Lesson 2. Regular expressions
CSIE 3110 – Formal Languages and Automata Theory

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2. The equivalence between regular expressions and NFA
(Def.) Let $\Sigma$ be an alphabet.

Regular expressions (over $\Sigma$) are expressions built inductively as follows.

- $\emptyset$ is a regular expression.
- $a$ is a regular expression, for every symbol $a \in \Sigma$.
- If $e_1, e_2$ are regular expressions, so are $(e_1 \cdot e_2)$ and $(e_1 \cup e_2)$.
- If $e$ is a regular expression, so is $(e)^*$.
Regular expressions

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Regular expression is usually abbreviated as regex.
Some examples of regular expressions

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Let $\Sigma = \{a, b\}$.
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Let $\Sigma = \{a, b\}$.

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- $\emptyset \cdot \emptyset$ is a regular expression.
Some examples of regular expressions

- ∅ is a regular expression.
- a is a regular expression, for every symbol a ∈ Σ.
- If e₁, e₂ are regular expressions, so are (e₁ ⋅ e₂) and (e₁ ∪ e₂).
- If e is a regular expression, so is (e)*.

Let Σ = {a, b}.

- ∅ is a regular expression.
- ∅ ⋅ ∅ is a regular expression. ⇒ usually written as ∅∅
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- $(ab)^* \cup ab^*$ is a regular expression.
Examples that are not regular expressions

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- If $e_1, e_2$ are regular expressions, so are $(e_1 \cdot e_2)$ and $(e_1 \cup e_2)$.
- If $e$ is a regular expression, so is $(e)^\ast$.
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- $a \cap b$ is not a regular expression, because $\cap$ is not allowed.
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- $\ast$ is not a regular expression.
Examples that are not regular expressions

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Let $\Sigma = \{a, b\}$.

- $a \cap b$ is not a regular expression, because $\cap$ is not allowed.
- $c$ is not a regular expression over $\Sigma$, because $c \notin \Sigma$.
- $*$ is not a regular expression.
- (NOT $a$) is not a regular expression.
The meaning of a regular expression

Each regular expression $e$ represents/defines a language $L(e)$. 
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As analogy, a C++ program:

```cpp
Boolean myprog(String w)
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The meaning of a regular expression

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We can view `myprog` as defining the language:

$$L(\text{myprog}) := \{ w \mid \text{myprog outputs true on } w \}$$
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As analogy, a C++ program:

$$\text{Boolean } \text{myprog(String } w)$$

We can view myprog as defining the language:

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So we can say that myprog is a finite representation of a (possibly infinite) language $L(\text{myprog})$. 
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As analogy, a C++ program:

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$$L(\text{myprog}) := \{w \mid \text{myprog outputs true on } w\}$$

So we can say that myprog is a finite representation of a (possibly infinite) language $L(\text{myprog})$.

Similarly, we can say that a regular expression $e$ is a finite representation of the language $L(e)$. 
The formal definition of the language $L(e)$

(Def.) A regular expression $e$ over $\Sigma$ defines the language $L(e)$ over the same alphabet $\Sigma$ as follows.
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(Def.) A regular expression $e$ over $\Sigma$ defines the language $L(e)$ over the same alphabet $\Sigma$ as follows.

- If $e$ is $\emptyset$, then $L(e) = \emptyset$. 
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(Def.) A regular expression $e$ over $\Sigma$ defines the language $L(e)$ over the same alphabet $\Sigma$ as follows.

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- If $e$ is of the form $e_1e_2$, then $L(e) = L(e_1) \cdot L(e_2)$. 
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- If $e$ is of the form $(e_1)^*$, then $L(e) = L(e_1)^*$.
Some examples

• $L(\emptyset) = \emptyset$.
• $L(a) = \{a\}$.
• $L(ab) = L(a)L(b) = \{a\} \cdot \{b\} = \{ab\}$.
• $L((a \cup b)^\ast) = (L(a \cup b))^\ast = (L(a) \cup L(b))^\ast = \{a, b\}^\ast = \{a, b\}$.

That is, the language $\{w | w$ is a word that ends with $a\}$. 
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- $L((a \cup b)^*a) = (L(a \cup b)^* \cdot L(a) = \{a, b\}^* \cdot \{a\}$.

  That is, the language $\{w \mid w$ is a word that ends with $a\}$. 
The main theorem in this lesson

**Theorem 2.1**

Regular expressions define precisely the class of regular languages. More formally:

- For every regular expression $e$ over $\Sigma$, $L(e)$ is a regular language, i.e., there is an NFA $A$ such that $L(A) = L(e)$.
- For every NFA $A$, there is a regular expression $e$ such that $L(e) = L(A)$. 
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The main theorem in this lesson

**Theorem 2.1**

Regular expressions define precisely the class of regular languages.
More formally:

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- For every NFA $A$, there is a regular expression $e$ such that $L(e) = L(A)$. 
We will first prove the first item:

**Theorem (The first item of Theorem 2.1)**

- For every regular expression $e$ over $\Sigma$, $L(e)$ is a regular language, i.e., there is an NFA $A$ such that $L(A) = L(e)$.

**(Proof)** By induction on the regex $e$. The base case is when $e$ is either $\emptyset$ or a symbol $a \in \Sigma$. 
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- When $e$ is $\emptyset$, then $L(e) = \emptyset$.
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**Theorem (The first item of Theorem 2.1)**
- For every regular expression \(e\) over \(\Sigma\), \(L(e)\) is a regular language, i.e., there is an NFA \(A\) such that \(L(A) = L(e)\).

**Proof** By induction on the regex \(e\). The base case is when \(e\) is either \(\emptyset\) or a symbol \(a \in \Sigma\).
- When \(e\) is \(\emptyset\), then \(L(e) = \emptyset\).

The NFA that accepts \(\emptyset\) is:

\[
\begin{array}{c}
q \\
\end{array}
\]
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**Theorem (The first item of Theorem 2.1)**
- For every regular expression $e$ over $\Sigma$, $L(e)$ is a regular language, i.e., there is an NFA $A$ such that $L(A) = L(e)$.

**(Proof)** By induction on the regex $e$. The base case is when $e$ is either $\emptyset$ or a symbol $a \in \Sigma$.

- When $e$ is $\emptyset$, then $L(e) = \emptyset$.

  The NFA that accepts $\emptyset$ is:

  $\tau$

- When $e$ is $a$, for some symbol $a \in \Sigma$, then $L(e) = \{a\}$.
We will first prove the first item:

**Theorem (The first item of Theorem 2.1)**

- For every regular expression e over Σ, L(e) is a regular language, i.e., there is an NFA A such that L(A) = L(e).

**Proof** By induction on the regex e. The base case is when e is either ∅ or a symbol a ∈ Σ.

- When e is ∅, then L(e) = ∅.

  The NFA that accepts ∅ is:

  ![NFA for ∅](image1)

- When e is a, for some symbol a ∈ Σ, then L(e) = {a}.

  The NFA that accepts {a} is:

  ![NFA for {a}](image2)
(Proof continued)

For the induction step, suppose \( e \) is either of the form \( \alpha \cdot \beta \), \( \alpha \cup \beta \) or \( \alpha^* \).
(Proof continued)

For the induction step, suppose $e$ is either of the form $\alpha \cdot \beta$, $\alpha \cup \beta$ or $\alpha^*$. By the induction hypothesis, there are NFA $A_1$ and $A_2$ that accept the languages $L(\alpha)$ and $L(\beta)$, respectively.
(Proof continued)

For the induction step, suppose $e$ is either of the form $\alpha \cdot \beta$, $\alpha \cup \beta$ or $\alpha^*$. 

By the induction hypothesis, there are NFA $A_1$ and $A_2$ that accept the languages $L(\alpha)$ and $L(\beta)$, respectively.

Since regular languages are closed under concatenation, union and Kleene star, (See Remark 1.4 and Theorem 1.8 in Lesson 1), there are NFAs for all the languages $L(\alpha \cdot \beta)$, $L(\alpha \cup \beta)$ and $L(\alpha^*)$. 
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Since regular languages are closed under concatenation, union and Kleene star, (See Remark 1.4 and Theorem 1.8 in Lesson 1), there are NFAs for all the languages \( L(\alpha \cdot \beta) \), \( L(\alpha \cup \beta) \) and \( L(\alpha^*) \).

Recall that:

- \( L(\alpha \cdot \beta) = L(\alpha) \cdot L(\beta) \).
- \( L(\alpha \cup \beta) = L(\alpha) \cup L(\beta) \).
- \( L(\alpha^*) = L(\alpha)^* \).
We now prove the second item:

**Theorem (The second item of Theorem 2.1)**
- For every NFA $A$, there is a regular expression $e$ such that $L(e) = L(A)$.

**(Proof)** Let $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ be an NFA, where $Q = \{1, \ldots, n\}$. 
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For every $1 \leq i, j \leq n$ and $0 \leq k \leq n$, define the language $L(i, j, k)$:

$$L(i, j, k) := \left\{ w \in \Sigma^* \mid \text{there is a run of } A \text{ on } w \text{ from state } i \text{ to state } j \text{ without passing any states } \geq k + 1 \right\}$$
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That is, if $w \in L(i, j, k)$, there is a run of $A$ on $w$ from state $i$ to $j$ *without* passing through the states $k + 1, \ldots, n$. 
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**Theorem (The second item of Theorem 2.1)**

- For every NFA \( A \), there is a regular expression \( e \) such that \( L(e) = L(A) \).

**Proof** Let \( A = \langle \Sigma, Q, q_0, F, \delta \rangle \) be an NFA, where \( Q = \{1, \ldots, n\} \).

For every \( 1 \leq i, j \leq n \) and \( 0 \leq k \leq n \), define the language \( L(i, j, k) \):

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L(i, j, k) := \left\{ w \in \Sigma^* \mid \text{there is a run of } A \text{ on } w \text{ from state } i \text{ to state } j \text{ without passing any states } \geq k + 1 \right\}
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That is, if \( w \in L(i, j, k) \), there is a run of \( A \) on \( w \) from state \( i \) to \( j \) *without* passing through the states \( k + 1, \ldots, n \). 

![Diagram of NFA states and transitions]

- no states \( k+1, k+2, \ldots, n \)
Claim 1
For every $1 \leq i, j \leq n$ and $0 \leq k \leq n$, there is a regex $e$ such that $L(e) = L(i, j, k)$.

(Proof of Claim 1: By induction on $k$)
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For every $1 \leq i, j \leq n$ and $0 \leq k \leq n$, there is a regex $e$ such that $L(e) = L(i, j, k)$.

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(Base case $k = 0$) For every $1 \leq i, j \leq n$, we consider $L(i, j, 0)$:
Claim 1
For every $1 \leq i, j \leq n$ and $0 \leq k \leq n$, there is a regex $e$ such that $L(e) = L(i, j, k)$.

(Proof of Claim 1: By induction on $k$)

(Base case $k = 0$) For every $1 \leq i, j \leq n$, we consider $L(i, j, 0)$:

- If $i \neq j$ and there is no transition from $i$ to $j$:
  
  The language $L(i, j, 0) = \emptyset$, so the regex $e$ is $\emptyset$. 

\[\begin{array}{c}
  i \\
  j \\
\end{array}\]
Claim 1

For every $1 \leq i, j \leq n$ and $0 \leq k \leq n$, there is a regex $e$ such that $L(e) = L(i, j, k)$.

(Proof of Claim 1: By induction on $k$)

(Base case $k = 0$) For every $1 \leq i, j \leq n$, we consider $L(i, j, 0)$:

- If $i \neq j$ and there is no transition from $i$ to $j$:
  
  ![Diagram](image)

  The language $L(i, j, 0) = \emptyset$, so the regex $e$ is $\emptyset$.

- If $i \neq j$ and there are some transitions from $i$ to $j$:
  
  ![Diagram](image)

  The language $L(i, j, 0) = \{a_1, \ldots, a_t\}$, so the regex $e$ is $a_1 \cup \cdots \cup a_t$. 
(Base case $k = 0$ – continued)

- If $i = j$ and there is no transition from $i$ to $i$:
  The language $L(i, j, 0) = \{\varepsilon\}$, so the regex is $\emptyset^*$. 

- If $i = j$ and there are some transitions from $i$ to $j$:
  The language $L(i, j, 0) = \{a_1, \ldots, a_t, \varepsilon\}$, so the regex is $a_1 \cup \cdots \cup a_t \cup \emptyset^*$. 

(Base case $k = 0$ – continued)

- If $i = j$ and there is no transition from $i$ to $i$:

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(Base case $k = 0$ – continued)

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- If $i = j$ and there are some transitions from $i$ to $j$:

  $a_1, \ldots, a_t$

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(Induction step – proof of Claim 1)

We have the identity:

\[ L(i, j, k + 1) = L(i, j, k) \cup (L(i, k + 1, k) \cdot L(k + 1, k + 1, k) \ast \cdot L(k + 1, j, k)) \]

By induction hypothesis, there is regex for each of

\[ L(i, j, k), L(i, k + 1, k), L(k + 1, k + 1, k), \text{ and } L(k + 1, j, k). \]

Thus, there is regex for

\[ L(i, j, k + 1). \]
(Induction step – proof of Claim 1) We have the identity:

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By induction hypothesis, there is regex for each of \( L(i, j, k) \), \( L(i, k + 1, k) \), \( L(k + 1, k + 1, k) \), and \( L(k + 1, j, k) \).
(Induction step – proof of Claim 1) We have the identity:

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Thus, there is regex for \( L(i, j, k + 1) \).
(Finishing the proof of Claim 1)

The language $L(A)$ can be defined as:

$$L(A) = \bigcup_{q_f \in F} L(q_0, q_f, n)$$
(Finishing the proof of Claim 1)

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(Finishing the proof of Claim 1)

The language $L(\mathcal{A})$ can be defined as:

$$L(\mathcal{A}) = \bigcup_{q_f \in F} L(q_0, q_f, n)$$

By Claim 1, there is a regex that defines each $L(q_0, q_f, n)$.

Taking the union over all $q_f \in F$, we have a regex for $L(\mathcal{A})$. 
To conclude:

**Corollary 2.2**

Let $L$ be a language. The following are equivalent.

- $L$ is accepted by a DFA.
- $L$ is accepted by an NFA.
- $L$ is defined by a regular expression.

One nice implication of this corollary is that languages defined by regular expressions are also closed under intersection and complement. This is despite the fact that we are not allowed to use intersection or negation in regular expressions.
To conclude:

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End of Lesson 2