

Lesson 4: The polynomial hierarchy

Theme: The polynomial hierarchy.

For every integer $i \geq 1$, the class Σ_i^p is defined as follows. A language $L \subseteq \{0,1\}^*$ is in Σ_i^p , if there is a polynomial $q(n)$ and a polynomial time DTM \mathcal{M} such that for every $w \in \{0,1\}^*$, $w \in L$ if and only if the following holds.

$$\exists y_1 \in \{0,1\}^{q(|w|)} \forall y_2 \in \{0,1\}^{q(|w|)} \dots Q y_i \in \{0,1\}^{q(|w|)} \mathcal{M} \text{ accepts } (w, y_1, \dots, y_i) \quad (1)$$

where $Q = \exists$, if i is odd and $Q = \forall$, if i is even.

The class Π_i^p is defined as above, but the sequence of quantifiers in (1) starts with \forall . Alternatively, it can also be defined as $\Pi_i^p \stackrel{\text{def}}{=} \{\bar{L} : L \in \Sigma_i^p\}$. Note that $\mathbf{NP} = \Sigma_1^p$ and $\mathbf{coNP} = \Pi_1^p$.

Remark 4.1 The class Σ_i^p can also be defined as follows. A language L is in Σ_i^p , if there is a polynomial time ATM \mathcal{M} that decides L such that for every input word $w \in \{0,1\}^*$, the run of \mathcal{M} on w can be divided into i layers. Each layer consists of nodes of the same depth in the run. (Recall that the run of an ATM is a tree.) In the first layer all nodes are labeled with existential configurations, in the second layer with universal configurations, and so on. It is not difficult to show that this definition is equivalent to the one above.

The *polynomial time hierarchy* (or, in short, *polynomial hierarchy*) is defined as the following class.

$$\mathbf{PH} \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} \Sigma_i^p$$

Note that $\mathbf{PH} \subseteq \mathbf{PSPACE}$.

It is conjectured that $\Sigma_1^p \subsetneq \Sigma_2^p \subsetneq \Sigma_3^p \subsetneq \dots$. In this case, we say that *the polynomial hierarchy does not collapse*. We say that *the polynomial hierarchy collapses*, if there is i such that $\mathbf{PH} = \Sigma_i^p$, in which case we also say that *the polynomial hierarchy collapses to level i* .

We define the notion of hardness and completeness for each Σ_i^p as follows. For $i \geq 1$, a language K is Σ_i^p -hard, if for every $L \in \Sigma_i^p$, $L \leq_p K$. It is Σ_i^p -complete, if it is in Σ_i^p and it is Σ_i^p -hard. The same notion can be defined analogously for \mathbf{PH} and each Π_i^p .

Define the language Σ_i -SAT as consisting of true QBF of the form:

$$\exists \bar{x}_1 \forall \bar{x}_2 \dots Q \bar{x}_i \varphi(\bar{x}_1, \dots, \bar{x}_i)$$

where $\varphi(\bar{x}_1, \dots, \bar{x}_i)$ is quantifier-free Boolean formula and $Q = \exists$, if i is odd, and $Q = \forall$, if i is even. Here $\bar{x}_1, \dots, \bar{x}_i$ are all vectors of boolean variables. In other words, Σ_i -SAT is a subset of TQBF where the number of quantifier alternation is limited to $(i - 1)$. The language Π_i -SAT is defined analogously with the starting quantifiers being \forall .

Theorem 4.2

- For every $i \geq 1$, Σ_i -SAT is Σ_i^p -complete and Π_i -SAT is Π_i^p -complete.
- If $\Sigma_i^p = \Pi_i^p$ for some $i \geq 1$, then the polynomial hierarchy collapses.
- If there is language that is \mathbf{PH} -complete, then the polynomial hierarchy collapses.