Lesson 10: Distributional complexity

**Theme:** Distributional complexity and Yao’s minimax principle for establishing lower bounds.

Let $f : X \times Y \rightarrow \{0, 1\}$, and let $\mu$ be a probability distribution on $X \times Y$. The $(\mu, \epsilon)$-distributional communication complexity of $f$, denoted by $D^\mu_\epsilon(f)$, is the cost of the best deterministic protocol that gives the correct answer for $f$ on at least $(1 - \epsilon)$ fraction of all inputs in $X \times Y$, weighted by $\mu$. More formally,

$$D^\mu_\epsilon(f) = \min \left\{ \text{cost}(P) \middle| \Pr_{(x,y) \sim \mu}[P(x,y) = f(x,y)] \geq 1 - \epsilon \right\}$$

**Lemma 10.1** Let $f : X \times Y \rightarrow \{0, 1\}$. Let $P$ be a randomized public coin protocol for computing $f$ with error at most $\epsilon$, and let $\Pi$ be the set of its random strings. Then, for every probability distribution $\mu$ on $X \times Y$, there is $r \in \Pi$ such that:

$$\Pr_{(x,y) \sim \mu}[P(x,y,r) = f(x,y)] \geq 1 - \epsilon.$$

**Lemma 10.2** Let $f : X \times Y \rightarrow \{0, 1\}$. Then, for every $0 \leq \epsilon < 1$, $R^\mu_\epsilon(f) \leq \max_{\mu} D^\mu_\epsilon(f)$.

Combining Lemmas [10.1] and [10.2] we have the following.

**Theorem 10.3** Let $f : X \times Y \rightarrow \{0, 1\}$. Then, for every $0 \leq \epsilon < 1$, $R^\mu_\epsilon(f) = \max_{\mu} D^\mu_\epsilon(f)$.

Recall the function $\text{disj}(x,y) = 1$ if and only if $x$ and $y$ are disjoint, for $x,y \subseteq [n]$.

**Lemma 10.4** There exist:

- $A \subseteq \text{disj}^{-1}(1)$ and $B \subseteq \text{disj}^{-1}(0)$,
- a probability distribution $\mu$ on $X \times Y$, i.e., $X = Y = 2^{[n]}$,
- positive constants $\alpha$ and $\delta$,

such that the following holds.

- $\mu(A) = 3/4$.
- For every rectangle $R$, $\mu(R \cap B) \geq \alpha \cdot \mu(R \cap A) - 2^{-\delta n}$.

Furthermore, there is $0 \leq \epsilon < 1/2$ such that $D^\mu_\epsilon(\text{disj}) \geq \delta n - O(1)$.

**Theorem 10.5** $R^\mu_\epsilon(\text{disj}) = \Omega(n)$, for some $0 \leq \epsilon < 1/2$.
Appendix

Let $M$ be an $m \times n$ matrix with entries from real numbers. Two players $R$ and $C$, which stand for Row and Column, play a game according to $M$. Player $R$ chooses a strategy $i \in [m]$ and player $C$ a strategy $j \in [n]$. The payoff is $M_{i,j}$ for player $R$ and $-M_{i,j}$ for player $C$. In playing the game, player $R$ aims to maximize the payoff, while player $C$ to minimize the payoff.

We represent a probability distribution $p$ on $[m]$ by a column vector $p \in \mathbb{R}^m$:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix}$$

In our setting, it intuitively means that the probability that player $R$ chooses strategy $i$ is $p_i$. A probability distribution on $[n]$ can be represented by $q \in \mathbb{R}^n$ similarly. Thus, note that $p^t M q$ is the expectation of the payoff of the game (for player $R$) according to probability distributions $p$ and $q$.

**Theorem 10.6 (von Neumann’s minimax theorem)** Let $M$ be an $m \times n$ real number matrix.

$$\max_p \min_q p^t M q = \min_q \max_p p^t M q,$$

where $p$ and $q$ range over all the probability distributions on $[m]$ and $[n]$, respectively.

**Theorem 10.7 (Loomis’ theorem)** Let $M$ be an $m \times n$ real number matrix.

$$\max_p \min_{1 \leq j \leq n} p^t M e_j = \min_q \max_{1 \leq i \leq m} e_i^t M q,$$

where $p$ and $q$ range over all the probability distributions on $[m]$ and $[n]$, respectively. (Here $e_i$ and $e_j$ denote the unit vectors.)

**Theorem 10.8 (Yao’s minimax principle)** Let $M$ be an $m \times n$ real number matrix. For every probability distributions $p$ and $q$ on $[m]$ and $[n]$, respectively,

$$\min_{1 \leq j \leq n} p^t M e_j \leq \max_{1 \leq i \leq m} e_i^t M q.$$

To prove a lower bound on randomized protocols/algorithms, we usually take $[m]$ as a collection of deterministic protocols/algorithms and $[n]$ a collection of inputs.