Question 1 (2 points). Consider the following Turing machine $M_1 = (\Sigma, \Gamma, q_0, q_{acc}, q_{rej}, \delta)$.

- $\Sigma = \{0, 1\}$.
- $\Gamma = \{\#, 0, 1, \sqcup\}$.
- $Q = \{q_0, p_0, p_1, s, t, r_0, r_1, q', q_{acc}, q_{rej}\}$.
- $q_0, q_{acc}, q_{rej}$ are the initial, accepting and rejecting states, respectively.
- $\delta$ is defined as follows.

$$
\begin{align*}
(q_0, \sqcup) \rightarrow (q_{rej}, \sqcup, \text{Stay}) & \quad (p_0, \sqcup) \rightarrow (q_{rej}, 0, \text{Stay}) & (p_1, \sqcup) \rightarrow (s, 1, \text{Stay}) \\
(q_0, 0) \rightarrow (p_0, <, \text{Right}) & \quad (p_0, 0) \rightarrow (p_0, 0, \text{Right}) & (p_1, 0) \rightarrow (p_0, 1, \text{Right}) \\
(q_0, 1) \rightarrow (p_1, <, \text{Right}) & \quad (p_0, 1) \rightarrow (p_1, 0, \text{Right}) & (p_1, 1) \rightarrow (p_1, 1, \text{Right}) \\
(q_0, <) \rightarrow (q_{rej}, <, \text{Stay}) & \quad (p_0, <) \rightarrow (q_{rej}, <, \text{Stay}) & (p_1, <) \rightarrow (q_{rej}, <, \text{Stay})
\end{align*}
$$

\begin{align*}
(s, \sqcup) \rightarrow (q_{rej}, \sqcup, \text{Stay}) & \quad (t, \sqcup) \rightarrow (q_{rej}, \sqcup, \text{Stay}) & (q', \sqcup) \rightarrow (q', 0, \text{Left}) \\
(s, 0) \rightarrow (t, 1, \text{Left}) & \quad (t, 0) \rightarrow (t, 0, \text{Left}) & (q', 0) \rightarrow (r_0, 0, \text{Left}) \\
(s, 1) \rightarrow (s, 0, \text{Left}) & \quad (t, 1) \rightarrow (t, 1, \text{Left}) & (q', 1) \rightarrow (r_1, 0, \text{Left}) \\
(s, <) \rightarrow (r_1, <, \text{Right}) & \quad (t, <) \rightarrow (q_{acc}, <, \text{Stay}) & (q', <) \rightarrow (q_{acc}, <, \text{Right})
\end{align*}

Determine the run of $M$ on each of the following input words: $\epsilon$, $011$, $100$, $111$.

Question 2 (2 points). In the following, for a Turing machine $M$, we denote by $L(M)$ the language that consists of all words accepted by $M$. That is, $L(M) = \{w \mid M \text{ accepts } w\}$.

Consider the following Turing machine $A$ that works as follows.

INPUT: $|M|\$w$.
- Construct a TM $K_{M,w}$ that works as follows.
  - INPUT: $u \in \Sigma^*$.
    - Simulate $M$ on $w$.
    - If $M$ accepts $w$, ACCEPT.
    - If $M$ rejects $w$:
      * If $u \in \{0^n1^n \mid n \geq 0\}$, ACCEPT.
      * Else, REJECT.
  - Output $|K_{M,w}|$. 
Answer each of the following questions

(a) If $M$ accepts $w$, what is the language $L(K_{M,w})$?

(b) If $M$ rejects $w$, what is the language $L(K_{M,w})$?

(c) If $M$ does not halt on $w$, what is the language $L(K_{M,w})$?

(d) Recall the language $L_5 := \{[M] \mid L(M)$ is a regular language$\}$ in the Note 11.

Is the following true?

$$[M] \# w \in \text{HALT} \text{ if and only if } [K_{M,w}] \in L_5$$

Justify your answer.

**Question 3 (1 points).** Consider the following language:

$$L_\infty := \{[M] \mid L(M) \text{ is infinite}\}.$$ 

That is, $L_\infty$ consists of all descriptions of Turing machines that accepts infinitely many words. Prove that $L_\infty$ is undecidable.

**Question 4 (1 points).** Prove that the following problem is undecidable.

**Input:** A CFG $G$ and a DFA $A$.

**Task:** Decide whether $L(G) = L(A)$. That is, return True, if $L(G) = L(A)$. Otherwise, return False.

**Question 5 (2 points).** Prove that the class $\text{coNP}$ is closed under union and intersection. That is,

- if $L_1, L_2 \in \text{coNP}$, then $L_1 \cup L_2 \in \text{coNP}$,
- if $L_1, L_2 \in \text{coNP}$, then $L_1 \cap L_2 \in \text{coNP}$.

**Hint:** Use the definition of $\text{coNP}$.

**Question 6 (2 points).** Prove that if $\text{SAT} \in \text{coNP}$, then $\text{NP} \subseteq \text{coNP}$, and hence, $\text{NP} = \text{coNP}$.

**Hint:** Use the fact that $\text{SAT}$ is $\text{NP}$-hard. For this question, any solution that contains the term “$\text{coNP}$ algorithm” will be penalized immediately.