Homework 1
Due on Monday, 10:30 am, 25 March 2019 (2019/03/25)

Name : 
Student ID : 

Note:
1. Write down clearly your name and student ID in the space above.
2. There are FIVE questions altogether.
3. Write your solution for each question in the space provided.
Question 1. Recall the function $\text{MED}(x, y)$: For subsets $x, y \subseteq \{1, \ldots, n\}$, $\text{MED}(x, y)$ is the median of the elements in the multiset $x \cup y$.

If the multiset $x \cup y$ has an even number of elements, say $2k$, then the median is the $k$th smallest element. When it has an odd number of elements, say $2k + 1$, then the median is the $(k + 1)$th smallest element.

We have discussed in the class that $D(\text{MED}) = O(\log n)$, by assuming that $|x| = |y| = 2^t$, for some $t$.

(a) Prove that the assumption that $|x| = |y| = 2^t$ does not effect the generality of our result.

(b) We have also discussed that the missing step in the sample solution in the textbook. Provide that missing step.

Solution.
Question 2. Recall the function $\text{disj}$: For every $x, y \subseteq \{1, \ldots, n\}$,

$$\text{disj}(x, y) = \begin{cases} 
1, & \text{if } x \cap y = \emptyset \\
0, & \text{otherwise}
\end{cases}$$

Prove that the size of any 1-monochromatic rectangle of $\text{disj}$ is at most $2^n$. Then, deduce that $D(\text{disj}) = \Omega(n)$.

Solution.
Question 3. Recall the function $\text{avg}$: For every subsets $x, y \subseteq \{1, \ldots, n\}$, $\text{avg}(x, y)$ is the average of the elements in the multiset $x \cup y$. Prove that $D(\text{avg}) = \Theta(\log n)$.

As a reminder, to show $D(\text{avg}) = \Theta(\log n)$, you have to show both upper and lower bound: $D(\text{avg}) = O(\log n)$ and $D(\text{avg}) = \Omega(\log n)$.

Solution.
Question 4. Let $X, Y \subseteq 2^{\{1,\ldots,n\}}$ such that for every $x \in X$ and $y \in Y$, the following holds:

$$|x \cap y| \leq 1$$

Define the function $f : X \times Y \to \{0, 1\}$ as follows:

$$f(x, y) = |x \cap y|$$

Prove that $D(f) = O(\log^2 n)$.

Solution.
Question 5. In the following, let \([n] = \{1, \ldots, n\}\) and \(\bar{z} = (z_1, \ldots, z_n)\) be a vector of \(n\) variables. For integers \(0 \leq k \leq n\), we define \(N_{n, k} = \sum_{i=0}^{k} \binom{n}{i}\).

(a) For \(x \subseteq [n]\) and \(x \neq \emptyset\), we define the (multi-variate) monomial \(q_x(\bar{z}) := \prod_{i \in x} z_i\). When \(x = \emptyset\), \(q_x(\bar{z}) = 0\).

For \(y \subseteq [n]\), define the vector \(\bar{c}_y = (c_1, \ldots, c_n)\) where each \(c_i\) is as follows.

\[
c_i = \begin{cases} 1, & \text{if } i \notin y \\ 0, & \text{if } i \in y \end{cases}
\]

Note that \(q_x(\bar{c}_y) = 1\) if and only if \(x \cap y = \emptyset\).

Consider the following (multi-linear) polynomial \(p(\bar{z})\).

\[
p(\bar{z}) := \alpha_0 + \alpha_1 q_{x_1}(\bar{z}) + \cdots + \alpha_m q_{x_m}(\bar{z}),
\]

where \(x_1 \subseteq x_2 \subseteq \cdots \subseteq x_m\) and \(|x_m| \leq k\), and each \(\alpha_i\) is non-zero real number.

Prove that there is \(y \subseteq [n]\) such that \(|y| \leq k\) and \(p(\bar{c}_y) \neq 0\).

(b) Consider the following polynomial \(p(\bar{z})\).

\[
p(\bar{z}) := \alpha_1 q_{x_1}(\bar{z}) + \cdots + \alpha_m q_{x_m}(\bar{z}),
\]

where \(x_1, \ldots, x_m\) are all the subsets of \([n]\) with cardinality \(\leq k\) and not all of \(\alpha_i\) is zero.

Prove that there is \(y \subseteq [n]\) such that \(|y| \leq k\) and \(p(\bar{c}_y) \neq 0\).

(c) Define the following function \(\text{DISJ}_{\leq k}:\) For every \(x, y \subseteq [n]\) such that \(|x|, |y| \leq k\),

\[
\text{DISJ}_{\leq k}(x, y) = \begin{cases} 1, & \text{if } x \cap y = \emptyset \\ 0, & \text{otherwise} \end{cases}
\]

Prove that \(\text{rank}(\text{DISJ}_{\leq k}) = N_{n, k}\), and hence, \(D(\text{DISJ}_{\leq k}) \geq \log N_{n, k}\).

Hint: Consider the matrix of \(\text{DISJ}_{\leq k}\), and also what the polynomial \(p(\bar{z})\) in (b) represents.

Solution.