Lesson 1. Finite state automata

CSIE 3110 – Formal Languages and Automata Theory

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In this case, we will say that "*$\mathcal{A}$ is a DFA over alphabet Σ*," or that "*the alphabet of $\mathcal{A}$ is Σ*."

Example 1

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{a, b\}$
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $F = \{p, q\}$ is the set of *accepting* states.
- The transition function δ is defined as:

$$
\begin{aligned}
\delta(p, a) &= p \qquad & \delta(p, b) &= r \\
\delta(q, a) &= p \qquad & \delta(q, b) &= p \\
\delta(r, a) &= q \qquad & \delta(r, b) &= r
\end{aligned}
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$$\begin{aligned} \delta(p,a) &= p & \qquad \delta(p,b) &= r \\ \delta(q,a) &= p & \qquad \delta(q,b) &= p \\ \delta(r,a) &= q & \qquad \delta(r,b) &= r \end{aligned}$$

This is a valid DFA.

Example 2

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{a, b\}$
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $\underline{F = \emptyset}$, i.e., it does not have any accepting state.
- The transition function δ is defined as:

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\begin{array}{ll}
\delta(p, a) = p & \delta(p, b) = r \\
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Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\underline{\Sigma = \{a\}}$.
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This is also a valid DFA.

Example 4

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\underline{\Sigma = \emptyset}$, i.e., the alphabet does not contain any symbol.
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $F = \{p, q\}$, i.e., it does not have any accepting state.
- The transition function δ is not defined since $\Sigma = \emptyset$.

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- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $F = \{p, q\}$, i.e., it does not have any accepting state.
- The transition function δ is not defined since $\Sigma = \emptyset$.

This is not a valid DFA, since the alphabet Σ must contain at least one symbol.

Example 5

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{0, 1\}$.
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $F = \{p, q\}$.
- The transition function δ is defined as:

$$\begin{aligned} \delta(p, a) &= p & \qquad \delta(p, b) &= r \\ \delta(q, a) &= p & \qquad \delta(q, b) &= p \\ \delta(r, a) &= q & \qquad \delta(r, b) &= r \end{aligned}$$

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This is not a valid DFA, since the transition function δ is defined on $Q \times \{a, b\}$, but the alphabet should be $\{0, 1\}$.

Example 6

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{0, 1\}$.
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
- $F = \{p, q\}$.
- The transition function δ is defined as:

$$\begin{aligned} \delta(p, 0) &= p & \qquad \delta(p, 1) &= r \\ \delta(q, 0) &= p & \qquad \delta(q, 1) &= p \\ \delta(r, 0) &= q \end{aligned}$$

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This is not a valid DFA, since δ is not defined on $(r, 1)$.

Example 7

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{0, 1\}$.
- $Q = \{q, p, r\}$ is the set of states.
- There is no initial state.
- $F = \{p, q\}$.
- The transition function δ is defined as:

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- $Q = \{q, p, r\}$ is the set of states.
- There is no initial state.
- $F = \{p, q\}$.
- The transition function δ is defined as:

$$\begin{array}{ll} \delta(p, 0) = p & \delta(p, 1) = r \\ \delta(q, 0) = p & \delta(q, 1) = p \\ \delta(r, 0) = q & \delta(r, 1) = q \end{array}$$

This is not a valid DFA, because DFA must have the initial state.

Example 8

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{0, 1\}$.
- $Q = \{q, p, r\}$ is the set of states.
- p and r are the initial states.
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- The transition function δ is defined as:

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This is not a valid DFA, because DFA must have exactly one initial state.

Visualizing DFA

Consider the following DFA $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ over $\Sigma = \{a, b\}$, where $Q = \{q, p, r\}$, r is the initial state, $F = \{p\}$ and δ is defined as:

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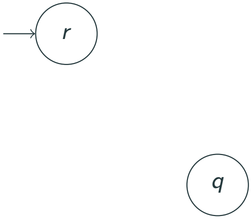
We can visualize it as a directed graph:



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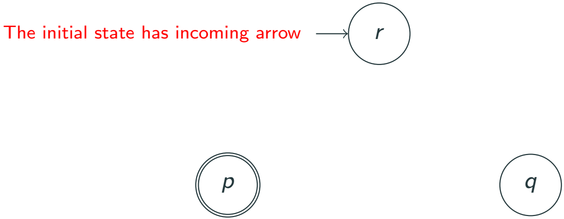

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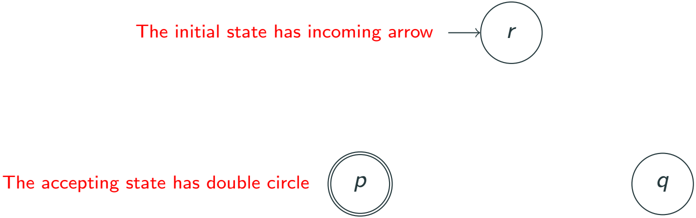

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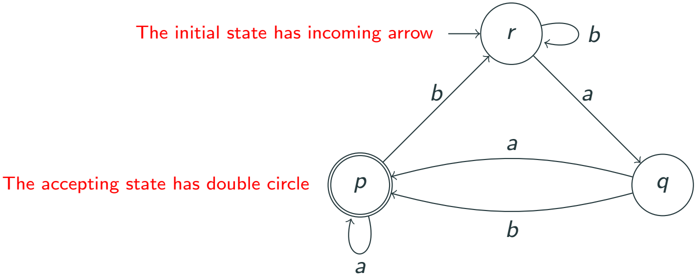

Important note!

In your solution for homework and exams, don't write DFA like this:

$\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ over $\Sigma = \{a, b\}$, where $Q = \{q, p, r\}$, r is the initial state, $F = \{p\}$ and δ is defined as:

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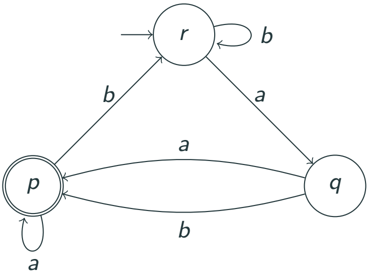
But draw the graph representation of DFA like this:



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It starts from the initial state.

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A DFA either accepts/rejects its input.

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Its input is always a finite string over its alphabet.

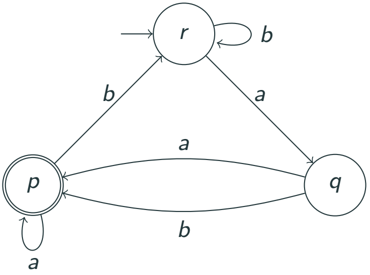
It moves from state to state depending on the input symbol that it reads.

It starts from the initial state.

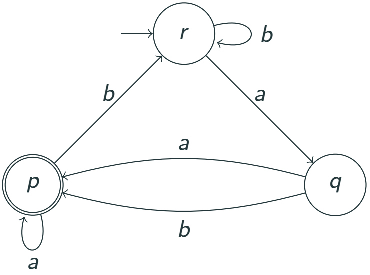
A DFA either accepts/rejects its input.

We can view “accept” as returning `True` and “reject” as returning `False`.

Example

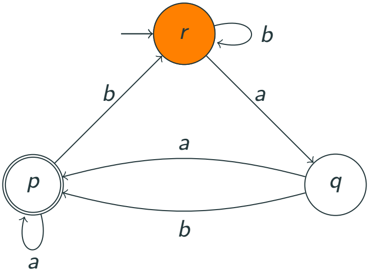


Example



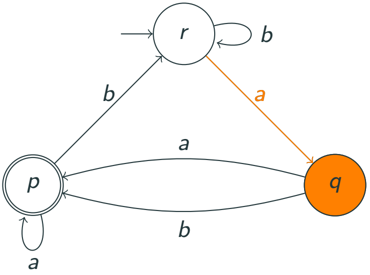
On input string *aba*:

Example



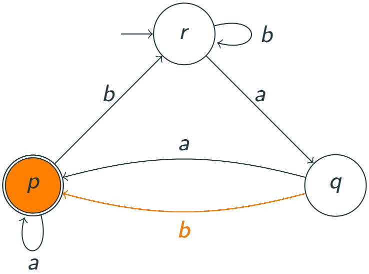
On input string *aba*: r

Example



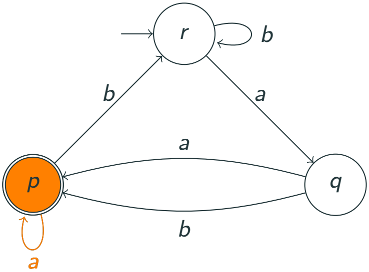
On input string *aba*: $r \quad \underline{a} \quad q$

Example



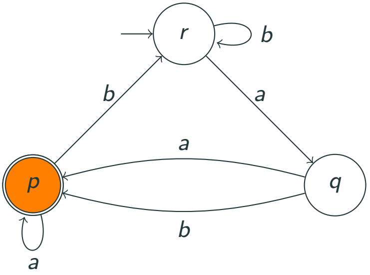
On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p$

Example



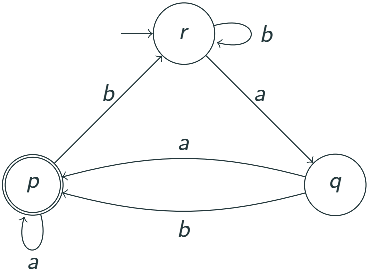
On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$

Example



On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$ (accepted by DFA)

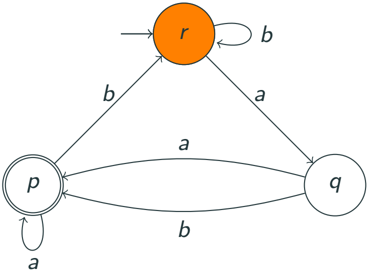
Example



On input string *aba*: $r \ \underline{a} \ q \ \underline{b} \ p \ \underline{a} \ p$ (accepted by DFA)

On input string *aab*:

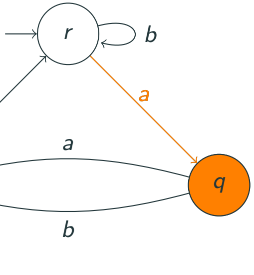
Example



On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$ (accepted by DFA)

On input string *aab*: r

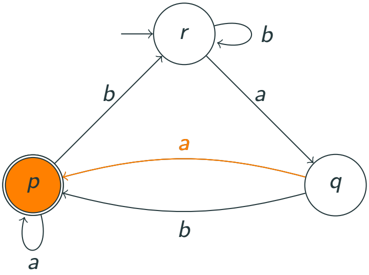
Example



On input string *aba*: $r \ \underline{a} \ q \ \underline{b} \ p \ \underline{a} \ p$ (accepted by DFA)

On input string *aab*: $r \ \underline{a} \ q$

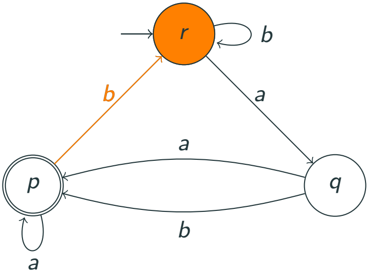
Example



On input string *aba*: r $\underline{a}$ q $\underline{b}$ p $\underline{a}$ p (accepted by DFA)

On input string *aab*: r $\underline{a}$ q $\underline{a}$ p

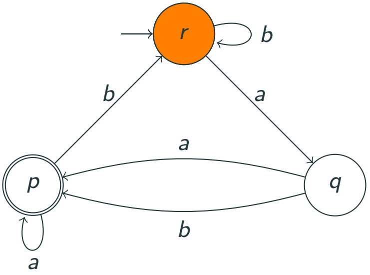
Example



On input string *aba*: $r \ \underline{a} \ q \ \underline{b} \ p \ \underline{a} \ p$ (accepted by DFA)

On input string *aab*: $r \ \underline{a} \ q \ \underline{a} \ p \ \underline{b} \ r$

Example



On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$ (accepted by DFA)

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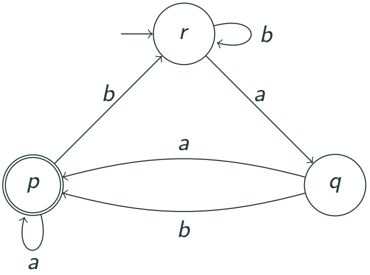
Example



On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$ (accepted by DFA)

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On input string ε:

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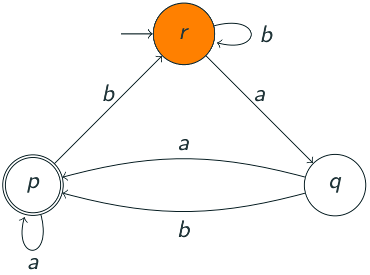

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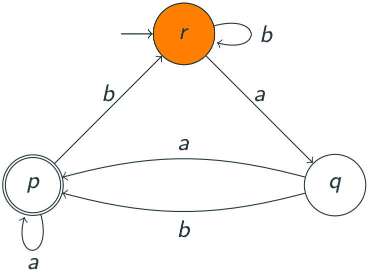
On input string ε: r

Example



On input string *aba*: $r \quad \underline{a} \quad q \quad \underline{b} \quad p \quad \underline{a} \quad p$ (accepted by DFA)

On input string *aab*: $r \quad \underline{a} \quad q \quad \underline{a} \quad p \quad \underline{b} \quad r$ (not accepted by DFA)

On input string ε: r (not accepted by DFA)

The formal definition of acceptance/rejection of words by DFA

Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$.

(Def.) On input word $w = a_1 \cdots a_n$, *the run of $\mathcal{A}$ on w* is the sequence:

$$p_0 \ a_1 \ p_1 \ a_2 \ p_2 \ \cdots \ a_n \ p_n,$$

where $p_0 = q_0$ and $\delta(p_i, a_{i+1}) = p_{i+1}$, for each $i = 0, \ldots, n-1$.

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(Def.) A run is called an *accepting* run, if $p_0 = q_0$ and $q_n \in F$.

The language accepted by DFA

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Some observations on DFA

(Rem. 1.2) Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ be a DFA.

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(Rem. 1.2) Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ be a DFA.

- For every word w, there is exactly *one* run of $\mathcal{A}$ on w.
- The empty string ε is accepted by $\mathcal{A}$ if and only if $q_0 \in F$.

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We will show that L_0 is a regular language.

Constructing a DFA for $L_0 := \{w \mid [\![w]\!] \equiv 0 \pmod 3\}$

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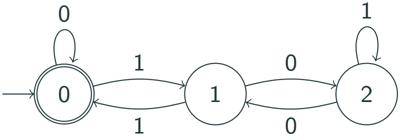

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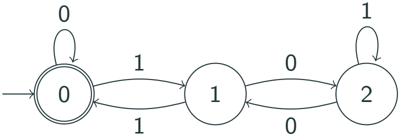

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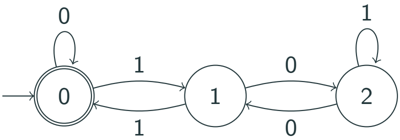

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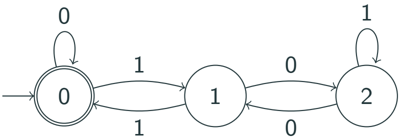

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So $L(\mathcal{A}) = L_0$.

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See Note 1 for the formal proof of Theorem 1.3.

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Note: In DFA, δ is a function $\delta : Q \times \Sigma \to Q$.

In NFA, δ is any subset of $Q \times \Sigma \times Q$.

Example 1

Consider the following $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$:

- $\Sigma = \{a, b\}$
- $Q = \{q, p, r\}$ is the set of states.
- r is the initial state.
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- The transition relation δ is a function defined as:

$$\begin{aligned} \delta(p, a) &= p & \qquad \delta(p, b) &= r \\ \delta(q, a) &= p & \qquad \delta(q, b) &= p \\ \delta(r, a) &= q & \qquad \delta(r, b) &= r \end{aligned}$$

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A DFA is a special case of NFA, because function is a special case of relation. (See Note 0.)

Visualizing NFA

Consider an DFA $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ over $\Sigma = \{a, b\}$, where $Q = \{q, p, r\}$, r is the initial state, $F = \{p\}$ and δ is as follows.

$$\delta = \{(p, a, p), (p, a, q), (p, b, q), (q, b, r), (r, a, q), (r, b, r)\}$$

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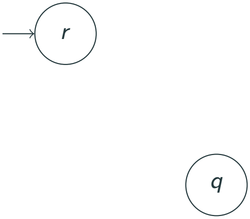

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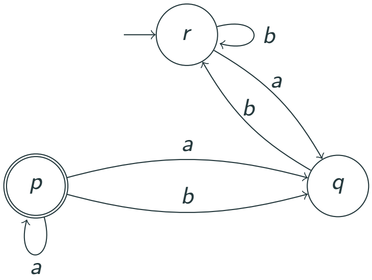

Acceptance/rejection of words by NFA

Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ be an NFA.

(Def.) On input word $w = a_1 \cdots a_n$, *a run of $\mathcal{A}$ on w* is the sequence:

$$p_0 \ a_1 \ p_1 \ a_2 \ p_2 \ \cdots \ a_n \ p_n,$$

where $p_0 = q_0$ and $(p_i, a_{i+1}, p_{i+1}) \in \delta$, for each $i = 0, \ldots, n-1$.

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(Def.) *A run of $\mathcal{A}$ on w starting from state q* is defined as the sequence above, but with condition $p_0 = q$.

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(Def.) *A run of* $\mathcal{A}$ *on* w *starting from state* q is defined as the sequence above, but with condition $p_0 = q$.

(Def.) A run is called an *accepting* run, if $p_0 = q_0$ and $q_n \in F$.

The language accepted by NFA

Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$.

(Def.) We say that $\underline{\mathcal{A}\ \textit{accepts}\ w}$, if <u>there is</u> an accepting run of $\mathcal{A}$ on w.

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The language accepted by NFA

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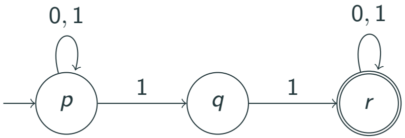
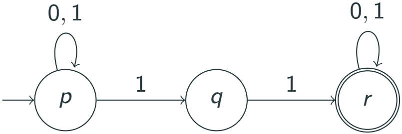
(Def.) The language of all words accepted by $\mathcal{A}$ is denoted by $L(\mathcal{A})$.

(Def.) A language L is called an *NFA language*, if there is a NFA $\mathcal{A}$ such that $L(\mathcal{A}) = L$.

Example 5

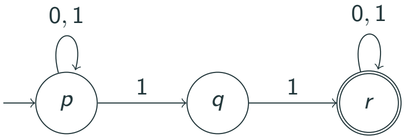


Example 5



On input string 10110, there are many possible runs:

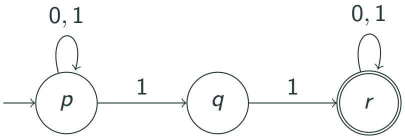
Example 5



On input string 10110, there are many possible runs:

- p 1 p 0 p 1 p 1 p 0 p. (not an accepting run).

Example 5



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- p 1 p 0 p 1 p 1 p 0 p. (not an accepting run).
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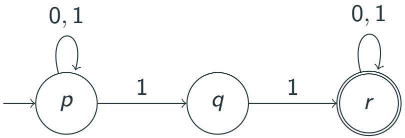

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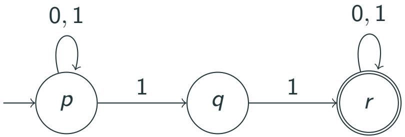

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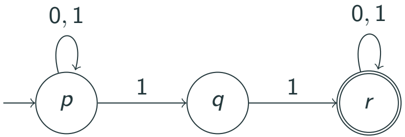

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- $p\ 1\ p\ 0\ p\ 1\ q\ 1\ r\ 0\ r$. (an accepting run).
- ... (there are many other runs)

There is an accepting run so $\mathcal{A}$ accepts 10110.

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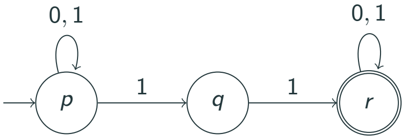

On Input word: 10110

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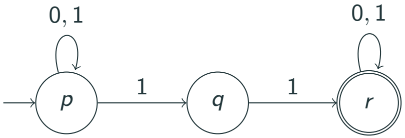

On Input word: 10110

p

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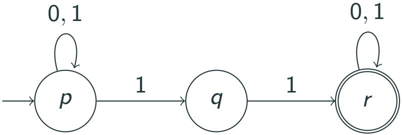

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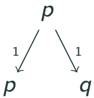

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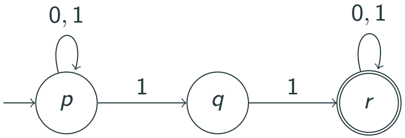

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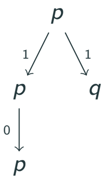

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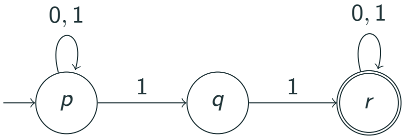

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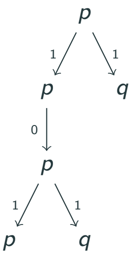

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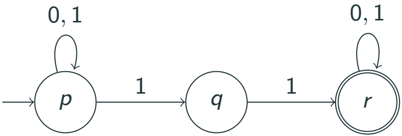

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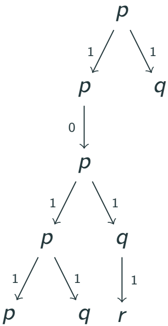

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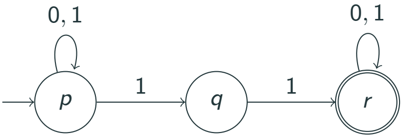

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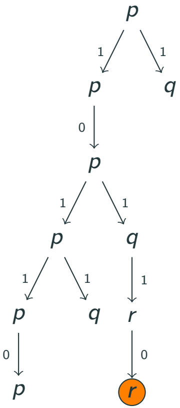

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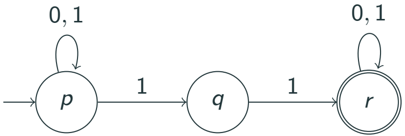

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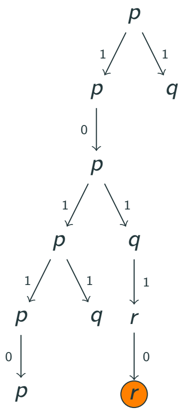

p

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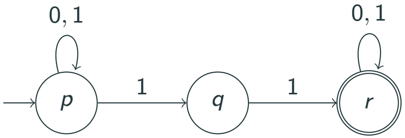

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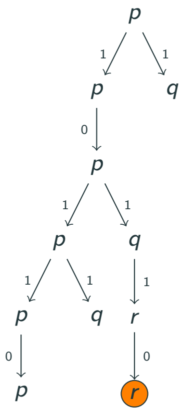

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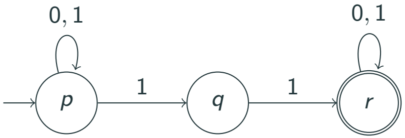

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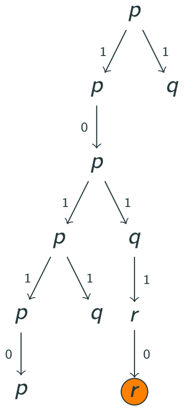

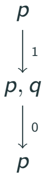

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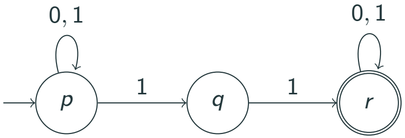

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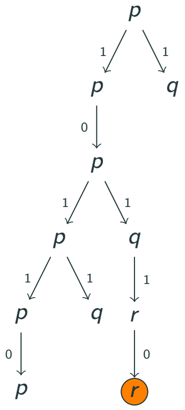

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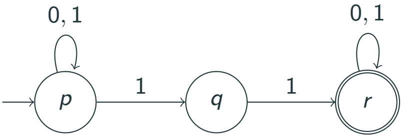

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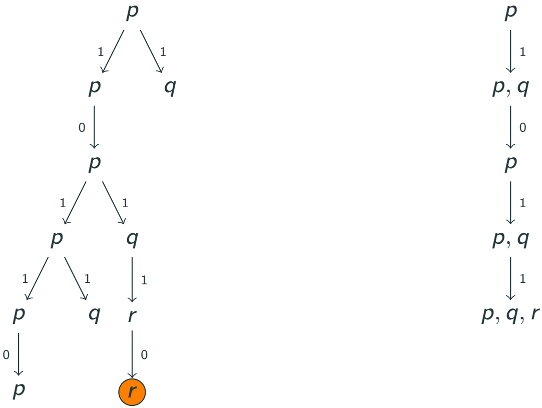

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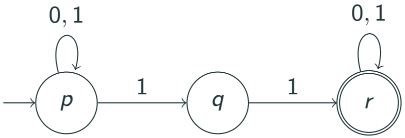

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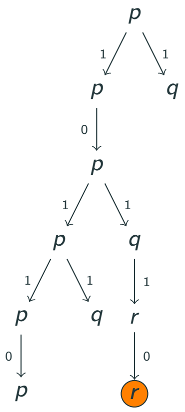

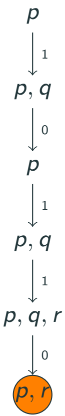

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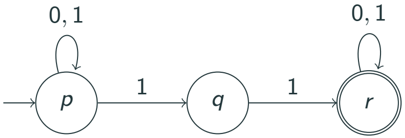
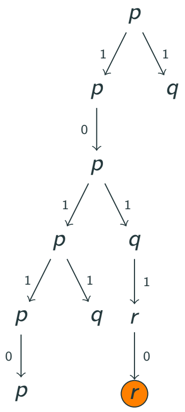
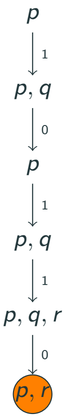


On Input word: 10110 (accepted)





Closure under union and intersection

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More formally, it can be stated as follows.

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The proof is the same as the one for DFA.

NFA can be converted to DFA

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$$\delta'(S, a) \quad = \quad \{p \mid \text{ there is } q \in S \text{ such that } (q, a, p) \in \delta\}$$

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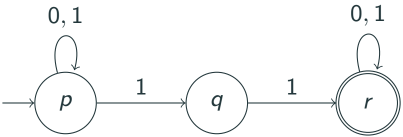
It can be shown that $L(\mathcal{A}') = L(\mathcal{A})$. See Note 1 for more details.

The intuitive idea



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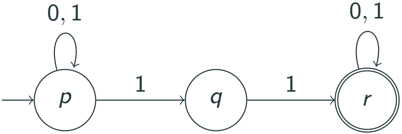
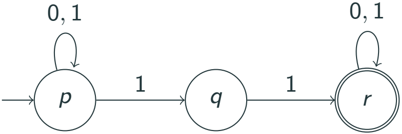


On input 10110:

The intuitive idea

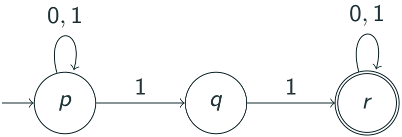


On input 10110:

$$\begin{array}{c} p \\ \big\downarrow {\scriptstyle 1} \\ p, q \\ \big\downarrow {\scriptstyle 0} \\ p \\ \big\downarrow {\scriptstyle 1} \\ p, q \\ \big\downarrow {\scriptstyle 1} \\ p, q, r \\ \big\downarrow {\scriptstyle 0} \\ p, r \end{array}$$

On input w, the set of states it can get to is a subset of $\{p, q, r\}$

The intuitive idea



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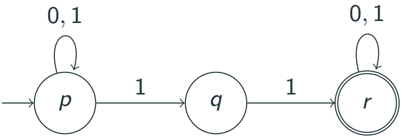

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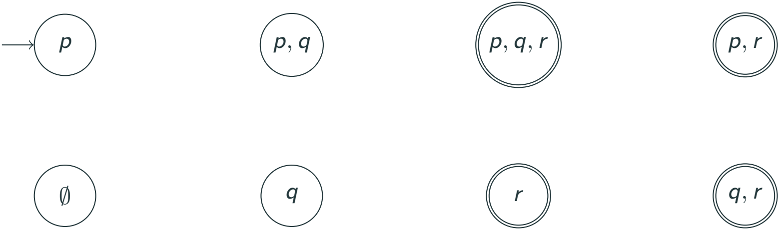

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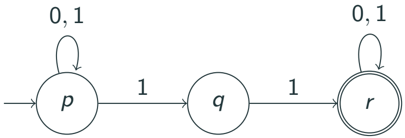

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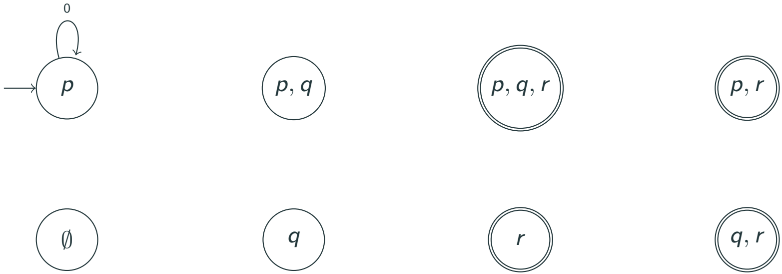

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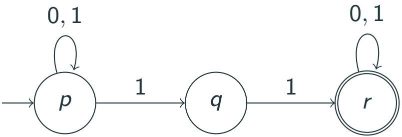

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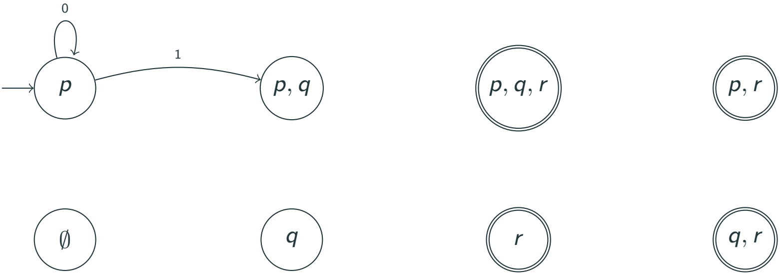

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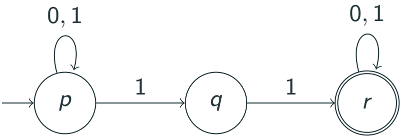

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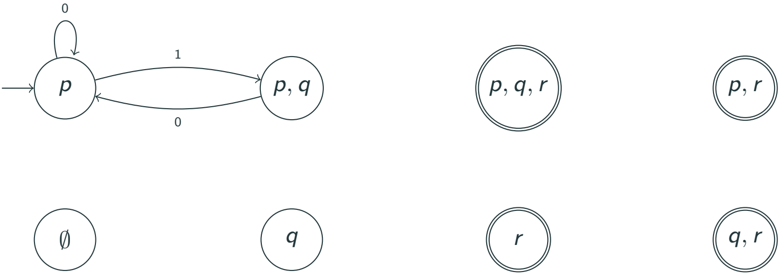

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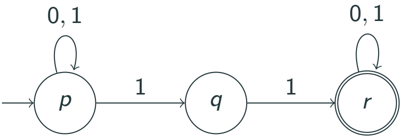

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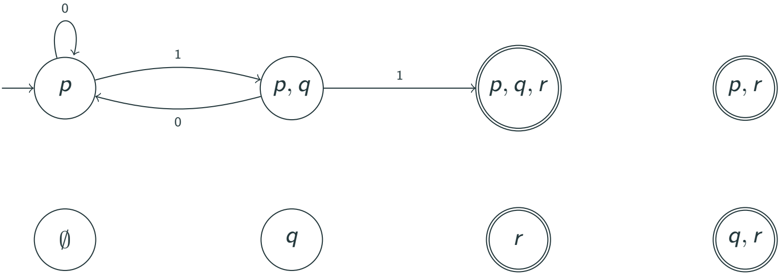

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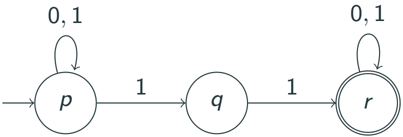

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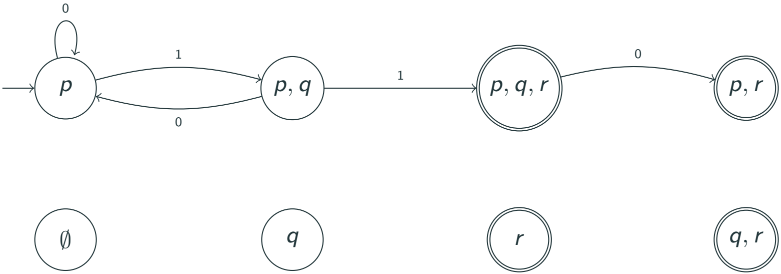

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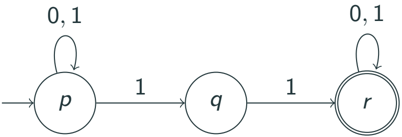

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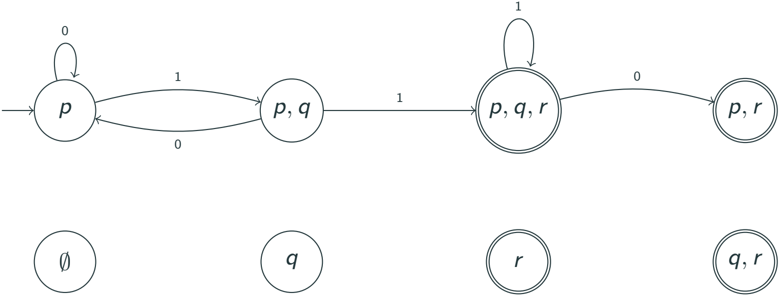

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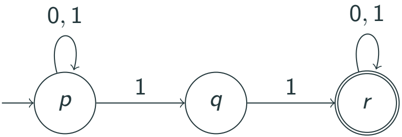

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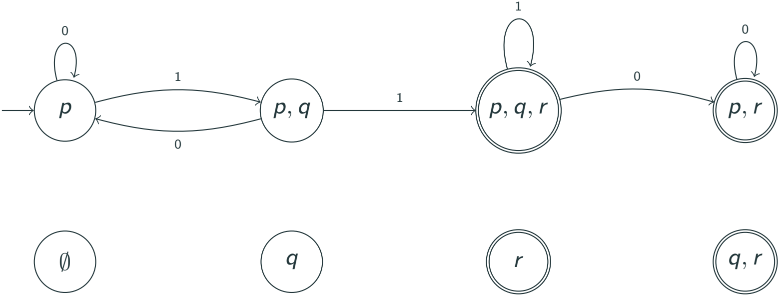

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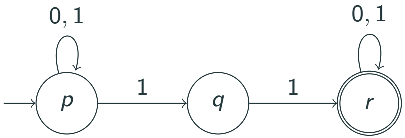

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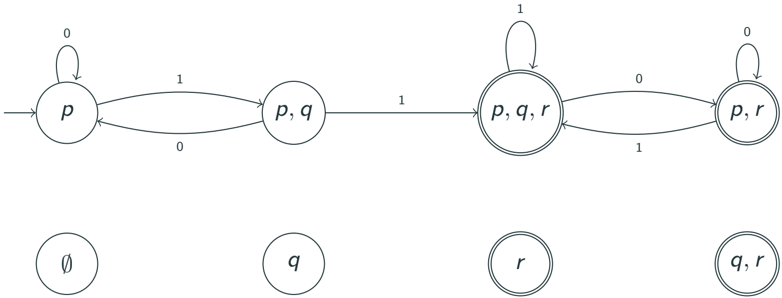

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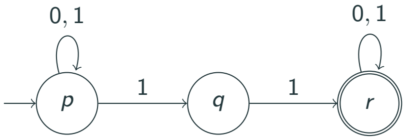

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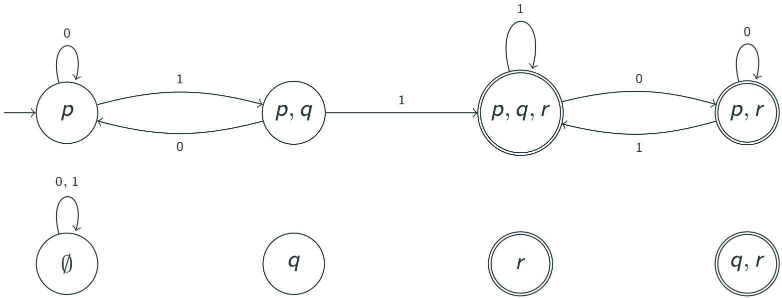

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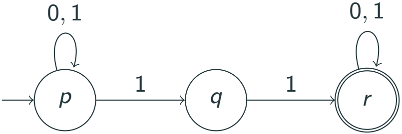

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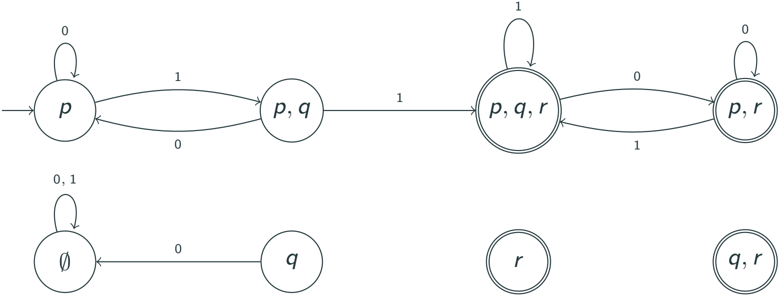

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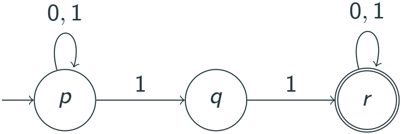

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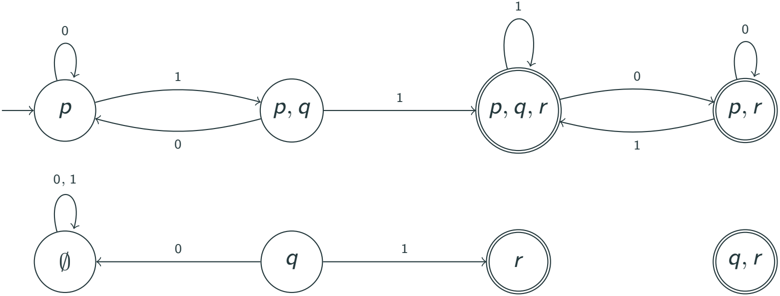

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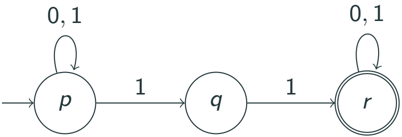

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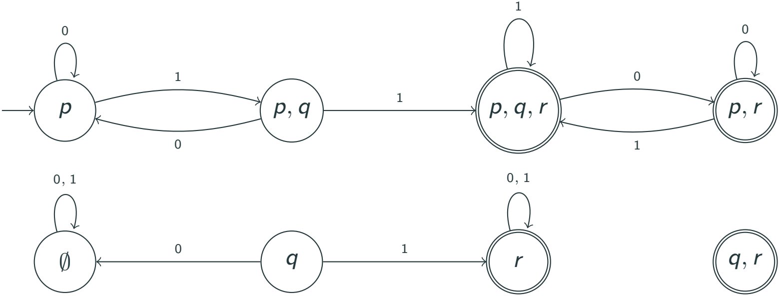

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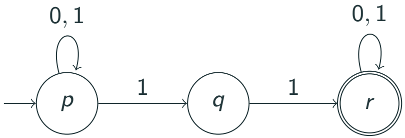

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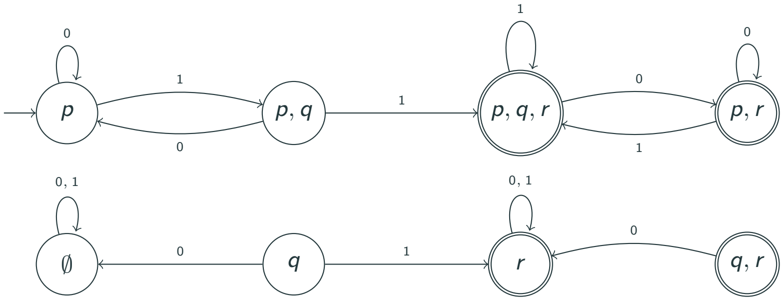

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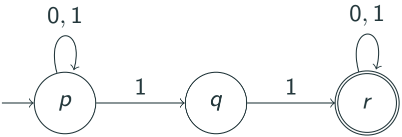

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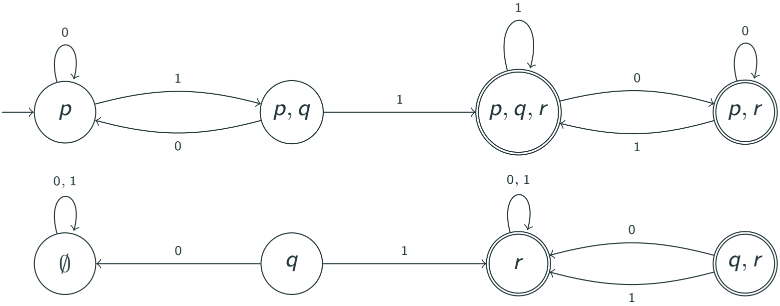

NFA and DFA

Theorem 1.5

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NFA languages are closed under complement.

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Corollary 1.6

NFA languages are closed under complement.

More precisely, we can say that for every NFA $\mathcal{A}$ over alphabet Σ, there is a DFA $\mathcal{A}'$ over the same alphabet Σ such that $L(\mathcal{A}') = \Sigma^* - L(\mathcal{A})$.

Concatenation and Kleene star

(Def.) For two words u and v, $u \cdot v$ denotes the word obtained by *concatenating* v at the end of u.

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$$L_1 \cdot L_2 := \{uv \mid u \in L_1 \text{ and } v \in L_2\} \qquad \text{(Concatenation)}$$

$$L^n := \{u_1 \cdots u_n \mid \text{each } u_i \in L\}$$

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$L_1 L_2$ reads as L_1 concatenates with L_2.

By default, for any set $X \subseteq \Sigma^*$, $X^0 = \{\epsilon\}$.

Thus, $\emptyset^* = \{\epsilon\}$.

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The proof can be found in Note 1.

Table of contents

Pumping lemma – A tool for showing non-regularity of a language

(Def.) For a word w and an integer $n \geqslant 0$, w^n is a word where w is repeated n number of times, i.e.,

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Lemma 1.9 (pumping lemma)
Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ be an NFA. Let $x \in L(\mathcal{A})$ be a word such that $|x| \geqslant |Q|$. Then, the word x can be divided into three parts u, v, w, i.e., $x = uvw$, such that $|v| \geqslant 1$ and for every integer $k \geqslant 0$, $uv^k w \in L(\mathcal{A})$.

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Let $u = a_1 \cdots a_i$, $v = a_{i+1} \cdots a_j$ and $w = a_{j+1} \cdots a_n$.

Then, for every integer $k \geqslant 0$, the following is an accepting run of $\mathcal{A}$ on $uv^k w$:

$$p_0 \ a_1 \ p_1 \ a_2 \ p_2 \ \cdots \ a_i \ p_i \ \underbrace{a_{i+1} \ p_{i+1} \ \cdots \ a_j \ p_j}_{\text{repeat } k \text{ times}} \ a_{j+1} \ p_{j+1} \ \cdots \ a_n \ p_n$$

Variations of pumping lemma

Lemma 1.11 (more refined pumping lemma)
Let $\mathcal{A} = \langle \Sigma, Q, q_0, F, \delta \rangle$ *be an NFA. Let* $x \in L(\mathcal{A})$ *be a word and* $x = szt$*, where* $|z| \geqslant |Q|$*. Then, the word* z *can be divided into three parts* u, v, w *such that* $|v| \geqslant 1$ *and for every positive integer* $k \geqslant 0$*,* $suv^k wt \in L(\mathcal{A})$*.*

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Pumping lemma can also be stated more elegantly as follows.

Lemma 1.10 (pumping lemma)
For every regular language L, there is an integer $n \geqslant 1$ such that for every word $x \in L$ with length $|x| \geqslant n$, there are u, v, w where $x = uvw$ and $|v| \geqslant 1$ and for every integer $k \geqslant 0$, $uv^kw \in L$.

Using pumping lemma to prove non-regularity

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By (more refined) pumping lemma, we can divide a^k into three parts u, v, w such that:

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Therefore, there is no NFA that accepts L_1 and L_1 is not regular.

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If we put $\ell = |u| + |w|$, we have:

$$|u\ v^\ell\ w| \ = \ (|u| + |w|)(|v| + 1) \qquad \text{which is not prime}$$

Using pumping lemma to prove non-regularity

We would like to show that $L_2 = \{w \mid |w| \text{ is a prime number}\}$ is not regular, i.e., there is no NFA that accepts L_2.

Suppose there is an NFA $\mathcal{A}$ that accepts L_2 where Q is the set of states.

Consider the following word: a^k where $k \geqslant |Q|$.

By pumping lemma, we can divide a^k into three parts u, v, w such that:

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So this contradicts the assumption that $\mathcal{A}$ accepts L_2.

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So this contradicts the assumption that $\mathcal{A}$ accepts L_2.

Therefore, there is no NFA that accepts L_1, i.e., L_1 is not regular.

End of Lesson 1