Lesson 7: CFG = PDA

Theme: The equivalence between context-free grammars (CFG) and push-down automata (PDA).

We are going to show that CFG and PDA define precisely the same class of languages. More precisely, we are going to show the following.

• For every CFG $G$, there is a PDA $A$ such that $L(A) = L(G)$.

• Vice versa, for every PDA $A$, there is a CFG $G$ such that $L(A) = L(G)$.

1 From CFG to PDA

Allowing the PDA to push a string of symbols. First, we note that we can modify the definition of CFG to one that allows it to push a string of symbols to its stack. That is, the transitions can be of the form:

$$(p, x, \text{pop}(y)) \rightarrow (q, \text{push}(z))$$

where $z \in \Gamma^*$

Allowing such transition does not change the capability of a CFG. We can simply add “new” states $t_1, \ldots, t_m$, where $|z| = m$ and $a_1 \ldots a_m$, and each $t_i$ is used to push the symbol $a_i$ into the stack. More formally, the transition $(p, x, \text{pop}(y)) \rightarrow (q, \text{push})$ can be replaced with the following transitions:

$$(p, x, \text{pop}(y)) \rightarrow (t_1, \text{push}(a_1))$$
$$(t_1, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_2, \text{push}(a_2))$$
$$(t_2, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_3, \text{push}(a_3))$$

... ...

$$(t_{i-1}, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_i, \text{push}(a_i))$$

... ...

$$(t_{m-1}, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_m, \text{push}(a_m))$$

$$(t_m, \epsilon, \text{pop}(\epsilon)) \rightarrow (q, \text{push}(\epsilon))$$

Left-most substitution property. Let $G = (\Sigma, V, R, S)$ be a CFG. Let $z_1 \Rightarrow z_2 \Rightarrow \cdots \Rightarrow z_n$ be a derivation. We say that the derivation has the left-most substitution property, if for every $i \in \{1, \ldots, n-1\}$, if $z_i \Rightarrow z_{i+1}$ is obtained by applying a rule $A \rightarrow w$, where $A$ is the left most variable in $z_i$. That is, $z_i$ is of the form $xAy$, where $x$ does not contain any variable, and $z_{i+1} = xwy$. Intuitively, it means that we only substitute the left-most variable in our derivation.

For example, suppose we have grammars with the following rules: $A \rightarrow aABa$, $A \rightarrow SS$, $B \rightarrow aab$, $B \rightarrow SA$. The derivation $aABAaa \Rightarrow aSSBAaa$ has the left-most substitution property, because we substitute the left-most variable which is $A$. On the other hand, $aABAaa \Rightarrow aAAa$ does not, because we substitute the variable $B$, which is not the left-most variable, and $aABAaa \Rightarrow aABaa$ does not either, because we substitute the second $A$, which is not left-most.

Remark 7.1 Note that without loss of generality, we only need to consider derivations that has left-most substitution property, by simply substituting the left-most variable first.

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Constructing a PDA from a CFG. Let \( G = (\Sigma, V, S) \) be a CFG. Consider the following PDA \( A = (\Sigma, \Gamma, Q, q_0, F, \delta) \) where each component is as follows.

- \( \Gamma = \Sigma \cup V \cup \{ \bot \} \).
- \( Q = \{ p, q, r \} \).
- \( p \) is the initial state.
- \( F = \{ r \} \).
- \( \delta \) consists of the following transition. (Here \( w^r \) denotes the reverse of \( w \).)

\[
\begin{align*}
(p, \epsilon, \text{pop}(\epsilon)) & \rightarrow (q, \text{push}(\bot S)) \\
(q, a, \text{pop}(a)) & \rightarrow (q, \text{push}(\epsilon)) \\
(q, \epsilon, \text{pop}(A)) & \rightarrow (q, \text{push}(w)) \\
(q, \epsilon, \text{pop}(\bot)) & \rightarrow (r, \text{push}(\epsilon))
\end{align*}
\]

Notice that in the first and third transitions, \( A \) pushes a string of symbols to its stack.

We have the following lemma.

**Lemma 7.2** For every \( u \in (\Sigma \cup V)^* \), for every \( w \in \Sigma^* \), if there is a derivation

\[
\begin{align*}
z_0 & \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_m
\end{align*}
\]

such that \( z_0 = u \) and \( z_m = w \), then there is a run:

\[
(q, v_0\bot) \vdash_{b_1} (q, v_1\bot) \vdash_{b_2} \cdots \vdash_{b_m} (q, v_n\bot),
\]

where \( v_0 = u \), \( v_n = \epsilon \) and \( b_1 \cdots b_n = w \). (Note that here some \( b_i \) may be \( \epsilon \).)

**Proof.** The proof is by induction on \( m \).

The base case is \( m = 0 \) which means that \( z_0 = z_m = \Sigma^* \), and hence, \( u = z_0 = z_m = w \). By the construction of our PDA \( A \), there is a transition \( (q, a, \text{pop}(a) \rightarrow (q, \text{push}(\epsilon))) \), for every \( a \in \Sigma \). Denoting \( w = a_1 \cdots a_k \), we have a run of length \( k = |w| \):

\[
(q, z_0\bot) \vdash_{a_1} (q, z_1\bot) \vdash_{a_2} \cdots \vdash_{a_k} (q, z_k\bot),
\]

where \( z_i = a_{i+1} \cdots a_k \), for each \( i \in \{1, \ldots, k\} \).

For the induction hypothesis, we assume that the lemma holds for \( m \). We will prove the \( m + 1 \) case for the induction step. Suppose we have derivation:

\[
z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_{m+1},
\]

where \( z_0 = u \) and \( z_{m+1} = w \). We may assume that it has the left-most substitution property, thus, there is a rule \( \Delta \rightarrow w \), variable \( \Delta \) is the left-most variable in \( z_0 \). So, we can denote \( z_0 \) by \( xAy \), where \( x \in \Sigma^* \), and hence, \( z_1 = xwy \). Let \( x = c_1 \cdots c_k \), where each \( c_i \in \Sigma \). Since \( x \) contains only terminals, there is a run:

\[
(q, c_1c_2 \cdots c_kAy\bot) \vdash_{c_1} (q, c_2 \cdots c_kAy\bot) \vdash_{c_{k-1}} (q, c_kAy\bot) \vdash_{c_k} (q, Ay\bot)
\]

(1)

By our construction of \( A \), there is a transition \( (q, \epsilon, \text{pop}(A)) \rightarrow (q, \text{push}(w)) \). So, we have:

\[
(q, Ay\bot) \vdash_{\epsilon} (q, wy\bot)
\]

(2)
Moreover, $x$ will not change during the derivation $z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_{m+1}$ (because $x$ contains only terminals). Thus, the string $z_i$ has prefix $x$, for every $i \in \{1,\ldots, m+1\}$. That is, $z_i = xz'_i$, for some $z'_i \in (\Sigma \cup V)^*$. So, the derivation is of the form:

$$ xAy \Rightarrow xz'_1 \Rightarrow \cdots \Rightarrow xz'_{m+1}. $$

Ignoring the first part of the derivation, we have:

$$ xz'_1 \Rightarrow \cdots \Rightarrow xz'_{m+1}, $$

which means we also have derivation (since $x$ contains only terminals):

$$ z'_1 \Rightarrow \cdots \Rightarrow z'_{m+1}, $$

which has length $m$. By the induction hypothesis, there is a run:

$$ (q, w_0\perp) \vdash d_1 \cdots d_l (q, w_l\perp), $$

where $w_0 = z'_1, w_l = \epsilon$ and $d_1 \cdots d_l = z'_{m+1}$. Now, recall that $z_1 = xz'_1 = xwy$, and hence, $w_0 = wy$. Thus, the desired run can be obtained by combining the runs 1, 2 and 3. This completes the proof of Lemma 7.2. □

**Lemma 7.3** For every $u \in (\Sigma \cup V)^*$, for every $w \in \Sigma^*$, if there is a run:

$$ (q, v_0\perp) \vdash b_1 \cdots b_n (q, v_n\perp), $$

where $v_0 = u, v_n = \epsilon$ and $b_1 \cdots b_n = w$, then there is a derivation:

$$ z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_m $$

such that $z_0 = u$ and $z_m = w$.

**Proof.** The proof is by induction on $n$.

The base case is $n = 0$, which means $v_0 = v_n = \epsilon$. This is trivial, since $\epsilon \Rightarrow^* \epsilon$ by definition.

For the induction hypothesis, we assume that the lemma holds for $n$. We will prove the $n + 1$ case for the induction step. Suppose there is a run:

$$ (q, v_0\perp) \vdash b_1 \cdots b_{n+1} (q, v_{n+1}\perp), $$

where $v_0 = u, v_{n+1} = \epsilon$ and $b_1 \cdots b_{n+1} = w$. There are two cases:

- $b_1 \neq \epsilon$, i.e., $b_1$ is a symbol from $\Sigma$.

  By the construction of the PDA $A$, it has to use the transition $(q, b_1, \text{pop}(b_1)) \rightarrow (q, \text{push}(\epsilon))$, which means that the top of the stack $v_0$ is $b_1$. Therefore, $v_0 = b_1v_1$, and there is a run of length $n$:

  $$ (q, v_1\perp) \vdash b_2 \cdots b_{n+1} (q, v_{n+1}\perp). $$

  By the induction hypothesis, we have the derivation:

  $$ z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_m $$

  such that $z_0 = v_1$ and $z_m = b_2 \cdots b_{n+1}$. Adding $b_1$ in front of each $z_i$, we have:

  $$ b_1z_0 \Rightarrow b_1z_1 \Rightarrow \cdots \Rightarrow b_1z_m $$

  Now, $b_1z_0 = b_1v_1 = v_0 = u$ and $b_1z_m = b_1b_2 \cdots b_n = w$. □
• \( b_1 = \epsilon \), hence, \( b_2 \cdots b_m = w \).

By the construction of the PDA \( \mathcal{A} \), it has to use the transition \((q, \epsilon, \text{pop}(A)) \rightarrow (q, \text{push}(w))\), which means that the top of the stack \( u_0 \) is \( A \) and there is a rule \( A \rightarrow x \). We denote \( v_0 = Ay \) and \( v_1 = xy \), for some \( y \).

Consider the run:

\[
(q, v_1 \bot) \vdash_{b_1} \cdots \vdash_{b_{n+1}} (q, v_{n+1} \bot).
\]

By the induction hypothesis, we have the derivation:

\[
v_1 \Rightarrow^* b_2 \cdots b_{n+1},
\]

Now, \( Ay \Rightarrow xy \). Thus, we have the derivation:

\[
v_0 \Rightarrow^* w,
\]

since by our notation, \( v_0 = Ay \), \( v_1 = xy \) and \( w = b_1 b_2 \cdots b_{n+1} = b_2 \cdots b_{n+1} \).

This completes our proof.

Using the two lemmas above, we can show that \( L(\mathcal{A}) = L(\mathcal{G}) \) as stated below.

**Theorem 7.4** \( L(\mathcal{G}) = L(\mathcal{A}) \).

**Proof.** We first prove \( L(\mathcal{G}) \subseteq L(\mathcal{A}) \). Let \( S \Rightarrow^* w \). By Lemma \( \ref{lem:run} \), there is a run: then there is a run:

\[
(q, v_0 \bot) \vdash_{b_1} (q, v_1 \bot) \vdash_{b_2} \cdots \vdash_{b_n} (q, v_n \bot),
\]

where \( v_0 = S \), \( v_n = \epsilon \) and \( b_1 \cdots b_n = w \). By the construction of \( \mathcal{A} \), we have the following accepting run on \( w \):

\[
(p, \epsilon) \vdash_{\epsilon} (q, v_0 \bot) \vdash_{b_1} (q, v_1 \bot) \vdash_{b_2} \cdots \vdash_{b_n} (q, v_n \bot) \vdash_{\epsilon} (r, \epsilon).
\]

Now we prove \( L(\mathcal{G}) \supseteq L(\mathcal{A}) \). Let \( w \in L(\mathcal{A}) \). So, there is an accepting run of \( \mathcal{A} \) on \( w \), which must start from the initial state \( p \) and end with the accepting state \( r \). Thus, it has to start by using the transition \((p, \epsilon, \text{pop}(\epsilon)) \rightarrow (q, \text{push}(\bot))\) and end by using the transition \((q, \epsilon, \text{pop}(\bot)) \rightarrow (r, \text{push}(\epsilon))\). This means the run is of the form:

\[
(p, \epsilon) \vdash_{\epsilon} (q, v_0 \bot) \vdash_{b_1} (q, v_1 \bot) \vdash_{b_2} \cdots \vdash_{b_n} (q, v_n \bot) \vdash_{\epsilon} (r, \epsilon),
\]

where \( v_0 = S \), \( v_n = \epsilon \) and \( w = b_1 \cdots b_n \). By Lemma \( \ref{lem:run} \), \( S \Rightarrow^* w \). This completes the proof of our theorem.

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2 From PDA to CFG

Let \( \mathcal{A} = (\Sigma, \Gamma, Q, q_0, F, \delta) \) be a PDA. Without loss of generality, we can assume the following.

- It has only one final state, say \( q_f \). That is, \( F = \{ q_f \} \).
- The stack is empty before accepting an input word. More precisely, on every word \( w \), if \( \mathcal{A} \) accepts \( w \), there is an accepting run of \( \mathcal{A} \) on \( w \) from the initial configuration \((q_0, \epsilon)\) to a final configuration \((q_f, \epsilon)\) where the content of the stack is empty.
• In each transition, $A$ either pushes a symbol into the stack or pops one from the stack, but it cannot do both. More precisely, every transition can only be of the forms:

\[
(p, x, \text{pop}(y)) \rightarrow (q, \text{push}(\epsilon)) \\
(p, x, \text{pop}(\epsilon)) \rightarrow (q, \text{push}(z))
\]

Consider the following CFG $G = (\Sigma, V, R, S)$ where each component is as follows.

• $V = \{A_{p,q} \mid p, q \in Q\}$.
• $A_{q_0, q_f}$ is the start variable.
• $R$ consists of the following rules:
  
  - For every state $p, q, r, s \in Q$ and every symbol $z \in \Gamma$ and every symbol $a, b \in \Sigma \cup \{\epsilon\}$, if the following transitions are in $\delta$:
    
    \[
    (p, a, \text{pop}(\epsilon)) \rightarrow (r, \text{push}(z)) \\
    (s, b, \text{pop}(z)) \rightarrow (q, \text{push}(\epsilon))
    \]

    then the following rule is in $R$:
    
    \[
    A_{p,q} \rightarrow a A_{r,s} b
    \]

  - For every state $p, q, r \in Q$ and every symbol $a \in \Sigma \cup \{\epsilon\}$, if the following transition is in $\delta$:
    
    \[
    (p, a, \text{pop}(\epsilon)) \rightarrow (r, \text{push}(\epsilon))
    \]

    then the following rule is in $R$:
    
    \[
    A_{p,q} \rightarrow a A_{r,q}
    \]

  - For every state $p, q, r \in Q$, we have the following rule in $R$:
    
    \[
    A_{p,q} \rightarrow A_{p,r} A_{r,q}
    \]

  - For every $p \in Q$, we have the following rule in $R$:
    
    \[
    A_{p,p} \rightarrow \epsilon
    \]

We can show that $L(A) = L(G)$ by proving the following lemma.

**Lemma 7.5** If there is a derivation:

\[z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_m\]

such that $z_0 = A_{p,q}$ and $z_m = w \in \Sigma^*$, then for every $v \in \Gamma^*$, there is a run:

\[(p_0, v_0 \perp) \vdash b_1 (q, v_1 \perp) \vdash b_2 \cdots \vdash b_n (p_n, v_n \perp)\]

such that $p_0 = p$, $p_n = q$, $v_0 = v_n = v$, $w = b_1 \cdots b_n$, and the following condition (C) holds:

(C) For every $i \in \{1, \ldots, n\}$, $v$ is a prefix of $v_i$, i.e., the $v$ part is left “untouched” during the run.
Proof. The proof is by induction on m. The base case is m = 1, in which case z_m = ε. Thus, z_0 = A_p,p, and the lemma holds trivially.

For the induction hypothesis, we assume that the lemma holds for m. We will prove the case m + 1. Suppose there is a derivation:

\[ z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_{m+1} \]

such that \( z_0 = A_{p,q} \) and \( z_{m+1} = w \in \Sigma^* \). Let \( v \in \Gamma^* \).

There are three cases:

- \( z_1 = aA_{r,s}b \), for some \( a, b \in \Sigma \cup \{\varepsilon\} \).

This means that every \( z_i \) starts with \( a \) and ends with \( b \), for every \( i \in \{1, \ldots, m + 1\} \). We denote each \( z_i = a z'_i b \), for some \( z'_i \). Thus, there is a run \( A_{r,s} \Rightarrow z'_2 \Rightarrow \cdots \Rightarrow z'_{m+1} \). By the induction hypothesis, for every \( v \in \Gamma^* \), there is a run:

\[
(p_0, v_0 \bot) \vdash b_1 (q, v_1 \bot) \vdash b_2 \cdots \vdash b_n (p_n, v_n \bot)
\]

such that \( p_0 = r, p_n = s, v_0 = v_n = v, z'_{m+1} = b_1 \cdots b_n \), and for every \( i \in \{1, \ldots, n\} \), \( v \) is a prefix of \( v_i \).

Now, since \( z_1 = aA_{r,s}b \), there are the following transitions in \( \delta \), for some \( z \):

\[
\begin{align*}
(p, a, \text{pop}(\varepsilon)) & \rightarrow (r, \text{push}(z)) \\
(s, b, \text{pop}(\varepsilon)) & \rightarrow (q, \text{push}(\varepsilon))
\end{align*}
\]

This means for every \( v \in \Gamma^* \), we have:

\[
(p, v \bot) \vdash_a (r, zv \bot) \quad \text{and} \quad (s, zv \bot) \vdash_b (q, v \bot)
\]

Since the run holds for every \( v \in \Gamma^* \), and in particular for every \( zv \in \Gamma^* \). Thus, for every \( v \in \Gamma^* \), there is a run:

\[
(p, v \bot) \vdash_a (p_0, v_0 \bot) \vdash b_1 (q, v_1 \bot) \vdash b_2 \cdots \vdash b_n (p_n, v_n \bot) \vdash_b (q, v \bot)
\]

where \( v_0 = v_n = zv \) and each \( v_i \) has prefix \( zv \), and hence prefix \( v \), so condition (C) holds.

Now, \( z_{m+1} = a z'_{m+1} b = w \). Therefore, the run is as desired.

- \( z_1 = A_{p,r}A_{r,q} \), for some \( r \in Q \).

This means we can partition \( w = xy \) such that \( A_{p,r} \Rightarrow^* x \) and \( A_{r,q} \Rightarrow^* y \). Both have derivation length \( \leq m \). By induction hypothesis, for every \( v \in \Gamma^* \),

- there is a run:

\[
(p_0, v_0 \bot) \vdash b_1 (q, v_1 \bot) \vdash b_2 \cdots \vdash b_n (p_n, v_n \bot)
\]

such that \( p_0 = p, p_n = r, v_0 = v_n = v, x = b_1 \cdots b_n \), and for every \( i \in \{1, \ldots, n\} \), \( v \) is a prefix of \( v_i \);

- there is a run:

\[
(q_0, u_0 \bot) \vdash c_1 (q_1, u_1 \bot) \vdash c_2 \cdots \vdash c_k (q_k, u_k \bot)
\]

such that \( q_0 = r, q_k = q, u_0 = u_k = v, y = c_1 \cdots c_k \), and for every \( i \in \{1, \ldots, k\} \), \( v \) is a prefix of \( u_i \);
Combining the run 5 and 6, we have the desired run.

- \( z_1 = aA_{r,q} \) for some \( r \in Q \) and \( a \in \Sigma \cup \{ \epsilon \} \).

This means that every \( z_i \) starts with \( a \), for every \( i \in \{1, \ldots, m+1\} \). We denote each \( z_i = az'_i \), for some \( r \in Q \) and \( a \in \Sigma \cup \{ \epsilon \} \).

This means for every \( v \in \Gamma^* \), we have:

\[
(p_0, v_0 \perp) \vdash b_1 (q, v_1 \perp) \vdash b_2 \cdots \vdash b_n (p_n, v_n \perp) \tag{7}
\]

such that \( p_0 = r, p_n = q, v_0 = v_n = v, z'_{m+1} = b_1 \cdots b_n \), and for every \( i \in \{1, \ldots, n\} \), \( v \) is a prefix of \( v_i \).

Now, since \( z_1 = aA_{r,q} \), there is the following transitions in \( \delta \):

\[
(p, a, \text{pop}(\epsilon)) \rightarrow (r, \text{push}(\epsilon))
\]

This means for every \( v \in \Gamma^* \), we have:

\[
(p, v \perp) \vdash_a (r, v \perp),
\]

which can be added into the front of the run 7. Thus, for every \( v \in \Gamma^* \), there is a run:

\[
(p, v \perp) \vdash_a (p_0, v_0 \perp) \vdash b_1 (q, v_1 \perp) \vdash b_2 \cdots \vdash b_n (p_n, v_n \perp) \vdash_b (q, v \perp)
\]

where \( v_0 = v_n = v \) and each \( v_i \) has prefix \( v \). Now, \( z_{m+1} = az'_{m+1} = w \). Therefore, the run is as desired.

The following lemma is the converse direction of the previous lemma.

**Lemma 7.6** If for every \( v \in \Gamma^* \), there is a run:

\[
(p_0, v_0 \perp) \vdash b_1 (q, v_1 \perp) \vdash b_2 \cdots \vdash b_n (p_n, v_n \perp)
\]

such that \( p_0 = p, p_n = q, v_0 = v_n = v, w = b_1 \cdots b_n \), and the following condition (C) holds:

(C) for every \( i \in \{1, \ldots, n\} \), \( v \) is a prefix of \( v_i \), i.e., the \( v \) part is left “untouched” during the run,

then there is a derivation:

\[
 z_0 \Rightarrow z_1 \Rightarrow \cdots \Rightarrow z_m
\]

such that \( z_0 = A_{p,q} \) and \( z_m = w \in \Sigma^* \).

**Proof.** The proof is by induction on \( n \). The base case \( n = 0 \) is trivial.

For the induction hypothesis, we assume that the lemma holds for \( n \). We will now prove it for \( n + 1 \).

Suppose for every \( v \in \Gamma^* \), there is a run:

\[
(p_0, v_0 \perp) \vdash b_1 (q, v_1 \perp) \vdash b_2 \cdots \vdash b_{n+1} (p_{n+1}, v_{n+1} \perp)
\]

such that \( p_0 = p, p_{n+1} = q, v_0 = v_n = v, w = b_1 \cdots b_{n+1} \), and the condition (C) holds.

There are three cases:
• \(v_1 = v\).

This means that on \((p_0, v_0, \bot) \vdash b_1 (q, v_1, \bot)\), the PDA pops and pushes nothing, hence, there is a transition \((p_0, b_1, \text{pop}(\epsilon)) \rightarrow (p_1, \text{push}(\epsilon))\). By the construction of \(G\), there is a rule \(A_{p_0, p_{n+1}} \rightarrow b_1 A_{p_1, p_{n+1}}\).

By induction hypothesis on the runs from \((p_1, v_1, \bot)\) to \((p_{n+1}, v_{n+1}, \bot)\), which has length \(n\) now, there is a derivation:

\[
A_{p_1, p_{n+1}} \Rightarrow \cdots \Rightarrow b_2 \cdots b_{n+1} \tag{8}
\]

Therefore, there is a derivation \(A_{p_0, p_{n+1}} \Rightarrow b_1 A_{p_1, p_{n+1}} \Rightarrow \cdots \Rightarrow b_1 \cdots b_{n+1} \).

• For some \(j \in \{2, \ldots, n\}\), \(v_j = v\). Let \(j\) be such that \(v_j = v\).

By induction hypothesis on the runs from \((p_0, v, \bot)\) to \((p_j, v, \bot)\) and from \((p_j, v, \bot)\) to \((p_{n+1}, \bot)\) (both have length \(\leq n\) now), there are derivations:

\[
A_{p_0, p_j} \Rightarrow \cdots \Rightarrow b_j \tag{9}
\]

\[
A_{p_j, p_{n+1}} \Rightarrow \cdots \Rightarrow b_{j+1} \cdots b_{n+1} \tag{10}
\]

By construction of \(G\), there is a rule \(A_{p, q} \rightarrow A_{p, p_j} A_{p_j, q}\). Therefore, \(A_{p, q} \Rightarrow b_1 \cdots b_{n+1} \).

• For every \(i \in \{1, \ldots, n\}\), \(v_i \neq v\).

Since \(v_0 = v_{n+1} = v\), \(v_i\) is obtained by pushing some \(z\) to \(v_0\) and in which case \(z\) is popped from \(v_n\) to obtain \(v_{n+1}\). That is, there is \(z\) such that \(v_1 = v_n = z v\). This means that there are the following transitions: there are the following transitions in \(\delta\), for some \(z\):

\[
(p_0, a, \text{pop}(\epsilon)) \rightarrow (p_1, \text{push}(z))
\]

\[
(p_n, b, \text{pop}(z)) \rightarrow (p_{n+1}, \text{push}(\epsilon))
\]

where \(b_1 = a\) and \(b_{n+1} = b\).

By definition, there is a rule \(A_{p, q} \rightarrow a A_{p_1, p_n} b\). By the induction hypothesis, there is a derivation: \(A_{p_1, p_n} \Rightarrow \cdots \Rightarrow b_n\). Thus, there is a derivation:

\[
A_{p, q} \Rightarrow a A_{p_1, p_n} b \Rightarrow \cdots \Rightarrow a b_2 \cdots b_n b = w.
\]

This completes the proof of our lemma.

\[\square\]

**Theorem 7.7** \(L(A) = L(G)\).

**Proof.** We first prove \(L(G) \supseteq L(A)\). Let \(w \in L(A)\). So, there is an accepting run of \(A\) on \(w\), which must start from the initial state \(q_0\) and end with the accepting state \(q_f\):

\[
(p_0, \epsilon) \vdash b_1 (q, v_1, \bot) \cdots b_n (p_n, \epsilon)
\]

such that \(p_0 = q_0, p_n = q_f\) and \(b_1 \cdots b_n = w\). Now, since the PDA never goes below empty stack, we can add any \(v \in \Gamma^*\) in the bottom of the stack. So, for every \(v \in \Gamma^*\), there is a run:

\[
(p_0, v_0, \bot) \vdash b_1 (q, v_1, \bot) \cdots b_n (p_n, v_{n+1})
\]

such that \(v_0 = v_n = v\) and condition (C) holds. By Lemma 7.6, \(A_{p_0, p_n} = A_{q_0, q_f} \Rightarrow \cdots \Rightarrow b_1 \cdots b_n\).

Now we prove that \(L(G) \subseteq L(A)\). Let \(A_{q_0, q_f} \Rightarrow \cdots \Rightarrow w\). By Lemma 7.5, for every \(v \in \Gamma^*\), there is a run:

\[
(p_0, v_0) \vdash b_1 (p_1, v_1) \cdots b_n (p_n, v_n)
\]

such that \(p_0 = q_0, p_n = q_f, v_0 = v_n = v, b_1 \cdots b_n = w\), and condition (C) holds. In particular, it holds also for \(v = \epsilon\), which is an accepting run. Thus, \(w \in L(A)\). This completes the proof of our theorem.

\[\square\]