Lesson 8: Logical consequences and theories

Theme: Logical consequences and first-order theories.

1 Logical consequences

Definition 8.1 Let $X$ be a set of formulas. We write $(\mathcal{A}, \text{val}) \models X$, if $(\mathcal{A}, \text{val}) \models \varphi$, for every $\varphi \in X$.

Definition 8.2 A formula $\beta$ is a logical consequence of a formula $\alpha$, denoted by $\alpha \models \beta$, if every model of $\alpha$ is also a model of $\beta$. If $\alpha \models \beta$ and $\beta \models \alpha$, we write $\alpha \equiv \beta$, or $\alpha \equiv \beta$.

One example is $\forall x \varphi \models \exists x \varphi$. (Recall that the domain of a structure is never empty.)

Definition 8.3 We say that $\alpha$ is a logical consequence of a set $X$ of formulas, denoted by $X \models \alpha$, if every model of $X$ is also a model of $\alpha$. More formally, $X \models \alpha$ means that for every model $(\mathcal{A}, \text{val})$, if $(\mathcal{A}, \text{val}) \models X$, then $(\mathcal{A}, \text{val}) \models \alpha$.

We write $X \not\models \alpha$, if it is not the case that $X \models \alpha$.

Definition 8.4 A sentence $\varphi$ is valid, if $\models \varphi$. In other words, $\varphi$ is valid, if $\mathcal{A} \models \varphi$, for every structure $\mathcal{A}$.

Some conventions to read the notations:

- $(\mathcal{A}, \text{val}) \models X$ is read as “$(\mathcal{A}, \text{val})$ is a model of $X$.”
- $\alpha \models \beta$ is also read as “$\alpha$ implies $\beta$."
- $\alpha \equiv \beta$ is also read as “$\alpha$ and $\beta$ are equivalent.”

Theorem 8.5 $X \models \varphi$ if and only if $X \cup \{\neg \varphi\}$ is not satisfiable.

Proposition 8.6 For every formulas $\alpha$ and $\beta$, the following holds.

$$
\neg \forall x \alpha \equiv \exists x \neg \alpha \\
\neg \exists x \alpha \equiv \forall x \neg \alpha \\
\alpha \land \forall x \beta \equiv \forall x(\alpha \land \beta) \quad \text{when } x \text{ is not free in } \alpha \\
\alpha \land \exists x \beta \equiv \exists x(\alpha \land \beta) \quad \text{when } x \text{ is not free in } \alpha
$$

Definition 8.7 Every formula is in Prenex Normal Form (PNF), if is is of the form:

$$Q_1x_1 \cdot \cdot \cdot Q_nx_n \varphi,$$

where $\varphi$ is quantifier-free, and each $Q_i \in \{\forall, \exists\}$.

Theorem 8.8 Every formula is equivalent to another formula in PNF.

*Recall that $\models \varphi$ is the abbreviation for $\emptyset \models \varphi$.}
2 First-order theories

Definition 8.9

• A set $T$ of sentences is called a *theory*, if it is closed under logical consequences, i.e., for every sentence $\varphi$, if $T \models \varphi$, then $\varphi \in T$.

• A theory $T$ is *complete*, if for every sentence $\varphi$, either $\varphi \in T$ or $\neg \varphi \in T$.

Definition 8.10

• For a set $X$ of sentences, $\text{Model}(X) := \{A \mid A \models X\}$.

• For a set $X$ of sentences, $\text{Cn}(X) := \{\varphi \mid X \models \varphi\}$.

• For a set $K$ of structures, $\text{Th}(K) := \{\varphi \mid \varphi \text{ holds in every structure in } K\}$.

Theorem 8.11 For a set $K$ of structures, and a set $X$ of sentences, the following holds.

• $K \subseteq \text{Model}(\text{Th}(K))$.

• $\text{Th}(K)$ is a theory.

• $\text{Cn}(X) = \text{Th}(\text{Model}(X))$.

Definition 8.12 A theory $T$ is *finitely axiomatizable*, if there is a finite set $\Sigma$ such that $T = \text{Cn}(\Sigma)$.

Remark 8.13 If $\text{Cn}(T)$ is finitely axiomatizable, then there is a finite subset $T_0 \subseteq T$ such that $\text{Cn}(T_0) = \text{Cn}(T)$. 
Exercises

(1) Show that $\exists x \forall y \varphi \not\models \forall x \exists y \varphi$.

That is, give a model $\mathcal{A}$ and a formula $\varphi$ such that $\mathcal{A} \models \exists x \forall y \varphi$, but $\mathcal{A} \not\models \forall x \exists y \varphi$.

(2) Give a set $\mathcal{K}$ of sentences such that $\mathcal{K} \not\models \text{Model}(\text{Th}(\mathcal{K}))$.

(3) Let $\mathcal{K} = \{A\}$, i.e., it consists of only one structure $\mathcal{A}$. Prove that $\text{Th}(\mathcal{K})$ is complete.

(4) Give a set $\mathcal{K}$ of structures such that $\text{Th}(\mathcal{K})$ is not complete.

(5) Let $T$ be a complete theory and let $\mathcal{A} \models T$. Prove that for every sentence $\alpha$, $\mathcal{A} \models \alpha$ if and only if $T \models \alpha$.

We denote by $\mathcal{A} \cong \mathcal{B}$, if $\mathcal{A}$ is isomorphic to $\mathcal{B}$, i.e., there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, written as $\mathcal{A} \equiv \mathcal{B}$, if for every sentence $\varphi$,

$\mathcal{A} \models \varphi$ if and only if $\mathcal{B} \models \varphi$.

(6) Prove that if $\mathcal{A} \equiv \mathcal{B}$, then $\mathcal{A} \equiv \mathcal{B}$.

(7) Let $\mathcal{K}$ be a set of structures such that for every $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, we have $\mathcal{A} \cong \mathcal{B}$. Prove that $\text{Th}(\mathcal{K})$ is complete.

Appendix

The converse of question (6) does not hold in general. That is, $\mathcal{A} \equiv \mathcal{B}$ does not necessarily imply $\mathcal{A} \cong \mathcal{B}$. Consider, for example, the following two structures.

- $\mathcal{R} = (\mathbb{R}, <^\mathbb{R})$, where $<^\mathbb{R}$ is the standard ordering in $\mathbb{R}$.
- $\mathcal{Q} = (\mathbb{Q}, <^\mathbb{Q})$, where $<^\mathbb{Q}$ is the standard ordering in $\mathbb{Q}$.

It is known that $\mathcal{R} \equiv \mathcal{Q}$, but $\mathcal{R}$ is not isomorphic to $\mathcal{Q}$, since $\mathbb{R}$ is uncountable, but $\mathbb{Q}$ is countable.

In general it is not a trivial matter to determine whether two structures are elementarily equivalent. It usually involves a technique called Ehrenfeucht-Fraïssé game, which we will not cover in this course.