Lesson 9: Gödel’s completeness theorem

Theme: Consistent set, Henkin set and the equivalence between the notions of ⊢ and |=.

1 Consistent sets

Let $L$ be a vocabulary, and let $X \subseteq \text{FO}[L]$. The set $X$ is inconsistent, if there is a formula $\alpha$ such that $X \vdash \alpha$ and $X \vdash \neg \alpha$. By the contradiction rule, this also means that $X$ is inconsistent if $X \vdash \beta$, for every formula $\beta$.

We say that $X$ is consistent, if $X$ is not inconsistent. It is maximally consistent, if it is consistent and for every set $Y \subseteq \text{FO}[L]$ and $Y \supseteq X$, $Y$ is inconsistent.

2 Constants elimination

Let $c$ be a constant symbol and $z$ be a variable. For a formula $\alpha$, we write $\alpha z$ to denote the formula obtained by replacing every constant symbol $c$ in $\alpha$ by $z$. For a set $X$, we write $X z$ to denote the set $\{\alpha z | \alpha \in X\}$.

Lemma 9.1 Suppose $X \vdash_L \alpha$. Let $c$ be a constant in $L$, and $L'$ denote $L - \{c\}$. Then, there is a finite subset $X_0 \subseteq X$ and a variable $z /\notin \text{var}(X_0) \cup \text{var}(\alpha)$,

$$X_0 z \vdash_{L'} \alpha z.$$  

Proof. (Sketch) Suppose $X \vdash_L \alpha$. By the finiteness theorem of $\vdash$, there is a finite set $X_0 \subseteq X$ such that $X_0 \vdash L \alpha$. Let $z /\notin \text{var}(X_0) \cup \text{var}(\alpha)$.

Claim 1 $X_0 z \vdash_{L'} \alpha z$.

The claim can be proved by induction on the length of the proof of $X_0 \vdash_L \alpha$. ■

Lemma 9.2 Suppose $X \vdash \alpha[c/x]$ and $c$ does not appear in $X$ and $\alpha$. Then, $X \vdash \forall x \alpha$.

Proof. Suppose $X \vdash \alpha[c/x]$, where $c$ does not appear in $X$ and $\alpha$.

By Lemma 9.1 there is a finite subset $X_0 \subseteq X$ such that $X_0 z \vdash \alpha[c/x] z$, where $z /\notin \text{var}(X_0) \cup \text{var}(\alpha[c/x])$.

Now, since $c$ does not appear in $X$, $X_0 z = X_0$. So,

$$X_0 \vdash \alpha[c/x] z.$$  

Moreover, $c$ does not appear in $\alpha$. So $\alpha[c/x] z = \alpha[z/x]$. Thus,

$$X_0 \vdash \alpha[z/x].$$  

Since $z$ does not appear in $X_0$ and $\alpha$, by generalisation rule, we have $X_0 \vdash \forall x \alpha$. Lemma 9.2 follows immediately by monotonicity rule. ■

For a variable $x \in \text{VAR}$ and $\alpha \in \text{FO}[L]$, we define a “new” constant $c_{x, \alpha} /\notin L$. We define the following formula $\alpha^x \in \text{FO}[L \cup \{c_{x, \alpha}\}]$.

$$\alpha^x := \neg \forall x \alpha \land \alpha[c_{x, \alpha}/x]$$

*Similar material can be obtained from Section 3.2 in the textbook A Concise Introduction to Mathematical Logic (3rd ed.) by Wolfgang Rautenberg.
Lemma 9.3  Let $L$ be a vocabulary. Define the set $\Gamma_L$ of formulas as follows:

$$\Gamma_L := \{-\alpha^x \mid x \in \text{VAR and } \alpha \in \text{FO}[L]\}$$

If a set $X$ is consistent, then so is $X \cup \Gamma_L$.

Proof. Let $X$ be a consistent set. Suppose to the contrary that $X \cup \Gamma_L$ is inconsistent. That is, there is $\varphi$ such that

$$X \cup \Gamma_L \vdash \varphi \quad \text{and} \quad X \cup \Gamma_L \vdash \neg \varphi.$$  

Thus, $X \cup \Gamma_L \vdash F$, where $F$ denotes $\varphi \land \neg \varphi$. By finiteness theorem, there is a finite subset $X_0 \subseteq X$ such that

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}}, -\alpha_n^{x_n} \vdash F.$$  

(1)

We can assume that $n$ is minimal in the sense that $X_0, -\alpha_1^{x_1}, \ldots, -\alpha_i^{x_i} \not\vdash F$, for every $i < n$. By Contradiction Rule on (1),

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}}, -\alpha_n^{x_n} \vdash \alpha_n^{x_n}.$$  

(2)

By Initial Rule and Monotonicity Rule,

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}}, \alpha_n^{x_n} \vdash \alpha_n^{x_n}.$$  

(3)

By Negation Rule on (2) and (3),

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}} \vdash \neg \forall x \alpha.$$  

(4)

Let us denote by $x := x_n$, $\alpha := \alpha_n$ and $c := c_{x, \alpha}$. Thus,

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}} \vdash \neg \forall x \alpha \land \alpha[c_{x, \alpha}/x].$$  

(5)

By And Split Rule on (5)

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}} \vdash \neg \forall x \alpha$$  

(6)

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}} \vdash \alpha[c_{x, \alpha}/x].$$  

(7)

Since $c_{x, \alpha}$ does not appear in $X_0$ and in each of $\alpha_i^{x_i}$, by Lemma 9.2 on (7), we have

$$X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}} \vdash \forall x \alpha.$$  

(8)

But (6) and (8) imply that $X_0, -\alpha_1^{x_1}, \ldots, -\alpha_{n-1}^{x_{n-1}}$ is inconsistent, which contradicts the assumption that $n$ is minimal.  

\[\text{Note that } \Gamma_L \text{ is a set of formulas over the vocabulary } L \cup \{c_{x, \alpha} \mid \alpha \in \text{FO}[L], x \in \text{VAR}\}.\]
3 Henkin sets

Definition 9.4 A set $X \subseteq \text{FO}[L]$ is called a Henkin set, if it satisfies the following properties.

(H1) $X \vdash \neg \alpha$ if and only if $X \not\models \alpha$. Or, equivalently, $X \vdash \alpha$ if and only if $X \not\models \neg \alpha$.

(H2) $X \vdash \forall x \alpha$ if and only if $X \vdash \alpha[c/x]$ for every constant $c \in L$.

Proposition 9.5 If $X$ is a Henkin set over vocabulary $L$, then for each $L$-term $t$, there is a constant $c \in L$ such that $X \vdash t \approx c$.

Proof. Let $X$ be a Henkin set over vocabulary $L$. By Example 8.10, we have $\vdash \neg \forall x t \not\approx x$, when $x \notin \text{var}(t)$. By Monotonicity Rule, $X \vdash \neg \forall x t \not\approx x$. Since $X$ is Henkin, by (H1), we have $X \not\models \forall x t \not\approx x$.

By (H2), for some constant $c$,

$X \not\models t \not\approx c$.

By (H1),

$X \vdash t \approx c$.

This completes our proof of Proposition 9.5. ■

Lemma 9.6 For every consistent set $X \subseteq \text{FO}[L]$, there is a Henkin set $Y \supseteq X$, where $Y \subseteq \text{FO}[L \cup C]$, for some set $C$ of “new” constants not in $L$.

Proof. Let $X \subseteq \text{FO}[L]$ be a consistent set. For each integer $i \geq 0$, we define the sets $\Gamma_i$, $\Delta_i$, $L_i$ and $C_i$ as follows.

$\Delta_0 := X \quad L_0 := L \quad C_0 := \emptyset \quad \Gamma_0 := \emptyset$

For each $i > 0$,

$C_i := \{c_{x,\alpha} \mid x \in \text{VAR} \text{ and } \alpha \in \text{FO}[L_{i-1}]\}$

$L_i := L_{i-1} \cup C_i$

$\Gamma_i := \{\neg \alpha^x \mid \alpha^x := \neg \forall x \alpha \land \alpha[c_{x,\alpha}/x] \text{ where } \alpha \in \text{FO}[L_{i-1}] \text{ and } c_{x,\alpha} \in C_i\}$

$\Delta_i := \Delta_{i-1} \cup \Gamma_i$

Now, let $\Delta := \bigcup_{i>0} \Delta_i$ and $L' := \bigcup_{i>0} L_i$.

Consider the poset $(\mathcal{F}, \subseteq)$, where

$\mathcal{F} := \{Z \mid \Delta \subseteq Z \subseteq \text{FO}[L'] \text{ and } Z \text{ is consistent}\}$.

Claim 2 Let $K$ be a chain in $(\mathcal{F}, \subseteq)$. Then, $\bigcup K$ is consistent.

Proof. (of Claim 2) Proceeds like the one in propositional calculus. ■

By Zorn’s lemma, there is a maximal consistent set $Y \in \mathcal{F}$. We will now show that that $Y$ is Henkin.
Claim 3 $Y$ satisfies (H1), i.e., $Y \vdash \neg \alpha$ if and only if $Y \nvdash \alpha$.

**Proof.** (of Claim 3) For the “only if” direction, suppose $Y \vdash \neg \alpha$. Since $Y$ is consistent, $Y \nvdash \alpha$.

For the “if” direction, suppose $Y \nvdash \alpha$, which means that $\alpha \notin Y$. Since $Y$ is maximal, $Y \cup \{\alpha\}$ is not consistent. So,

$$Y, \alpha \vdash \neg \alpha.$$  

By Initial Rule,

$$Y, \neg \alpha \vdash \neg \alpha.$$  

By Negation Rule,

$$Y \vdash \neg \alpha.$$  

This completes our proof of Claim 3. $lacksquare$

Claim 4 $Y$ satisfies (H2), i.e., $Y \vdash \forall x \alpha$ if and only if $Y \vdash \alpha[c/x]$ for every constant $c \in L'$.

**Proof.** (of Claim 4) For the “only if” direction, suppose $Y \vdash \forall x \alpha$. Let $c \in L'$. Now $[c/x]$ is collision-free in $\alpha$. By Specialisation Rule, $Y \vdash \alpha[c/x]$.

For the “if” direction, suppose $Y \vdash \alpha[c/x]$ for every constant $c \in L'$. Let $\alpha \in \text{FO}[L_\alpha]$. So, in particular for $c \in C_n$,

$$Y \vdash \alpha[c/x]. \quad (9)$$

Now, suppose to the contrary that $Y \nvdash \forall x \alpha$. By (H1),

$$Y \vdash \neg \forall x \alpha. \quad (10)$$

By And Combine Rule on (9) and (10),

$$Y \vdash \neg \forall x \alpha \land \alpha[c/x] \quad (11)$$

Note that the right side of (11) is simply $\alpha^x$. So, $Y \vdash \alpha^x$.

However, $\neg \alpha^x \in Y$. So, $Y \vdash \neg \alpha^x$, which means $Y$ is inconsistent. This contradicts the fact that $Y \in \mathcal{F}$, which means that $Y$ is consistent. Therefore, $Y \vdash \forall x \alpha$, and this completes the proof of Claim 4. $lacksquare$

Claims 3 and 4 state that $Y$ is Henkin, and this completes our proof of Lemma 9.6. $lacksquare$

**Lemma 9.7** Every Henkin set is satisfiable.

**Proof.** This will be proved in the tutorial. $lacksquare$

## 4 The completeness theorem for FO

**Theorem 9.8** (Gödel’s completeness theorem) $X \models \alpha$ if and only if $X \vdash \alpha$.

**Proof.** The “if” direction is the soundness theorem. For the “only if” direction, we show if $X \nvdash \alpha$, then $X \not\models \alpha$. Suppose $X \nvdash \alpha$. Then, $X \cup \{\neg \alpha\}$ is consistent. By Lemmas 9.6 and 9.7, there is a Henkin set $Y \supseteq X \cup \{\neg \alpha\}$ and $Y$ is satisfiable. This means $X \cup \{\neg \alpha\}$ is satisfiable, and therefore, $X \not\models \alpha$. $lacksquare$

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4Lemma 4.2 can be easily proved for a set $X$ of first-order formulas.
Exercises

In questions (1)-(8) below we are going to show that every Henkin set is satisfiable. Let \( Y \) be a Henkin set and \( C \) be the set of all the constants that appear in \( Y \). We associate each constant \( c \in C \) with an element \( a_c \). Different constants \( c \neq c' \) are associated with different elements \( a_c \neq a_{c'} \). Consider the set \( U \).

\[
U := \{ a_c \mid c \in C \}
\]

Define a relation \( \sim \) on \( U \) as follows.

\[
a_c \sim a_{c'} \text{ if and only if } Y \vdash c \approx c'
\]

(1) Prove that \( \sim \) is an equivalence relation on \( U \). (Note this is not a trivial question.)

Let \( [a_c] \) denote the equivalence class of \( a_c \) w.r.t. \( \sim \). The structure \( A = (A, R_1, \ldots, f_1, \ldots, c_1, \ldots) \) is defined as follows.

- \( A = \{ [a_c] \mid a_c \in U \} \).
- \( c_i = [a_{c_i}] \).
- \( R_i([a_{c_1}], \ldots, [a_{c_n}]) \) if and only if \( Y \vdash R(c_1, \ldots, c_n) \).
- \( f_i([a_{c_1}], \ldots, [a_{c_n}]) = [a_c] \), if \( Y \vdash f_i(c_1, \ldots, c_n) \approx c \).

(2) Prove that the definition of \( R_i \) is well defined.

That is, if \( ([a_{c_1}], \ldots, [a_{c_n}]) = ([a_{d_1}], \ldots, [a_{d_n}]) \), then,

\[
Y \vdash R(c_1, \ldots, c_n) \text{ if and only if } Y \vdash R(d_1, \ldots, d_n)
\]

(3) Prove that the definition of \( f_i \) is well defined.

That is,

- for every \( c_1, \ldots, c_n \in C \), there is \( c \) such that \( Y \vdash f_i(c_1, \ldots, c_n) \approx c \), and
- if \( ([a_{c_1}], \ldots, [a_{c_n}]) = ([a_{d_1}], \ldots, [a_{d_n}]) \), then \( f_i([a_{c_1}], \ldots, [a_{c_n}]) = f_i([a_{d_1}], \ldots, [a_{d_n}]) \).

Consider the following valuation \( \text{val} : \text{VAR} \to A \), where \( \text{val}(x) = [a_c] \), where \( Y \vdash x \approx c \).

(4) Prove that \( \text{val} \) is well defined.

(5) Prove that for every term \( t \), if \( Y \vdash t \approx c \), then \( t^A[\text{val}] = [a_c] \).

Next, we will show that \( Y \) is satisfiable, i.e., \( (A, \text{val}) \models \alpha \), for every \( \alpha \in Y \).

(6) Prove that \( (A, \text{val}) \models s \approx t \), for every atomic formula \( s \approx t \in Y \).

(7) Prove that \( (A, \text{val}) \models R(s_1, \ldots, s_n) \), for every atomic formula \( R(s_1, \ldots, s_n) \in Y \).

(8) Prove that \( (A, \text{val}) \models \alpha \), for every \( \alpha \in Y \), and hence, \( Y \) is satisfiable.

Compactness theorem states that \( X \) is satisfiable if and only if \( X \) is finitely satisfiable.

(9) Use the completeness theorem to prove the compactness theorem for \( \text{FO} \).