Homework 3: due 10:30 am, 3 May 2018

Question 1. Construct a structure $A$ and a formula $\varphi$ such that $A \models \forall x \exists y \varphi$, but $A \not\models \exists x \forall y \varphi$.

Question 2. There are many ways to “encode” propositional calculus in first-order logic. One way is to view a propositional variable $p$ as a relation symbol of arity 0 as explained in the class.

Here is another way. Let $L = \{c\}$ be a vocabulary that consists of only one constant symbol $c$. Given a propositional formula $\alpha$ using variables $p_1, \ldots, p_k$, consider the FO formula $\Phi$ obtained from $\alpha$ by changing every $p_i$ with an atomic $x_i \approx c$. For example, if $\alpha$ is

$$p_1 \land \neg p_2,$$

then $\Phi$ is

$$(x_1 \approx c) \land \neg (x_2 \approx c).$$

Prove that $\alpha$ is satisfiable (in the sense of propositional calculus) if and only if $\Phi$ is.

Question 3. Consider a sentence of the following form:

$$\Phi := \exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \varphi,$$

where $\varphi$ is quantifier free and does not contain any function and constant symbol, and $n, m \geq 1$.

Prove that if $\Phi$ is satisfiable, then there is a structure $A$ that satisfies $\Phi$ with $|A| \leq n$.

Question 4. Let $h : A \to B$ be a homomorphism. Define a relation $\sim_h$ on $A$ as follows: $a \sim_h a'$ if and only if $h(a) = h(a')$. Prove that $\sim_h$ is an equivalence relation on $A$, and that $\sim_h$ is, in fact, a congruence in $A$.

Question 5. Let $h : A \to B$ be a strong and surjective homomorphism, and let $A/\sim_h$ be the factor of $A$ modulo $\sim_h$. Define a function $\xi : A/\sim_h \to B$, where $\xi([a]_{\sim_h}) = h(a)$. Prove that $\xi$ is an isomorphism.