Lesson 9: Variants of Turing machines

**Theme:** Some variations of Turing machines.

1 Multi-tape Turing machines

A multi-tape Turing machine is a Turing machine that has a few tapes. On each tape, the Turing machine has one head. Formally, it is defined as follows. Let $k \geq 1$. A $k$-tape Turing machine is $M = \langle \Sigma, \Gamma, Q, q_0, q_{\text{acc}}, q_{\text{rej}}, \delta \rangle$, where $\delta$ is a function

$$\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma^k \rightarrow Q \times \Gamma \times \left\{ \text{Left, Right, Stay} \right\}^k$$

As before, an element of $\delta$ is written in the form:

$$(q, a_1, \ldots, a_k) \rightarrow (p, b_1, \ldots, b_k, \alpha_1, \ldots, \alpha_k).$$

Intuitively, it means that if the TM is in state $q$, and on each $i = 1, \ldots, k$, the head on tape $i$ is reading $a_i$, then it enters state $p$, and for $i = 1, \ldots, k$, the head on tape $i$ writes the symbol $b_i$ and moves according to $\alpha_i$.

A configuration of $M$ is of the form $(q, u_1, \ldots, u_k)$, where $q \in Q$ and each $u_i$ is a string over $\Gamma \cup \{\bullet\}$ and the symbol $\bullet$ appears exactly once in each $u_i$. The symbol $\bullet$ is to denote the position of the head.

The input is always written in the first tape. All the other tapes are initially blank. Formally, the initial configuration on input word $w$ is $(q_0, \bullet w, \bullet, \ldots, \bullet)$.

The notion of “one step computation” $C \vdash C'$ is defined similarly as in the standard Turing machine. Likewise, the conditions of acceptance and rejection are defined as when the Turing machines enter the accepting and rejecting states, respectively.

**Theorem 9.1** For every language $L$, the following holds.

- If $L$ is recognized by a $k$-tape TM $M$, then there is a single tape TM $M'$ that recognizes $L$.
- If $L$ is decided by a $k$-tape TM $M$, then there is a single tape TM $M'$ that decides $L$.

2 Non-deterministic Turing machines

A non-deterministic Turing machine (NTM) $M = \langle \Sigma, \Gamma, Q, q_0, q_{\text{acc}}, q_{\text{rej}}, \delta \rangle$ is defined as the standard Turing machine, with the exception that $\delta$ is now a relation:

$$\delta \subseteq (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \times Q \times \Gamma \times \left\{ \text{Left, Right, Stay} \right\}$$

As before, we write an element of $\delta$ is in the form:

$$(q, a) \rightarrow (p, b, \alpha).$$

The initial configuration of $M$ on input word $w$ is $q_0w$. For two configurations $C, C'$, the notion of “one step computation” $C \vdash C'$ is defined similarly as in the standard Turing machine. A run of $M$ on input $w$ is a sequence:

$$C_0 \vdash C_1 \vdash \cdots ,$$

where $C_0$ is the initial configuration on $w$. A run is accepting/rejecting, if it ends up in an accepting/rejecting configurations, respectively. However, due to non-determinism, for each $C$ there can be a few configuration $C'$ such that $C \vdash C'$, thus, there can be many runs. Some are accepting, some are rejecting, and some other do not halt.
Important definitions.

- An NTM $M$ accepts $w$, if there is an accepting run of $M$ on $w$.
- An NTM $M$ rejects $w$, if all runs of $M$ on $w$ are rejecting.
- A language $L$ is decided by an NTM $M$, if
  - for every $w \in L$, $M$ accepts $w$;
  - for every $w \notin L$, $M$ rejects $w$.
- A language $L$ is recognized by an NTM $M$, if
  - for every $w \in L$, $M$ accepts $w$;
  - for every $w \notin L$, $M$ does not accept $w$.

Recall that the standard TM is always deterministic. To avoid potential confusion, we will use the abbreviation DTM to mean deterministic Turing machine.

**Theorem 9.2** For every language $L$, the following holds.

- If $L$ is recognized by an NTM $M$, then there is a DTM $M'$ that recognizes $L$.
- If $L$ is decided by an NTM $M$, then there is a DTM $M'$ that decides $L$.

The computation of an NTM $M$ on input $w$ can be pictured as a tree whose nodes are configurations of $M$ defined as follows.

- The root node is the initial configuration $q_0w$.
- The children of a node $C$ are all possible $C'$ where $C \vdash C'$.

3 Some theorems on recognizable and decidable languages

**Theorem 9.3**

- If a language $L$ is decidable, so is its complement $\Sigma^* - L$.
- If both a language $L$ and its complement $\Sigma^* - L$ are recognizable, then $L$ is decidable.

**Theorem 9.4**

- Recognizable languages are closed under union, intersection, concatenation and Kleene star.
- Decidable languages are closed under union, intersection, complement, concatenation and Kleene star.