Sample solution for midterm exam

In the following, the alphabet is always $\Sigma = \{a, b\}$.

**Question 1.**

(a) $L_1 = \{a^{2n} \mid n \geq 1\}$ is defined by the regex: $aa(aa)^*$.

(b) $L_2 = \{w \mid$ every $a$ in $w$ is followed immediately by $b\}$ is accepted by the following DFA:

(c) $L_3 = \{w \mid w$ contains three consecutive $a$’s$\}$ is defined by the regex: $\Sigma^*aaa\Sigma^*$.

Alternatively, one can also construct the following DFA:

(d) $L_4 = \{w \mid w$ does not contain three consecutive $a$’s$\}$ is accepted by the complement of the DFA in (c):

**Question 2.** Abusing the notation, for two regex $e_1$ and $e_2$, we write $e_1 = e_2$ to denote $L(e_1) = L(e_2)$. Likewise, $e_1 \subseteq e_2$ denotes $L(e_1) \subseteq L(e_2)$ and $e_1 \neq e_2$ denotes $L(e_1) \neq L(e_2)$.

(a) $r^* \cup s^* \neq (r \cup s)^*$, when $r = a$ and $s = b$.

(b) $(r^* \cdot s^*)^* \neq (r \cdot s)^*$, when $r = a$ and $s = b$.

(c) $(r^* \cup s)^* = (r \cup s)^*$.

**Proof:** Note that $r \cup s \subseteq r^* \cup s$, hence, $(r \cup s)^* \subseteq (r^* \cup s)^*$. The other direction follows from the following.

$$ (r^* \cup s)^* \subseteq ((r \cup s)^* \cup s)^* = ((r \cup s)^*)^* = (r \cup s)^* $$

The inclusion comes from the fact that $r \subseteq r \cup s$, and hence, $r^* \subseteq (r \cup s)^*$. The first equality comes from the fact that $s \subseteq (r \cup s)^*$, and hence, $(r \cup s)^* \cup s = s$, whereas the second equality from the fact that $(A^*)^* = A^*$, for every set $A$. 

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(d) \( (r^* \cup s^*)^* = (r \cup s)^* \).

**Proof:** Applying the equality in (c), we have the following.

\[
(r \cup s)^* = (r^* \cup s^*)^* = (r^* \cup s^*)^*.
\]

**Question 3.** In the following, capital letters \( X, S \) are variables and \( S \) is always the start variable.

(a) \( K_1 = \{ a^n b^n \mid n \geq 1 \} \) is generated by the following grammar:

\[
S \to aSb \mid ab
\]

(b) \( K_2 = \{ a^n x b^n \mid x \in \Sigma^* \text{ and } n \geq 1 \} \) is generated by the following grammar:

\[
\begin{align*}
S & \to aSb \mid X \\
X & \to aX \mid bX \mid \epsilon
\end{align*}
\]

(c) \( K_3 = \{ a^n b^n a^m b^m \mid n, m \geq 1 \} \) is generated by the following grammar:

\[
\begin{align*}
S & \to XX \\
X & \to aXb \mid ab
\end{align*}
\]

(d) \( K_4 = \{ a^n b^m a^m b^n \mid m, n \geq 1 \} \) is generated by the following grammar:

\[
\begin{align*}
S & \to aSb \mid X \\
X & \to bXa \mid ba
\end{align*}
\]

**Question 4.** Let \( G = \langle \Sigma, V, R, S \rangle \) be a left-linear grammar. Construct the following NFA \( A = \langle \Sigma, Q, q_0, F, \delta \rangle \), where the set of states \( Q \) is \( V \cup \{ q_f \} \), the initial state \( q_0 \) is \( S \), the set of final states is \( F = \{ q_f \} \), and the set of transitions \( \delta \) is as follows.

- For every rule \( A \to cB \) in \( R \), we have a transition \( (A, c, B) \) in \( \delta \).
- For every rule \( A \to c \) in \( R \), we have a transition \( (A, c, q_f) \) in \( \delta \).

To show that \( L(G) = L(A) \) holds, we can prove that for every word \( w \), for every variable \( A \), the following holds.

- \( S \Rightarrow^* wA \) if and only if there is a run of \( A \) from \( S \) to \( A \) on \( w \).
- \( S \Rightarrow^* w \) if and only if there is a run of \( A \) from \( S \) to \( q_f \) on \( w \).

The proof is by straightforward induction on the length of \( w \).

**Question 5.** Let \( A_1 = \langle \Sigma, Q_1, q_0, 1, F_1, \delta_1 \rangle \) be a PDA that accepts \( K \) and \( A_2 = \langle \Sigma, Q_2, q_0, 2, F_2, \delta_2 \rangle \) be an NFA that accepts \( L \).

Construct the following PDA \( A = \langle \Sigma, Q, q_0, F, \delta \rangle \) that simulates both \( A_1 \) and \( A_2 \) simultaneously.

- \( Q = Q_1 \times Q_2 \).
- \( q_0 = (q_{0,1}, q_{0,2}) \).
- \( F = F_1 \times F_2 \).
• $\delta$ is defined as follows.
  
  - For every $(p_1, x, \text{pop}(y) \rightarrow (q_1, \text{push}(z))) \in \delta_1$, where $x \neq \epsilon$ and $(p_2, x, q_2) \in \delta_2$, the following transition is in $\delta$:
    $((p_1, p_2), x, \text{pop}(y) \rightarrow ((q_1, q_2), \text{push}(z)))$

  - For every $(p_1, x, \text{pop}(y) \rightarrow (q_1, \text{push}(z))) \in \delta_1$, where $x = \epsilon$, for every $p_2 \in Q_2$, the following transition is in $\delta$:
    $((p_1, p_2), x, \text{pop}(y) \rightarrow ((q_1, p_2), \text{push}(z)))$

That $\mathcal{A}$ accepts precisely $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ can be proved in a similar manner as the fact that regular languages are closed under intersection.