Lesson 10: Löwenheim-Skolem theorem and categorical sets

Theme: Cardinals of first-order structures.

1 Cardinal numbers

- Two sets $A$ and $B$ have the same cardinality, if there is a bijection from $A$ to $B$, denoted by $|A| = |B|$.
- In the same spirit, $|A| \leq |B|$, if there is an injective function from $A$ to $B$.
- $|A| < |B|$, if $|A| \leq |B|$ and $|A| \neq |B|$.

For $i \in \{0, 1, 2, \ldots\}$, we define $\aleph_i$ and $\beth_i$ as follows.

- Both $\aleph_0$ and $\beth_0$ denote $\mathbb{N}$.
- For each $i \geq 1$, $\aleph_i$ denotes the minimal set such that $|\aleph_i| > |\aleph_{i-1}|$.
- For each $i \geq 1$, $\beth_i$ denotes $2^{\beth_{i-1}}$.

Abusing the notation, we will often regard each $\aleph_i$ and $\beth_i$ as “cardinalities.” So, when we write $A = \aleph_i$ and $A = \beth_i$, we mean $|A| = |\aleph_i|$ and $|A| = |\beth_i|$, respectively. Likewise, such abuse also applies for $<$ and $\leq$ comparisons.

**Theorem 10.1 (Cantor’s theorem)** $|A| < |2^A|$, for every set $A$.

Cantor’s theorem implies that the sequence $\beth_0, \beth_1, \beth_2, \ldots$ will never end, which in turn implies that the sequence $\aleph_0, \aleph_1, \aleph_2, \ldots$ will also never end. The so called *Continuum Hypothesis* (CH) states the following.

$$\aleph_1 = \beth_1$$

2 Löwenheim-Skolem theorem

**Theorem 10.2** (Löwenheim-Skolem theorem) If $X \subseteq \text{FO}[L]$ is satisfiable, and $L$ is countable, then $X$ is satisfied by a countable structure.

**Theorem 10.3** (Downward Löwenheim-Skolem theorem) If $X \subseteq \text{FO}[L]$ is satisfiable, and $L$ is of cardinality $\lambda$, then $X$ is satisfied by a structure with cardinality $\leq \lambda$.

**Theorem 10.4** (Upward Löwenheim-Skolem-Tarski theorem) If $X \subseteq \text{FO}[L]$ is satisfiable, and $L$ is of cardinality $\lambda$, then for every cardinal number $\kappa \geq \lambda$, there is a structure with cardinality $\kappa$ that satisfies $X$.

**Corollary 10.5**

(a) Let $X \subseteq \text{FO}[L]$, where $L$ is countable. If $X$ has an infinite model, then $X$ has models of every infinite cardinality.

(b) Let $A$ be an infinite structure for a countable vocabulary $L$. Then, for every infinite cardinal $\lambda$, there is a structure $B$ of cardinality $\lambda$, such that $A \equiv B$. 
3 Categorical sets

A set $X$ is **categorical**, if every two models of $X$ is isomorphic.

**Proposition 10.6** If $X$ has an infinite model, then $X$ is not categorical.

A theory $T$ is $\aleph_0$-**categorical**, if all infinite countable models of $T$ are isomorphic. A theory $T$ is $\kappa$-**categorical**, if all models of $T$ of cardinality $\kappa$ are isomorphic.

**Theorem 10.7 (Łoś-Vaught Test)** Let $T$ be a theory over a countable vocabulary. Assume that $T$ has no finite models.

(a) If $T$ is $\aleph_0$-categorical, then $T$ is complete.

(b) If $T$ is $\kappa$-categorical for some infinite cardinal $\kappa$, then $T$ is complete.

4 The ZFC system

The ZFC system (Zermelo-Fraenkel-Axiom of Choice) is a set of axioms that describe mathematics being founded entirely on set theory. The vocabulary has only one binary relation $\varepsilon$, which intuitively represents the standard relation $\in$.

The ZFC system consists of the following axioms.

**Extensionality axiom:** $\forall x \forall y \left( \forall z (z \varepsilon x \leftrightarrow z \varepsilon y) \rightarrow x \approx y \right)$.

Intuitively, this means that if $x$ and $y$ have the same members, then $x$ and $y$ are the same.

**Separation axioms:** $\forall x_1 \cdots \forall x_n \forall x \exists y \forall z \left( z \varepsilon y \leftrightarrow (z \varepsilon x \land \varphi(z, x_1, \ldots, x_n)) \right)$.

The formula $\varphi$ is over the vocabulary $\{\varepsilon\}$. Intuitively, it means that for a set $x$, and a “property” $\varphi$, there is a set $y$ that contains precisely the elements in $x$ that satisfies $\varphi$.

**Pairing axiom:** $\forall x \forall y \exists z \forall w \left( w \varepsilon z \leftrightarrow (w \approx x \lor w \approx y) \right)$.

Intuitively, it means that for every two sets $x$ and $y$, the set $\{x, y\}$ exists.

**Union axiom:** $\forall x \exists y \forall z \left( z \varepsilon y \leftrightarrow \exists w (w \varepsilon x \land z \varepsilon w) \right)$.

Intuitively, it means that for every set $x$, the set $\bigcup x$ exists.

**Power set axiom:** $\forall x \exists y \forall z \left( z \varepsilon y \leftrightarrow \forall w (w \varepsilon z \rightarrow w \varepsilon x) \right)$.

Intuitively, it means that for every set $x$, the set $2^x$ exists.

**Infinity axiom:** $\exists x \left( \emptyset \varepsilon x \land \forall y \left( y \varepsilon x \rightarrow y \cup \{y\} \varepsilon x \right) \right)$

Intuitively, it means that there is an infinite set containing $\emptyset, 1, 2, \ldots$, where $\emptyset$ stands for $\emptyset$, $\hat{1}$ stands for $\{\emptyset\}$, and $\hat{n} = \{1, \ldots, n-1\}$.

Note that both $\emptyset \varepsilon x$ and $y \cup \{y\} \varepsilon x$ are abbreviations, where $\emptyset \varepsilon x$ represents “$\emptyset \in x$,” i.e., $\exists y (\forall z (z \varepsilon y) \land y \varepsilon x)$, and $y \cup \{y\} \varepsilon x$ represents “$y \cup \{y\} \in x$,” which can be written in a similar manner.

**Replacement axioms:** $\forall x_1 \cdots \forall x_n \forall x \exists y \varphi(x, y, x_1, \ldots, x_n) \rightarrow \forall u \exists v \forall y \left( y \varepsilon v \leftrightarrow \exists x (\varphi(x, y, x_1, \ldots, x_n) \land x \varepsilon u) \right)$

Intuitively, this means that if for parameters $x_1, \ldots, x_n$, the formula $\varphi(x, y, x_1, \ldots, x_n)$ defines a map $x \mapsto y$, then the range of a set is again a set.
Axiom of choice: \( \forall x \left( \emptyset \not\in x \land \forall u \forall v \left( u \in x \land v \in x \land u \neq v \rightarrow u \cap v \approx \emptyset \right) \right) \rightarrow \exists y \forall w \left( w \in x \rightarrow \exists z^{=1} \left( z \in w \cap y \right) \right) \)

This states axiom of choice. As before, those underline represent abbreviations of first-order formula describing their respective intuitive meanings.

**Remark 10.8** Assuming the consistency of ZFC, the following holds.

- ZFC + CH is consistent (Gödel 1940).
- ZFC + \( \neg \)CH is consistent (Cohen 1963).

That is, both CH and its negation cannot be proved from ZFC, provided that ZFC is consistent.

5 Skolem paradox

It is generally accepted that ZFC is consistent, although there is no way to prove it. In the following we are going to show an application of Löwenheim-Skolem theorem that yields a seemingly absurd result, called *Skolem* paradox.

Assuming its consistency, by Löwenheim-Skolem theorem, ZFC has a countable structure \( \mathcal{A} = (A, \varepsilon^A) \). By the infinity axiom, there is an element \( x \in A \) such that \( x \) is an infinite set. By power set axiom, \( 2^x \in A \). Now, by Cantor’s theorem, we know that \( 2^x \) is uncountable. However, since \( A \) is countable, the set of elements related to \( 2^x \) (by relation \( \varepsilon \)) must be countable too (since they all must come from \( A \)). Does this mean that Cantor’s theorem and Löwenheim-Skolem theorem contradict each other? Or, that ZFC is inconsistent?