Lesson 2. Regular expressions

CSIE 3110 – Formal Languages and Automata Theory

Tony Tan
Department of Computer Science and Information Engineering
College of Electrical Engineering and Computer Science
National Taiwan University

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Regular expressions

(Def.) Let Σ be an alphabet.

Regular expressions (over Σ) are expressions built inductively as follows.

- $\emptyset$ is a regular expression.
- a is a regular expression, for every symbol $a \in \Sigma$.
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Regular expression is usually abbreviated as *regex*.

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Similarly, we can say that a regular expression e is a finite representation of the language $L(e)$.

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- $L((a \cup b)^* a) = (L(a \cup b)^* \cdot L(a) = \{a, b\}^* \cdot \{a\}$.
  That is, the language $\{w \mid w \text{ is a word that ends with } a\}$.

The main theorem in this lesson

Theorem 2.1

Regular expressions define precisely the class of regular languages.
More formally:

- *For every regular expression e over Σ, $L(e)$ is a regular language, i.e., there is an NFA $\mathcal{A}$ such that $L(\mathcal{A}) = L(e)$.*
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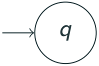

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  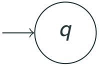

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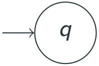

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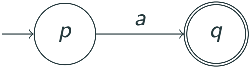

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By the induction hypothesis, there are NFA $\mathcal{A}_1$ and $\mathcal{A}_2$ that accept the languages $L(\alpha)$ and $L(\beta)$, respectively.

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Since regular languages are closed under concatenation, union and Kleene star, (See Remark 1.4 and Theorem 1.8 in Lesson 1), there are NFAs for all the languages $L(\alpha \cdot \beta)$, $L(\alpha \cup \beta)$ and $L(\alpha^*)$.

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Recall that:

- $L(\alpha \cdot \beta) = L(\alpha) \cdot L(\beta)$.
- $L(\alpha \cup \beta) = L(\alpha) \cup L(\beta)$.
- $L(\alpha^*) = L(\alpha)^*$.

We now prove the second item:

Theorem (The second item of Theorem 2.1)

- *For every NFA $\mathcal{A}$, there is a regular expression e such that $L(e) = L(\mathcal{A})$.*

(Proof) Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ be an NFA, where $Q = \{1, \ldots, n\}$.

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$$L(i,j,k) \quad := \quad \left\{ \; w \in \Sigma^* \; \middle| \; \begin{array}{l} \text{there is a run of } \mathcal{A} \text{ on } w \text{ from state } i \text{ to state } j \\ \textit{without} \text{ passing any states } \; \geqslant k+1 \end{array} \right\}$$

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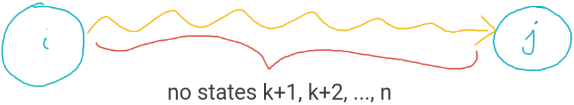

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- If $i \neq j$ and there is no transition from i to j:

The language $L(i, j, 0) = \emptyset$, so the regex e is $\emptyset$.

Claim 1

For every $1 \leqslant i, j \leqslant n$ and $0 \leqslant k \leqslant n$, there is a regex e such that $L(e) = L(i, j, k)$.

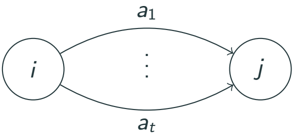
(Proof of Claim 1: By induction on k)

(Base case $k = 0$) For every $1 \leqslant i, j \leqslant n$, we consider $L(i, j, 0)$:

- If $i \neq j$ and there is no transition from i to j:

  The language $L(i, j, 0) = \emptyset$, so the regex e is $\emptyset$.

- If $i \neq j$ and there are some transitions from i to j:



  The language $L(i, j, 0) = \{a_1, \ldots, a_t\}$, so the regex e is $a_1 \cup \cdots \cup a_t$.

(Base case $k = 0$ – continued)

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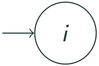
- If $i = j$ and there is no transition from i to i:



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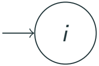

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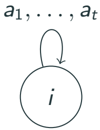

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(Induction step – proof of Claim 1)

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$$L(i,j,k+1) \quad = \quad L(i,j,k) \;\cup\; \Big(L(i,k+1,k)\cdot L(k+1,k+1,k)^{*}\cdot L(k+1,j,k)\Big)$$

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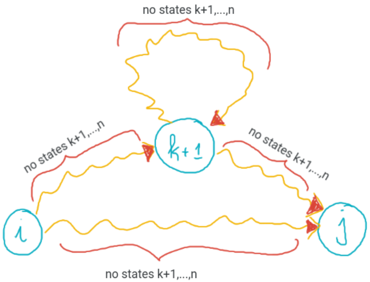

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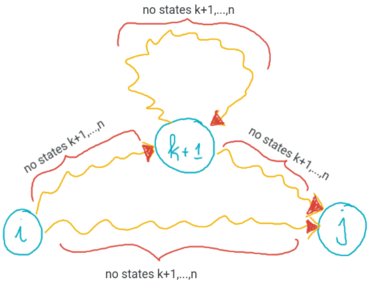

By induction hypothesis, there is regex for each of $L(i,j,k)$, $L(i,k+1,k)$, $L(k+1,k+1,k)$, and $L(k+1,j,k)$.

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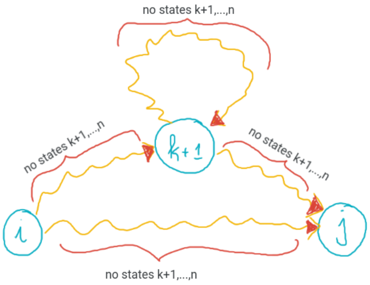

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Thus, there is regex for $L(i,j,k+1)$.

(Finishing the proof of Claim 1)

The language $L(\mathcal{A})$ can be defined as:

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By Claim 1, there is a regex that defines each $L(q_0, q_f, n)$.

Taking the union over all $q_f \in F$, we have a regex for $L(\mathcal{A})$.

To conclude:

Corollary 2.2

Let L be a language. The following are equivalent.

- *L is accepted by a DFA.*
- *L is accepted by an NFA.*
- *L is defined by a regular expression.*

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One nice implication of this corollary is that languages defined by regular expression are also closed under intersection and complement.

This is despite the fact that we are not allowed to use intersection or negation in regular expressions.

End of Lesson 2