Lesson 3: Proof system in propositional calculus

Theme: The notion of provability in propositional calculus.

1 Proofs in propositional calculus

Let \( X \) be a set of formulas and \( \alpha \) be a formula. We say that \( \alpha \) is provable/derivable from \( X \), denoted by \( X \vdash \alpha \), if it can be obtained inductively according to the following rules.

**Initial Segment (IS):**
\[
\alpha \vdash \alpha
\]

**Monotonicity Rule (MR):**
\[
\frac{X \vdash \alpha}{Y \vdash \alpha}
\text{ for every } Y \supseteq X
\]

**And Combine Rule (ACR):**
\[
\frac{X \vdash \alpha \text{ and } X \vdash \beta}{X \vdash \alpha \land \beta}
\]

**And Split Rule (ASR):**
\[
\frac{X \vdash \alpha \land \beta}{X \vdash \alpha \text{ and } X \vdash \beta}
\]

**Contradiction Rule (CR):**
\[
\frac{X \vdash \alpha \text{ and } X \vdash \neg \alpha}{X \vdash \beta}
\text{ for every formula } \beta
\]

**Negation Rule (NR):**
\[
\frac{X, \alpha \vdash \beta \text{ and } X, \neg \alpha \vdash \beta}{X \vdash \beta}
\]

Sometimes we will also say “\( \alpha \) can be proved from \( X \)” when \( X \vdash \alpha \). We write \( X \nvDash \alpha \), if it is not the case that \( X \vdash \alpha \).

**Remark 3.1** To avoid clutter, we write \( \alpha \vdash \alpha \) to denote \( \{\alpha\} \vdash \alpha \), whereas \( X, \alpha \vdash \beta \) means \( X \cup \{\alpha\} \vdash \beta \). We also write \( \{\alpha_1, \ldots, \alpha_n\} \vdash \alpha \) to denote \( \alpha_1, \ldots, \alpha_n \vdash \alpha \) and \( \vdash \alpha \) to denote \( \emptyset \vdash \alpha \).

**Remark 3.2** Note that in the proof system above, we only use the operators \( \neg \) and \( \land \). In the following a formula \( \alpha \rightarrow \beta \) is to be interpreted as an abbreviation for \( \neg(\alpha \land \neg \beta) \), and likewise, \( \alpha \lor \beta \) for \( \neg(\neg \alpha \land \neg \beta) \).

**Example 3.3 (Elimination of Negation)**
\[
\frac{X, \neg \alpha \vdash \alpha}{X \vdash \alpha}
\]

1. \( X, \neg \alpha \vdash \alpha \). (Supposition)
2. \( X, \alpha \vdash \alpha \). (Initial Segment and Monotonicity Rule)
3. \( X \vdash \alpha \). (Negation Rule on 1 and 2)
Example 3.4 (Reductio ad Absurdum) \[ X, \neg \alpha \vdash \beta \quad \text{and} \quad X, \neg \alpha \vdash \neg \beta \]
\[ X \vdash \alpha \]

1. \( X, \neg \alpha \vdash \beta \). (Supposition)
2. \( X, \neg \alpha \vdash \neg \beta \). (Supposition)
3. \( X, \neg \alpha \vdash \alpha \). (Contradiction Rule on 1 and 2)
4. \( X, \alpha \vdash \alpha \). (Initial Segment and Monotonicity Rule)
5. \( X \vdash \alpha \). (Negation Rule on 3 and 4)

Example 3.5 (Cut Rule) \[ X \vdash \alpha \quad \text{and} \quad X, \alpha \vdash \beta \]
\[ X \vdash \beta \]

1. \( X \vdash \alpha \). (Supposition)
2. \( X, \alpha \vdash \beta \). (Supposition)
3. \( X, \neg \alpha \vdash \neg \alpha \). (Initial Segment and Monotonicity Rule)
4. \( X, \neg \alpha \vdash \alpha \). (Monotonicity Rule on 1)
5. \( X, \neg \alpha \vdash \beta \). (Contradiction Rule on 3 and 4)
6. \( X \vdash \beta \). (Negation Rule on 2 and 5)

Example 3.6 (Elimination of \( \to \)) \[ X \vdash \alpha \to \beta \]
\[ X, \alpha \vdash \beta \]

1. \( X \vdash \alpha \to \beta \). (Supposition)
2. \( X, \alpha, \neg \beta \vdash \alpha \). (Initial Segment and Monotonicity Rule)
3. \( X, \alpha, \neg \beta \vdash \neg \beta \). (Initial Segment and Monotonicity Rule)
4. \( X, \alpha, \neg \beta \vdash \alpha \land \neg \beta \). (And Combine Rule on 2 and 3)
5. \( X, \alpha, \neg \beta \vdash \neg (\alpha \land \neg \beta) \). (Monotonicity Rule on 1)
6. \( X, \alpha, \neg \beta \vdash \beta \). (Cut rule on 4 and 5)
7. \( X, \alpha, \beta \vdash \beta \). (Initial Segment and Monotonicity Rule)
8. \( X, \alpha \vdash \beta \). (Negation Rule on 6 and 7)

Example 3.7 (Introduction of \( \to \)) \[ X, \alpha \vdash \beta \]
\[ X \vdash \alpha \to \beta \]

1. \( X, \alpha \vdash \beta \). (Supposition)
2. \( X, \alpha, \alpha \land \neg \beta \vdash \beta \). (Monotonicity Rule on 1)
3. \( X, \alpha \land \neg \beta \vdash \alpha \land \neg \beta \). (Initial Segment and Monotonicity Rule)
4. \( X, \alpha \land \neg \beta \vdash \alpha \). (And Split Rule on 3)
5. \( X, \alpha \land \neg \beta \vdash \beta \). (Cut rule on 4 and 2)
6. \( X, \alpha \land \neg \beta \vdash \neg \beta \). (And Split Rule on 3)
7. \( X \vdash \alpha \to \beta \). (Reductio ad Absurdum on 5 and 6)

Theorem 3.8 (Finiteness theorem for \( \vdash \)) If \( X \vdash \alpha \), then there is a finite set \( X_0 \subseteq X \) such that \( X_0 \vdash \alpha \).
Exercises

(1) Prove that \( X, \alpha \vdash \neg \alpha \). 

(2) Prove that \( X, \alpha \rightarrow \beta \). 

(3) Prove that \( X, \alpha \rightarrow \beta \). 

(4) Prove that \( X \vdash \neg \neg \alpha \). 

(5) Prove that \( X \vdash \alpha \rightarrow \beta \). 

(6) Prove that \( X, \alpha \rightarrow \beta \). 

(7) Prove that \( X, \neg \alpha \vdash \beta \). 

Note that \( \alpha \rightarrow \beta \) is an abbreviation for \( \neg (\alpha \land \neg \beta) \), whereas \( \neg \beta \rightarrow \neg \alpha \) for \( \neg (\neg \beta \land \neg \neg \alpha) \).