Lesson 1: Preliminaries

Theme: Review of some basic facts from discrete mathematics.

1 Sets

- A set is a collection of things, which are called its members or elements.

  \[ a \in X \] (read: \( a \) is in \( X \), or \( a \) belongs to \( X \)) means \( a \) is a member or an element of \( X \), whereas \( a \notin X \) means \( a \) is not a member of \( X \).

- An empty set is denoted by \( \emptyset \).

- \( X \) is a subset of \( Y \), denoted by \( X \subseteq Y \), if every element of \( X \) is also an element of \( Y \).

  \( X \) is a proper subset of \( Y \), denoted by \( X \subset Y \), if \( X \neq Y \) and \( X \subseteq Y \).

- For two sets \( X \) and \( Y \), we write \( X \cap Y \) and \( X \cup Y \) to denote their intersection and union, respectively.

- The cartesian product between two sets \( X \) and \( Y \) is the following.

  \[ X \times Y := \{(a, b) \mid a \in X \text{ and } b \in Y\} \]

  We write \( X^n \) to denote \( X \times \cdots \times X \), where \( X \) appears \( n \) time.

2 Relations

- A relation \( R \) over two sets \( X, Y \) is a subset of \( X \times Y \).

- A binary relation \( R \) over \( X \) is a subset of \( X \times X \).

- An \( n \)-ary relation \( R \) over \( X \) is a subset of \( X^n \).

3 Functions

- A relation \( R \) over \( X, Y \) is a function or a mapping, if for every \( x \in X \), there is exactly one \( y \in Y \) such that \((x, y) \in R \).

  In this case, we will say \( R \) is a function from \( X \) to \( Y \), or \( R \) maps \( X \) to \( Y \). We denote it by \( R : X \to Y \).

- We will usually use the letters \( f, g, h, \ldots \) to represent functions. As usual, we write \( f(x) \) to denote the element \( y \) in which \((x, y) \in f \).

- A function \( f : X \to Y \) is an injective function, if for every \( y \in Y \), there is at most one \( x \in X \) such that \( f(x) = y \). An injective functions is also called an injection.

- A function \( f : X \to Y \) is a surjective function, if for every \( y \in Y \), there is at least one \( x \in X \) such that \( f(x) = y \).

- A function \( f : X \to Y \) is a bijection, if it is both injective and surjective.
4 Equivalence relations

The symbol $\sim$ is reserved to denote a special relation, called equivalence relation. Using the standard notation, we write $x \sim y$ to mean that the pair $(x, y)$ belongs to the relation $\sim$.

Recall that $\sim$ being an equivalence relation (over some set, say, $X$) means it satisfies the following conditions.

- Reflexive: $x \sim x$, for every $x \in X$.
- Symmetric: $x \sim y$ if and only if $y \sim x$, for every $x, y \in X$.
- Transitive: for every $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

For $x \in X$, the equivalence class of $x$ in $\sim$ is defined as:

$$[x]_{\sim} := \{y \mid x \sim y\}$$

**Theorem 1.1** If $\sim$ is an equivalence relation over $X$, then the following holds.

- $[x]_{\sim} = [y]_{\sim}$ if and only if $x \sim y$.
- If $[x]_{\sim} \neq [y]_{\sim}$, then $[x]_{\sim} \cap [y]_{\sim} = \emptyset$.
- The equivalence classes of $\sim$ partition $X$, i.e., every member of $X$ belongs to exactly one equivalence class of $\sim$.

5 Countable and uncountable sets

Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, \ldots\}$. A set $X$ is countable, if there is an injective function from $X$ to $\mathbb{N}$. Otherwise, it is called an uncountable set.

**Theorem 1.2** The following sets are all countable.

- The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers.
- The set $\mathbb{N}^k$, for every integer $k \geq 1$.
- The set $\mathbb{N}^* := \bigcup_{k\geq1} \mathbb{N}^k$.

**Theorem 1.3** The set $2^{\mathbb{N}}$ is uncountable.

**Theorem 1.4** For every alphabet $\Sigma$, the set $\Sigma^*$ is countable.
Appendix

In this course it is important to be able to read mathematical/formal statements. It will take a while to get used to them. One important aspect of a formal statement is its use of “quantifiers.”

Consider the following statement.

Every student stays in a dormitory room. \hspace{1cm} (1)

If we want to write in strict logical form, we will have to write it in the following way.

For every student \( x \), there is a dormitory room \( y \) such that \( x \) stays in \( y \).

“For every” and “there exists” in the above sentence are called quantifiers.

The negation of statement (1) is:

There is student \( x \), for every dormitory room \( y \) such that \( x \) does not stay in \( y \). \hspace{1cm} (2)

Note also that neither (1) nor (2) are equivalent to the following sentence:

There is student \( x \), for every dormitory room \( y \) such that \( x \) stays in \( y \). \hspace{1cm} (3)