Homework 2

Due on Monday, 10:30 am, 25 April 2022 (111/04/25)

Question 1 (1 point). Consider the problem CIRCUIT-EVAL and Theorem 3.3 in Note 3. The main idea for the proof of hardness is as follows. Let $\mathcal{M}$ be a polynomial time DTM. On input word w, the reduction constructs a circuit C such that:

$$C(w) = 1 \qquad \text{if and only if} \qquad \mathcal{M} \text{ accepts } w$$

The construction of C is similar to Cook-Levin reduction for the proof of **NP**-hardness of SAT. Explain why we need the fact that $\mathcal{M}$ is DTM and *not* NTM in the proof of Theorem 3.3.

Question 2 (2 points). Prove that **PH** collapses if any one of the following is true.

- $\mathbf{\Sigma}_i^p = \mathbf{\Pi}_i^p$ for some $i \geqslant 1$.
- There is **PH**-complete language.
- **PH** = **PSPACE**.

Question 3 (2 points). Suppose that A is a language such that $\mathbf{P}^A = \mathbf{NP}^A$. Prove that $\mathbf{PH}^A \subseteq \mathbf{P}^A$.

Def: A language L is in the class $\mathbf{\Sigma}_i^p$ *with oracle access to* A, if there is a polynomial $q(n)$ and a polynomial time DTM $\mathcal{M}^A$ such that for every $w \in \Sigma^*$ the following holds.

$$w \in L \quad \text{iff} \quad \exists y_1 \in \{0,1\}^{q(|w|)}\ \forall y_2 \in \{0,1\}^{q(|w|)}\ \cdots\ Q y_i \in \{0,1\}^{q(|w|)}\ \mathcal{M}^A \text{ accepts } (w, y_1, \ldots, y_i)$$

As before, $\mathcal{M}^A$ denotes the TM $\mathcal{M}$ with oracle access to A. A language L is in $\mathbf{PH}^A$, if there is i such that L is in the class $\mathbf{\Sigma}_i^p$ with oracle access to A.

Question 4 (1 point). Consider the following two problems.

CYCLE-COVER	
Input:	A directed graph G.
Task:	Output a cycle cover of A, if exists. Otherwise, output 0.

MATCHING	
Input:	A bipartite (undirected) graph $H = (U, V, E)$, where $\|U\| = \|V\|$.
Task:	Output a matching of H, if exists. Otherwise, output 0.

Here a matching of $H = (U, V, E)$ is defined as a set $E_0 \subseteq E$ such that every vertex $u \in U \cup V$ is incident to exactly one edge in E_0. In other words, E_0 forms a "bijection" from U to V.

Prove that CYCLE-COVER and MATCHING are reducible to each other in polynomial time.

Question 5 (2 points). Prove Lemma 5.10 in Note 5.

Question 6 (2 points). Prove Lemma 5.11 in Note 5.