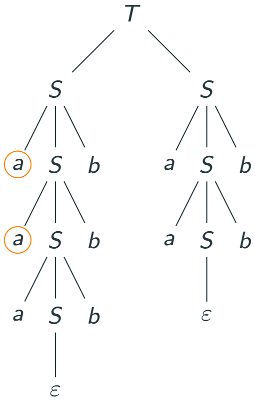

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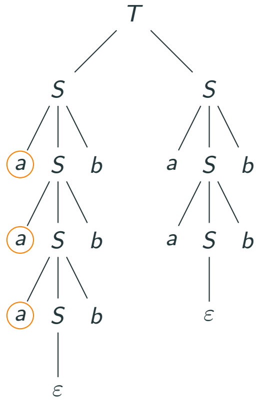

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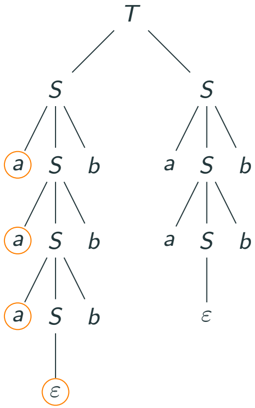

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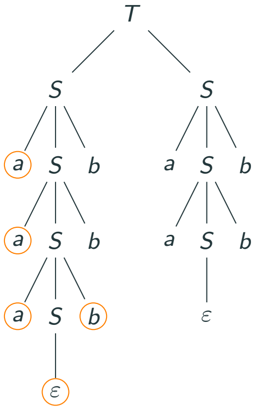

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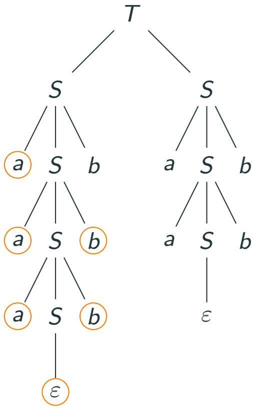

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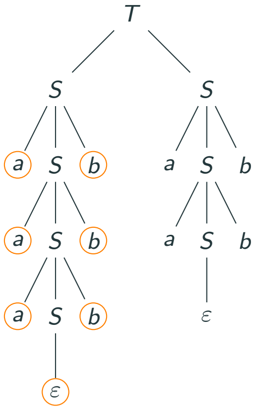

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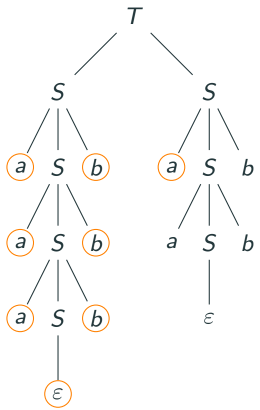

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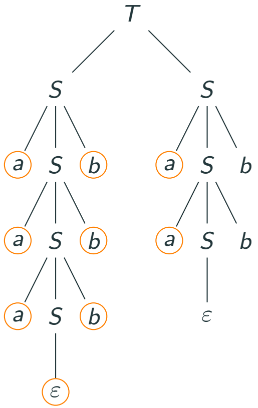

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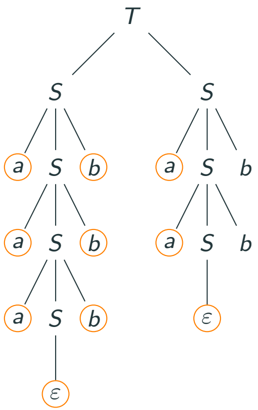

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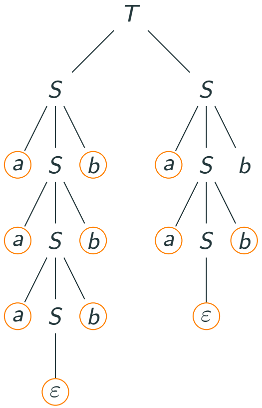

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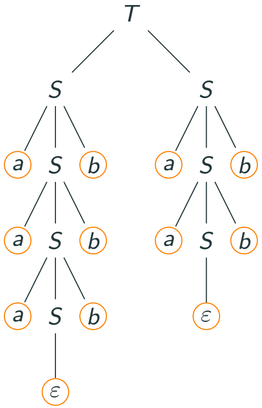

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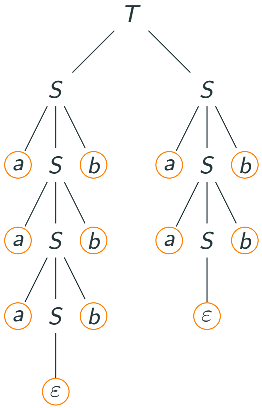

This is a derivation tree of the CFG on $a^3b^3a^2b^2$.

Derivation trees as an alternative condition for CFL membership

From the example, it is not difficult to see that derivation trees are just an alternative condition for CFL membership.

Theorem 3.5

Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ be a CFG. For every variable $A \in V$, for every word $w \in \Sigma^*$, the following holds.

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$$w \quad = \quad s\ x\ y\ z\ t$$

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- $|x| + |z| \geqslant 1$.
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Intuitively, this means that for a word of length $\geqslant N$, its derivation tree will have depth $\geqslant n + 1$.

Proof of pumping lemma (Continued)

Let $w \in L(\mathcal{G})$ and $|w| \geqslant N$. Recall that $N = m^n + 1$.

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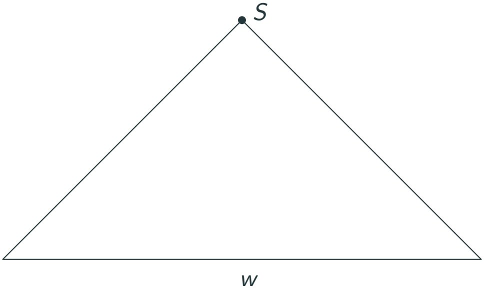
Consider its derivation tree T and its depth $\geqslant n + 1$.



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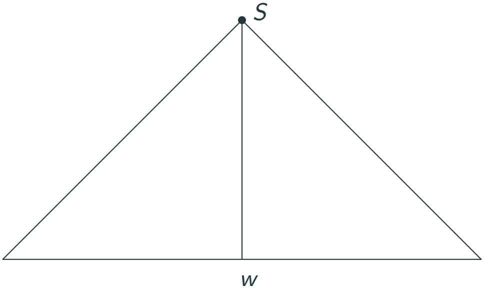

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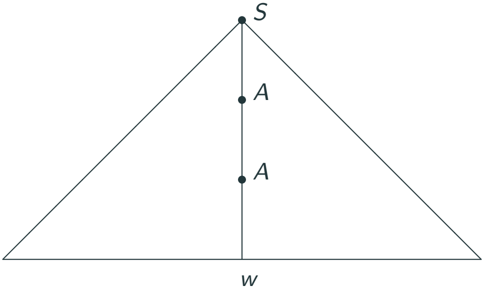

There exists a variable A that appears at least twice in the same path.

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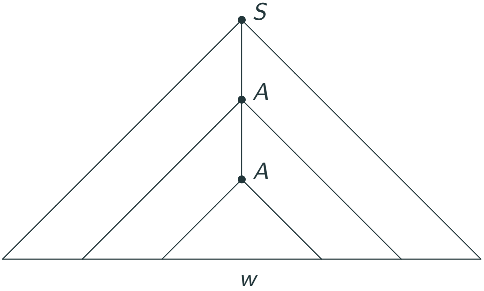

Pick the variable A such that in its subtree there is no variable that appears twice.

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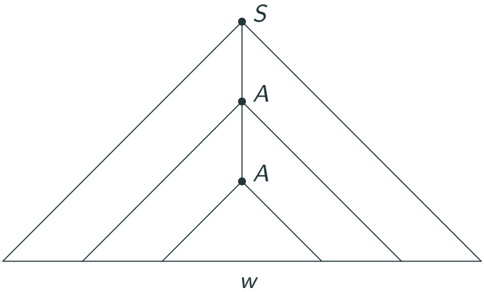

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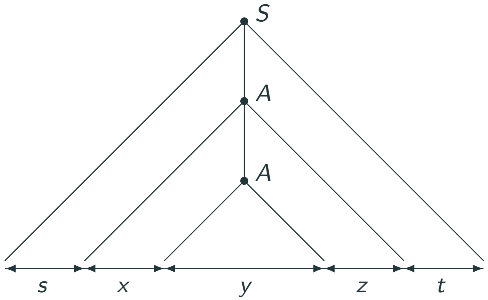

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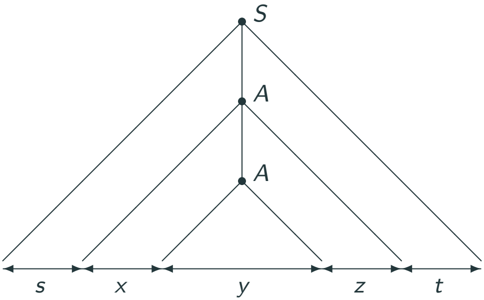

The three conditions hold.

- $|x| + |z| \geqslant 1$.
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- For every $i \geqslant 0$, $sx^i yz^i t \in L(\mathcal{G})$. $\Rightarrow$ By "pumping" variable A.

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The length N is usually called the pumping length of $\mathcal{G}$.

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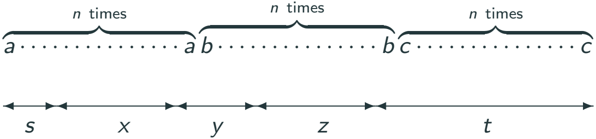

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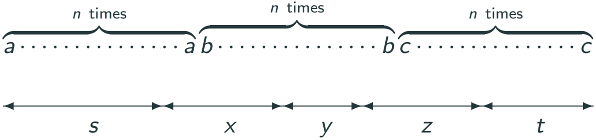

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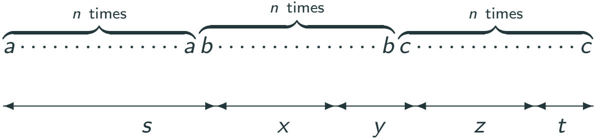

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For any other partition, pumping x and z will result in a word not in L. Thus, contradicting pumping lemma.

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Note: For a language L over alphabet Σ, $\overline{L} = \Sigma^* - L$, i.e., the complement of the language L.

End of Lesson 3