Homework 1
Due on Monday, 02:30 pm, 25 March 2019 (2019/03/25)

Name : 
Student ID : 

Note:
1. Write down clearly your name and student ID in the space above.
2. There are FIVE questions altogether.
3. Write your solution for each question in the space provided.
Question 1. In the following let $p, q, r$ be propositional variables. Decide which of the following formulas are tautology:

(a) $p \rightarrow (q \rightarrow p)$.
(b) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.
(c) $p \rightarrow (p \lor q)$.
(d) $p \rightarrow (\neg p \rightarrow q)$.

If you think that a formula is not a tautology, give an assignment under which it yields $F$, i.e., the false value.

Solution.
Question 2. For every integer \( n \geq 2 \), find a set of \( n \) propositional formulas \( \{ \alpha_1, \ldots, \alpha_n \} \), which is not satisfiable, but its every proper subset is satisfiable.

Solution.
Question 3. Prove that \( X \vdash \alpha \rightarrow \beta \) \\
\( X \vdash \neg \beta \rightarrow \neg \alpha \).

Note that \( \alpha \rightarrow \beta \) is an abbreviation for \( \neg (\alpha \land \neg \beta) \), whereas \( \neg \beta \rightarrow \neg \alpha \) for \( \neg (\neg \beta \land \neg \neg \alpha) \).

Solution.
Question 4. (Reproving compactness theorem for countable $X$) We will show the following.

*If $X$ is countable and finitely satisfiable, then $X$ is satisfiable.*

Let $X$ be a countable and finitely satisfiable set. Since $X$ is countable, there are only countably many variables used in $X$. Without loss of generality, we assume that there are infinitely many of them, which we denote by $p_1, p_2, p_3, \ldots$.

For every integer $i \geq 0$, define $\Delta_i$ as follows.

\[
\begin{align*}
\Delta_0 &:= X \\
\Delta_i &:= \begin{cases} 
\Delta_{i-1} \cup \{p_i\}, & \text{if } \Delta_{i-1} \cup \{p_i\} \text{ is finitely satisfiable} \\
\Delta_{i-1} \cup \{\neg p_i\}, & \text{otherwise}
\end{cases}
\end{align*}
\]

Define $\Delta$ to be the following set.

\[
\Delta := \bigcup_{i \geq 0} \Delta_i
\]

Prove the following.

(a) $\Delta$ is finitely satisfiable.
(b) $\Delta$ is satisfiable, and hence, $X$ is satisfiable.

Solution.
Question 5. In our proof above, we assume that the number of variables used in $X$ is countable infinite. What do you think of the case when the number of variables used is only finite?

Solution.