Lesson 4: Completeness of propositional calculus

**Theme:** The equivalence between provability and logical consequences (completeness of propositional calculus).

**Definition 4.1** A set $X$ is **inconsistent**, if there is $\alpha$ such that $X \vdash \alpha$ and $X \vdash \neg \alpha$. Otherwise, we say that $X$ is **consistent**.

**Lemma 4.2** For every set $X$ of formulas and for every formula $\alpha$, the following holds.

(a) $X \vdash \alpha$ if and only if $X \cup \{\neg \alpha\}$ is inconsistent.
(b) $X \vdash \neg \alpha$ if and only if $X \cup \{\alpha\}$ is inconsistent.

**Definition 4.3** A set $X$ is **maximally consistent**, if it is consistent and for every $Y \supseteq X$, $Y$ is inconsistent.

**Lemma 4.4** Every consistent set $X$ can be extended to a maximally consistent set. That is, for every consistent set $X$, there is a maximally consistent set $Y$ such that $Y \supseteq X$.

**Lemma 4.5** A maximally consistent set $X$ has the following property: For every $\alpha$,

$$X \vdash \neg \alpha \iff X \not\models \alpha.$$ 

**Lemma 4.6** A maximally consistent set $X$ is satisfiable.

**Proof.** (Sketch) Define the following assignment $w$, where for every atomic proposition $p$:

$$w(p) := \begin{cases} 
T, & \text{if } X \vdash p \\ 
F, & \text{if } X \vdash \neg p
\end{cases}$$

We have to show that for every $\alpha \in X$, $w(\alpha) = T$. It is sufficient to show the following.

$$X \vdash \alpha \iff w(\alpha) = T.$$ 

The proof is by induction on $\alpha$. □

**Theorem 4.7** (Completeness of propositional calculus) $X \vdash \alpha$ if and only if $X \models \alpha$.

**Proof.** The “only if” direction is straightforward. We prove the “if” direction by showing that $X \not\vdash \alpha$ implies $X \not\models \alpha$.

Suppose $X \not\vdash \alpha$. This means that $X \cup \{\neg \alpha\}$ is consistent. By Lemma 4.4, we can extend it to a maximally consistent set $Y$. Lemma 4.6 implies $Y$ is satisfiable, and hence, $X \cup \{\neg \alpha\}$ is also satisfiable, which further implies that $X \not\models \alpha$ (why?). This completes our proof. □

There are six rules in our proof system. Where do we use each of them in our proof of completeness theorem?