Sample solution HW 3

(5 points) Question 1. Consider the following Turing machine \( M_1 = (\Sigma, \Gamma, Q, q_0, q_{\text{acc}}, q_{\text{rej}}, \delta) \).

- \( \Sigma = \{0,1\} \), \( \Gamma = \{\langle,0,1,\rangle\} \), and \( Q = \{q_0,p_0,p_1,s,t,r_0,r_1,q',q_{\text{acc}},q_{\text{rej}}\} \).
  
  As usual, \( q_0 \), \( q_{\text{acc}} \), \( q_{\text{rej}} \) are the initial, accepting and rejecting states, respectively.

- \( \delta \) is defined as follows.

\[
\begin{align*}
(q_0, \langle) &\rightarrow (q_{\text{rej}}, \langle, \text{Stay}) \\
(q_0, 0) &\rightarrow (p_0, 0, \text{Right}) \\
(q_0, 1) &\rightarrow (p_1, \langle, \text{Right}) \\
(q_0, \langle) &\rightarrow (q_{\text{rej}}, \langle, \text{Stay}) \\
(s, \langle) &\rightarrow (q_{\text{rej}}, \langle, \text{Stay}) \\
(s, 0) &\rightarrow (t, 0, \text{Left}) \\
(s, 1) &\rightarrow (s, 0, \text{Left}) \\
(s, \langle) &\rightarrow (r_1, \langle, \text{Right}) \\
(r_0, \langle) &\rightarrow (t, 0, \text{Left}) \\
(r_0, 0) &\rightarrow (r_0, 0, \text{Right}) \\
(r_0, 1) &\rightarrow (r_1, 0, \text{Right}) \\
(r_0, \langle) &\rightarrow (q_{\text{rej}}, \langle, \text{Stay})
\end{align*}
\]

Determine the run of \( M \) on each of the following input words.

(a) \( e \).

\( q_0 \rightarrow q_{\text{rej}} \)

(b) \( 011 \).

\( q_0 011 \rightarrow \langle <p_0 11 \rightarrow \langle <0 p_1 1 \rightarrow \langle <0 p_1 \rightarrow \langle <0 1 s_1 \rightarrow \langle <s 000 \rightarrow t <1 00 \rightarrow q_{\text{acc}} <1 00 \).

(c) \( 111 \).

\( q_0 111 \rightarrow \langle p_1 11 \rightarrow \langle 1 p_1 1 \rightarrow \langle 1 p_1 \rightarrow \langle 1 1 0 1 \rightarrow \langle 0 1 0 1 \rightarrow \langle 0 1 0 1 \rightarrow \langle 0 1 0 1 \rightarrow q_{\text{acc}} <1 0 0 \).

(d) \( 1101 \).

\( q_0 1101 \rightarrow \langle p_1 011 \rightarrow \langle 1 p_1 01 \rightarrow \langle 1 1 p_0 1 \rightarrow \langle 1 1 0 p_1 \rightarrow \langle 1 1 0 s_1 \rightarrow \langle 1 1 0 s_0 \rightarrow q_{\text{acc}} <1 1 1 0 \rightarrow t <1 1 1 0 \rightarrow q_{\text{acc}} <1 1 1 0 \).

(e) \( 10100 \).

\( q_0 1101 \rightarrow \langle p_1 0100 \rightarrow \langle 1 p_0 100 \rightarrow \langle 1 0 1 0 p_0 \rightarrow q_{\text{acc}} <1 0 1 0 \rightarrow q_{\text{rej}} \).

Intuitively, the TM works as follows. The input is a number in binary representation. On input \( w \):

1. Write the marker \( \langle \) on the left-most cell, and shift the input one step to the right.

This is done by scanning from left to right, on reading each cell, the TM remembers in its state \( (p_0 \) and \( p_1 \) ) the bit read, and after moving right, it knows what to write.
2. If the input ends with 1 (which means the number is odd), the TM wants to “add one” into the number. It is done as follows.

   The head goes back (from right to left) to the left-most cell and on reading each cell, flips 1 to 0. (This is the purpose of state $s$.)

   The head continues doing so until it either finds bit 0 or the symbol $\prec$.

   (a) If it finds bit 0.

   Flips the bit into 1, and the head continues moving to the left until it finds the symbol $\prec$ and ACCEPT. (This is the purpose of state $t$.)

   (b) If it finds $\prec$.

   (This means the input $w$ is of the form $111 \cdots 1$. So, the TM has to change it into $1000 \cdots 0$.)

   The head moves right one step, write 1. Then, continues moving from left to right, writing 0 until it reads $\sqcap$, after which, writes one more 0. (This is the purpose of states $r_1$ and $r_0$. $r_1$ is to remember to write 1, and $r_0$ is to write 0 all the way to the end.)

   After this, the head moves back to the left-most cell and then ACCEPT.

3. If the input ends with 0 (which means the number is even), the TM writes 0 at end. After that, REJECT.
(2 points) Question 2. Let $L$ be a language over alphabet $\Sigma$. Prove that if both $L$ and its complement $\Sigma^* - L$ are recognizable, then $L$ is decidable.

Note: To prove this question properly, you have to present a precise definition of the Turing machine that decides $L$ (in terms of the Turing machines that recognizes $L$ and $\Sigma^* - L$).

Solution for Question 2. If $L$ is recognizable, then there is a DTM $M_1$ that recognizes it. Likewise, if $\Sigma^* - L$, then there is a DTM $M_2$ that recognizes it. We can assume that $M_1$ and $M_2$ have the same tape alphabet $\Gamma$.

We have a 2-tape DTM $M$ that decides $L$. It works as follows. On input $w$:

- Copy the input word $w$ onto tape-2.
- Move the heads in both tapes to the left-most cell.
- Run $M_1$ and $M_2$ in “parallel” where $M_1$ works on tape-1 and $M_2$ works on tape-2.
  Here run in parallel means the head on tape-1 works exactly like $M_1$ and the head on tape-2 works exactly like $M_2$.
- If $M_1$ accepts, then ACCEPT.
- If $M_2$ accepts, then REJECT.

To prove the correctness of the DTM $M$, note that every word $w$ is either in $L$ or $\Sigma^* - L$. Thus, by definition, either $w$ is accepted by $M_1$ or by $M_2$. Therefore, on every input word, the $M$ is guaranteed to stop (either ACCEPT or REJECT).
(3 points) Question 3. Recall that for a Turing machine $M$, we denote by $L(M)$ the language that consists of all words accepted by $M$. That is, $L(M) = \{ w \mid M \text{ accepts } w \}$.

Consider the following Turing machine $A$ that works as follows.

INPUT: $|M|w$.

- Construct a TM $K_{M,w}$ that works as follows.
  
  INPUT: $u \in \Sigma^*$.
  - Simulate $M$ on $w$.
  - If $M$ accepts $w$, ACCEPT.
  - If $M$ rejects $w$:
    * If $u \in \{0^n1^n \mid n \geq 0\}$, ACCEPT.
    * Else, REJECT.

- Output $|K_{M,w}|$.

Answer each of the following questions

(a) If $M$ accepts $w$, what is the language $L(K_{M,w})$?
(b) If $M$ rejects $w$, what is the language $L(K_{M,w})$?
(c) If $M$ does not halt on $w$, what is the language $L(K_{M,w})$?

Justify your answers.

Solutions for question 3.

(a) If $M$ accepts $w$, $L(K_{M,w}) = \Sigma^*$.
(b) If $M$ rejects $w$, $L(K_{M,w}) = \{0^n1^n \mid n \geq 0\}$.
(c) If $M$ does not halt on $w$, $L(K_{M,w}) = \emptyset$. 