Sample solution for homework 3

Question 1.
- On input $\epsilon$:
  $q_0 \vdash q_{\text{rej}}$.
- On input $011$:
  $q_0011 \vdash <p_011 \vdash <0p_11 \vdash <0p_11 \vdash 01s1 \vdash <0s10 \vdash <s000 \vdash t < 100 \vdash q_{\text{acc}} < 100$.
- On input $100$:
  $q_0100 \vdash <p_100 \vdash <1p_00 \vdash <10p_0 \vdash <10q_{\text{rej}}0$.
- On input $111$:
  $q_0111 \vdash <p_111 \vdash <1p_11 \vdash <1p_11 \vdash <11s1 \vdash <1s10 \vdash <s100 \vdash s < 000$
  $\vdash <r_10000 \vdash <1r_0000 \vdash <10r_00 \vdash <10r_00 \vdash <1r_0000 \vdash <t1000 \vdash t < 1000$
  $\vdash q_{\text{acc}} < 1000$.

Question 2.
(a) If $M$ accepts $w$, then the TM $K_{M,w}$ ignores its input $u$, and ACCEPTS immediately. In other words, it accepts everything. Therefore, $L(K_{M,w}) = \Sigma^*$.
(b) If $M$ rejects $w$, then the TM $K_{M,w}$ checks whether its input $u$ is of the form $0^n1^n$. If it is of the form $0^n1^n$, then $K_{M,w}$ ACCEPTS. Otherwise, it REJECTS. Therefore, when $M$ rejects $w$, $L(K_{M,w}) = \{0^n1^n \mid n \geq 0\}$.
(c) If $M$ does not halt on $w$, then the TM $K_{M,w}$ does not halt regardless of its input $u$. Therefore, $L(K_{M,w}) = \emptyset$.
(d) Note that there are three possibilities for $L(K_{M,w})$: $\Sigma^*$, $\{0^n1^n \mid n \geq 0\}$ and $\emptyset$. The first and the third are regular, but the second is not.

The following sentence is not true:

$\lfloor M \rfloor \mid w \in \text{HALT}$ if and only if $\lfloor K_{M,w} \rfloor \in L_5$

The part "if $\lfloor K_{M,w} \rfloor \in L_5$, then $\lfloor M \rfloor \mid w \in \text{HALT}" is not true, because from (c) we know that $L(K_{M,w}) = \emptyset$, which is regular, but $M$ does not accept $w$, i.e., $\lfloor M \rfloor \mid w \notin \text{HALT}$.

Note that the part "if $\lfloor M \rfloor \mid w \in \text{HALT}$, then $\lfloor K_{M,w} \rfloor \in L_5" is true, because from (a) we know that when $M$ accepts $w$, $L(K_{M,w}) = \Sigma^*$, which is regular.

Question 3. We will show that $\text{HALT} \leq_m L_\infty$. Consider the following reduction.

INPUT: $\lfloor M \rfloor \mid w$.
- Construct a TM $K_{M,w}$ that works as follows.
  INPUT: $u \in \Sigma^*$.
  - Simulate $M$ on $w$.
  - If $M$ accepts $w$, ACCEPT.
  - If $M$ rejects $w$, REJECT.
• Output $|\mathcal{K}_{M,w}|$.

We analyse the behavior of $\mathcal{K}_{M,w}$:

• If $\mathcal{M}$ accepts $w$, the TM $\mathcal{K}_{M,w}$ ignores its input $u$, and ACCEPTS immediately. That is, it accepts every input word. Therefore, $L(\mathcal{K}_{M,w}) = \Sigma^*$, which is infinite.

• If $\mathcal{M}$ rejects $w$, the TM $\mathcal{K}_{M,w}$ ignores its input $u$, and REJECTS immediately. That is, it rejects every input word. Therefore, $L(\mathcal{K}_{M,w}) = \emptyset$, which is finite.

• If $\mathcal{M}$ does not halt on $w$, the TM $\mathcal{K}_{M,w}$ does not halt regardless of its input $u$. Therefore, $L(\mathcal{K}_{M,w}) = \emptyset$, which is finite.

In other words, $|\mathcal{M}| \notin \text{HALT}$ if and only if $|\mathcal{K}_{M,w}| \in L_\infty$.

**Question 4.** We show a reduction from CFL-Universality.

On input CFG $G$, do the following.

• Construct a DFA $A$ that accepts $\Sigma^*$, i.e., $A$ has only one state which is accepting.

• Output $G$ and $A$.

Now, $L(G) = \Sigma^*$ if and only if $L(G) = L(A)$.

Thus, if there is an algorithm/TM for our problem, (i.e., to decide whether $L(G) = L(A)$), then there is an algorithm/TM for the problem CFL-Universality. Since CFL-Universality is undecidable, so is our problem.

**Question 5.** We first note that $\text{NP}$ are closed under union and intersection. That is, if $K_1, K_2 \in \text{NP}$, then $K_1 \cup K_2 \in \text{NP}$ and $K_1 \cap K_2 \in \text{NP}$. Indeed, suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are the NTMs for $K_1$ and $K_2$, respectively. Let $\mathcal{M}$ be a 2-tape NTM that on input $u$, run $\mathcal{M}_1$ on $u$ on tape 1, and run $\mathcal{M}_2$ on $u$ on tape 2.

• For language $K_1 \cup K_2$, $\mathcal{M}$ ACCEPTS if and only if at least one of $\mathcal{M}_1$ and $\mathcal{M}_2$ ACCEPT.

• For language $K_1 \cap K_2$, $\mathcal{M}$ ACCEPTS if and only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ ACCEPT.

Now, back to our question, suppose $L_1, L_2 \in \text{coNP}$. We will first prove that $L_1 \cup L_2 \in \text{coNP}$. Suppose $L_1, L_2 \in \text{coNP}$. By definition, $\Sigma^* - L_1$ and $\Sigma^* - L_2$ are in $\text{NP}$. By the following identity:

$$\Sigma^* - (L_1 \cup L_2) = (\Sigma^* - L_1) \cap (\Sigma^* - L_2)$$

Since both $\Sigma^* - L_1$ and $\Sigma^* - L_2$ are in $\text{NP}$, the intersection is also in $\text{NP}$. Thus, $L_1 \cup L_2$ is in $\text{coNP}$.

The proof for $L_1 \cap L_2 \in \text{coNP}$ is similar, except that we use the following identity:

$$\Sigma^* - (L_1 \cap L_2) = (\Sigma^* - L_1) \cup (\Sigma^* - L_2)$$

**Question 6.** Suppose $\text{SAT} \in \text{coNP}$. By definition of $\text{coNP}$, $\Sigma^* - \text{SAT} \in \text{NP}$. Let $\mathcal{M}_0$ be the polynomial time NTM that decides $\Sigma^* - \text{SAT}$.

Now, we will prove $\text{NP} \subseteq \text{coNP}$. Let $L \in \text{NP}$. Let $\mathcal{M}$ be an NTM for language $L$ that runs in polynomial time.

To show that $L \in \text{coNP}$, we have to show that $\Sigma^* - L \in \text{NP}$. 
Recall that SAT is \( \text{NP} \)-hard, so \( L \leq_p \text{SAT} \). That is, there is a polynomial time computable function \( F \) such that

\[
\text{if and only if } F(w) \in \text{SAT}
\]

Equivalently, \( w \notin L \) if and only if \( F(w) \notin \text{SAT} \), i.e., \( F(w) \in \Sigma^* - \text{SAT} \).

The following TM accepts \( \Sigma^* - L \) in polynomial time. On input \( w \):

1. Compute \( F(w) \).
2. Run \( M \) on \( F(w) \).
3. Accept, when \( M \) accepts \( F(w) \), i.e., when we reach the accepting state of \( M \).

Note that \( M \) is NTM, so our algorithm is also NTM.

*Complexity analysis:* Since \( F \) is polynomial time computable, the first step takes polynomial time. Now, by our assumption \( M \) is polynomial time NTM that accepts \( \Sigma^* - \text{SAT} \). So, overall our NTM above runs in polynomial time.

*Proof of correctness:* By definition, \( w \in \Sigma^* - L \) if and only if \( F(w) \in \Sigma^* - \text{SAT} \). Since \( M_0 \) accepts \( \Sigma^* - \text{SAT} \), we have that \( w \in \Sigma^* - L \) if and only if \( M_0 \) accepts \( F(w) \). Thus, our NTM above accepts \( \Sigma^* - L \).