Homework 4: due 10:30 am, 14 June 2018

In the following in writing FO formula you can use any logical operators: \( \land, \lor, \neg, \to, \leftrightarrow \), etc, as well as both quantifiers: \( \forall \) and \( \exists \).

**Question 1.** Prove that for every set \( X \) of sentences, \( X \) is satisfiable if and only if \( X \) is consistent.

**Question 2.** Prove the compactness theorem for first-order logic. That is, prove that a set \( X \) is satisfiable if and only if \( X \) is finitely satisfiable.

**Question 3.** For a set \( X \) of sentences, we say that \( X \) has arbitrarily large models, if for every positive integer \( n \), \( X \) has a finite model of cardinality \( \geq n \). Prove that if \( X \) has arbitrarily large models, then \( X \) has an infinite model.

**Question 4.** Let \( \Sigma \) be a set of sentences. Prove that if \( C_n(\Sigma) \) is finitely axiomatizable, then there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( C_n(\Sigma) = C_n(\Sigma_0) \).

**Question 5.** Consider a vocabulary \( L = \{E, s, t\} \), where \( E \) is a relation symbol of arity 2, and \( s, t \) are constant symbols. Every structure over \( L \) can be viewed as a graph \( \mathcal{A} = (A, E^\mathcal{A}, s^\mathcal{A}, t^\mathcal{A}) \) where \( A \) is the set of vertices, \( E^\mathcal{A} \) the set of edges, and \( s^\mathcal{A}, t^\mathcal{A} \) are two special nodes in \( \mathcal{A} \).

For integer \( k \geq 0 \), a path from \( s^\mathcal{A} \) to \( t^\mathcal{A} \) of length \( k \) in the graph \( \mathcal{A} \) is a sequence of vertices \( v_0, v_1, \ldots, v_k \) such that \( v_0 = s^\mathcal{A}, v_k = t^\mathcal{A} \) and for each \( i = 1, \ldots, k \), \((v_{i-1}, v_i) \in E^\mathcal{A} \). We say that a path is of finite length, if it is of length \( k \) for some integer \( k \geq 0 \).

- Show that for every integer \( k \geq 0 \), there is \( \phi_k \in \text{FO}[L] \) such that for every graph \( \mathcal{A} \):

  \[ \mathcal{A} \models \phi_k \text{ if and only if there is a path of length } \leq k \text{ from } s^\mathcal{A} \text{ to } t^\mathcal{A} \text{ in the graph } \mathcal{A}. \]

- Prove that there is no \( \Psi \in \text{FO}[L] \) such that for every graph \( \mathcal{A} \):

  \[ \mathcal{A} \models \Psi \text{ if and only if there is a path of finite length from } s^\mathcal{A} \text{ to } t^\mathcal{A} \text{ in the graph } \mathcal{A}. \]

In other words, there is no FO sentence that expresses graph reachability.

Hint: Use compactness theorem for FO.