Lesson 12: Linear lower bound for the disjoint function

Theme: A linear lower bound for the disjoint function

We will prove the following lemma.

**Lemma 12.1** There exist:
- $A \subseteq \text{DISJ}^{-1}(1)$ and $B \subseteq \text{DISJ}^{-1}(0)$,
- a probability distribution $\mu$ on $X \times Y$, i.e., $X = Y = 2^{[n]}$,
- positive constants $\alpha$ and $\delta$,

such that the following holds.
- $\mu(A) = 3/4$.
- For every rectangle $R$, $\mu(R \cap B) \geq \alpha \cdot \mu(R \cap A) - 2^{-\delta n}$.

Furthermore, there is $0 \leq \epsilon < 1/2$ such that $D^\epsilon_n(\text{DISJ}) \geq \delta n - O(1)$.

**Theorem 12.2** $R^\epsilon_n(\text{DISJ}) = \Omega(n)$, for some $0 \leq \epsilon < 1/2$.

1 Proof of Lemma [12.1]

In the following we fix $n = 4\ell - 1$, for some integer $\ell \geq 0$. As usual, $[n] = \{1, \ldots, n\}$. A partition $T = (T_1, T_2, \{i\})$ of $[n]$ is called legal, if $|T_1| = |T_2| = 2\ell - 1$.

**Step 1: Defining the probability distribution $\mu$.**

Consider the following algorithm $\mathcal{A}$:
- (a) Choose randomly a legal partition $T = (T_1, T_2, \{i\})$ of $[n]$, where each desirable partition $T$ is equally likely to be chosen.
- (b) Choose randomly $x \subseteq T_1 \cup \{i\}$ and $y \subseteq T_2 \cup \{i\}$ independently such that $|x| = |y| = \ell$.

The distribution here is also uniform, i.e., each such pair $(x, y)$ is equally likely to be chosen.
- (c) Output $(x, y)$.

The probability distribution $\mu$ on $2^{[n]} \times 2^{[n]}$ is defined as follows.

$$
\mu(x, y) := \text{the probability that } (x, y) \text{ is output by the algorithm } \mathcal{A}
$$

Note that $\mu(x, y) \neq 0$ if and only if $|x| = |y| = \ell$ and $|x \cap y| \leq 1$.

We will use the following notations.
- For a partition $T$, $e(T)$ denotes the event that $T$ is chosen in algorithm $\mathcal{A}$.
- Let $T \subseteq [n]$ be such that $|T| = 2\ell - 1$. For $i \in \{1, 2\}$, $e_i(T)$ denotes the event that the partition $T = (T_1, T_2, \{i\})$ chosen in algorithm $\mathcal{A}$ is such that $T_i = T$.

**Lemma 12.3** $\Pr_\mu[i \in x \mid e(T)] = \Pr_\mu[i \in y \mid e(T)] = 1/2$, for every legal partition $T$.

**Proof.** For a fixed $T = (T_1, T_2, \{i\})$, there are $\binom{2\ell}{\ell}$ number of possible $x$’s. Out of those, $\binom{2\ell-1}{\ell-1}$ are those $x$’s that contain $i$. Therefore, $\Pr_\mu[i \in x \mid e(T)] = \binom{2\ell-1}{\ell-1}/\binom{2\ell}{\ell} = 1/2$. The case $i \in y$ is symmetric. \[\square\]
Step 2: Defining the sets $A$ and $B$.

Let $A$ and $B$ be the following sets:

$$A := \{ (x, y) \mid \mu(x, y) \neq 0 \text{ and } x \cap y = \emptyset \}$$

$$B := \{ (x, y) \mid \mu(x, y) \neq 0 \text{ and } x \cap y \neq \emptyset \}$$

Thus, $A \subseteq \text{DISJ}^{-1}(1)$ and $B \subseteq \text{DISJ}^{-1}(0)$.

**Lemma 12.4** $\mu(B) = 1/4$, and hence, $\mu(A) = 3/4$.

**Proof.** Since $x$ and $y$ are chosen independently, for every legal $T = (T_1, T_2, \{i\})$:

$$\Pr_\mu[x \cap y \neq \emptyset \mid e(T)] = \Pr_\mu[i \in x \mid e(T)] \cdot \Pr_\mu[i \in y \mid e(T)] = 1/4$$

Thus, $\mu(B) = \Pr_\mu[(x, y) \in B] = 1/4$. \hfill \blacksquare

Step 3: Random variables connecting a partition $T$ and a rectangle $R$.

Let $R = C \times D$ be a rectangle and $T = (T_1, T_2, \{i\})$ a legal partition.

$$\text{Row}(T) := \Pr_\mu[x \in C \mid e(T)] \quad \text{Col}(T) := \Pr_\mu[y \in D \mid e(T)]$$

$$\text{Row}_0(T) := \Pr_\mu[x \in C \mid e(T) \land i \notin x] \quad \text{Col}_0(T) := \Pr_\mu[y \in D \mid e(T) \land i \notin y]$$

$$\text{Row}_1(T) := \Pr_\mu[x \in C \mid e(T) \land i \in x] \quad \text{Col}_1(T) := \Pr_\mu[y \in D \mid e(T) \land i \in y]$$

Note that all these can be viewed as random variables on the space of legal partitions.

Intuitively, the event $e(T)$ means the set $x$ is chosen from $T_1 \cup \{i\}$, $e(T) \land i \notin x$ means $x$ is from $T_1$, while $e(T) \land i \in x$ means $x = x' \cup \{i\}$ and $x'$ is chosen from $T_1$.

**Lemma 12.5** For every legal $T$:

- $\text{Row}(T) = (\text{Row}_0(T) + \text{Row}_1(T))/2$.
- $\text{Col}(T) = (\text{Col}_0(T) + \text{Col}_1(T))/2$.

Hence, $\text{Row}_0(T), \text{Row}_1(T) \leq 2\text{Row}(T)$ and $\text{Col}_0(T), \text{Col}_1(T) \leq 2\text{Col}(T)$.

**Proof.** Let $T = (T_1, T_2, \{i\})$.

$$\text{Row}(T) = \Pr_\mu[x \in C \mid e(T)] = \frac{\Pr_\mu[x \in C \land e(T)]}{\Pr_\mu[e(T)]} = \frac{\Pr_\mu[x \in C \land i \in x \land e(T)] + \Pr_\mu[x \in C \land i \notin x \land e(T)]}{\Pr_\mu[e(T)]} = \Pr_\mu[x \in C \land i \in x \land e(T)] \cdot \Pr_\mu[i \in x \land e(T)] + \Pr_\mu[x \in C \land i \notin x \land e(T)] \cdot \Pr_\mu[i \notin x \land e(T)]$$

$$\text{Row}(T) = \text{Row}_0(T) \cdot (1/2) + \text{Row}_1(T) \cdot (1/2) \quad (\text{By Lemma 12.3})$$

The case for $\text{Col}(T)$ is similar. \hfill \blacksquare
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Lemma 12.6

- For every legal $\mathcal{T} = (T_1, T_2, \{i\})$ and $\mathcal{T}' = (T'_1, T'_2, \{i'\})$ such that $T_2 = T'_2$:
  $$\text{Row}(\mathcal{T}) = \text{Row}(\mathcal{T}') \quad \text{and} \quad \text{Col}_0(\mathcal{T}) = \text{Col}_0(\mathcal{T}')$$

- Similarly, for every legal $\mathcal{T} = (T_1, T_2, \{i\})$ and $\mathcal{T}' = (T'_1, T'_2, \{i'\})$ such that $T_1 = T'_1$:
  $$\text{Col}(\mathcal{T}) = \text{Col}(\mathcal{T}') \quad \text{and} \quad \text{Row}_0(\mathcal{T}) = \text{Row}_0(\mathcal{T}')$$

Proof. The choice of $x$ comes from $[n] - T_2 = [n] - T'_2$, thus, $\text{Row}(\mathcal{T}) = \text{Row}(\mathcal{T}')$. Furthermore, the choice of $y$ that does not contain $i$ comes entirely from $T_2 = T'_2$. Therefore, $\text{Col}_0(\mathcal{T}) = \text{Col}_0(\mathcal{T}')$. The other case is symmetric.

Lemma 12.7 For every $T \subseteq [n]$ such that $|T| = 2\ell - 1$,
$$\mathbb{E}[\text{Row}_0(\mathcal{T}) \mid e_2(T)] = \mathbb{E}[\text{Row}(\mathcal{T}) \mid e_2(T)].$$

Proof. Let $T \subseteq [n]$ such that $|T| = 2\ell - 1$. The quantities are defined as:

\[
\mathbb{E}[\text{Row}_0(\mathcal{T}) \mid e_2(T)] = \sum_{\mathcal{T} = (T_1, T_2, \{i\}) \text{ s.t. } T_2 = T} \Pr_\mu[x \in C \mid e(\mathcal{T}) \land i \notin x] / |e_2(T)|
\]

\[
\mathbb{E}[\text{Row}(\mathcal{T}) \mid e_2(T)] = \sum_{\mathcal{T} = (T_1, T_2, \{i\}) \text{ s.t. } T_2 = T} \Pr_\mu[x \in C \mid e(\mathcal{T})] / |e_2(T)|
\]

We will prove that the two quantities above are the same. Without loss of generality, assume that $C = \{z\}$ where $|z| = \ell$.

For every legal $\mathcal{T} = (T_1, T_2, \{i\})$,

\[
\Pr_\mu[x = z \mid e(\mathcal{T})] = \frac{1}{\binom{2\ell}{\ell}} = \frac{1}{2^{2\ell-1}}
\]

\[
\Pr_\mu[x = z \mid e(\mathcal{T}) \land i \notin x] = \begin{cases} 0 & \text{if } i \in z \\ 1/(2^{2\ell-1}) & \text{if } i \notin z \\ \end{cases}
\]

Since $|z| = \ell$, the number of legal partitions $\mathcal{T} = (T_1, T_2, \{i\})$ such that $T_2 = T$ and $i \notin z$ is exactly half of all the possible legal partitions. The lemma follows.

Step 4: “Bad” legal partitions.

We fix a positive real number $\delta \geq 0$ (to be determined later). We also fix a rectangle $R = C \times D$.

- A legal partition $\mathcal{T}$ is called $x$-bad, if $\text{Row}_1(\mathcal{T}) < \text{Row}_0(\mathcal{T})/3 - 2^{-6n}$.
- A legal partition $\mathcal{T}$ is called $y$-bad, if $\text{Col}_1(\mathcal{T}) < \text{Col}_0(\mathcal{T})/3 - 2^{-5n}$.
- $\mathcal{T}$ is bad, if it is either $x$-bad or $y$-bad.

Recall that $\text{Row}$, $\text{Row}_0$, $\text{Row}_1$, $\text{Col}$, $\text{Col}_0$ and $\text{Col}_1$ are all defined with respect to a rectangle $R$.

Define the 0-1 random variables $\text{Bad}_x(\mathcal{T})$, $\text{Bad}_y(\mathcal{T})$ and $\text{Bad}(\mathcal{T})$ as indicator whether $\mathcal{T}$ is bad.

\[
\text{Bad}_x(\mathcal{T}) := \begin{cases} 1, & \text{if } \mathcal{T} \text{ is } x\text{-bad.} \\ 0, & \text{otherwise.} \\ \end{cases}
\]

$\text{Bad}_y(\mathcal{T})$ and $\text{Bad}(\mathcal{T})$ are defined similarly.
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Lemma 12.8 Let $T = (T_1, T_2, \{i\})$ be a legal partition. Let $S$ and $S'$ be the following sets.

\[
S := \{ x \mid x \in C \text{ and } |x| = \ell \text{ and } x \subseteq T_1 \cup \{i\} \}
\]

\[
S' := \{ x \mid x \in C \text{ and } |x| = \ell \text{ and } x \subseteq T_1 \}
\]

The following holds.

- $\text{Row}(T) = |S|/\left(\frac{2^\ell}{\ell}\right)$.
- $\text{Row}_0(T) = \frac{|S'|}{|S|} \cdot \text{Row}(T) \cdot 2$.
- If $T$ is $x$-bad, then $\Pr[i \in s] < 1/4$, where the probability is measured on $s$ being chosen randomly and uniformly from $S$.

Proof. The first bullet can be proved in a similar manner as Lemma 12.7. The second bullet is as follows.

\[
\text{Row}_0(T) = \Pr[i \notin s] \cdot \text{Row}(T) \cdot 2.
\]

Similarly, $\text{Row}_1(T) = \Pr[i \in s] \cdot \text{Row}(T) \cdot 2$. Therefore,

\[
\text{Row}_0(T) = \frac{\text{Row}_1(T)}{\Pr[i \in s]} \cdot \frac{\text{Row}(T)}{2}.
\]

If $T$ is $x$-bad, $\text{Row}_1(T) < \text{Row}_0(T)/3 - 2^{-\delta n} < \text{Row}_0(T)/3$. Now, $\Pr[i \in s] < 1/4$ follows immediately.

Lemma 12.9 There exists $\delta > 0$ such that for every set $T \subseteq [n]$ such that $|T| = 2\ell - 1$,

\[
\Pr[\text{Bad}_x(T) = 1 \mid e_2(T)] = \Pr[T \text{ is } x\text{-bad} \mid e_2(T)] < 1/5.
\]

Similarly, $\Pr[\text{Bad}_y(T) \mid e_1(T)] < 1/5$.

Proof. We assume that $\delta$ is fixed (to be determined later). Let $T \subseteq [n]$ be such that $|T| = 2\ell - 1$. By Lemma 12.6, for every $T = (T_1, T_2, \{i\})$ such that $T_2 = T$, the value $\text{Row}(T)$ is fixed. There are two cases: Either $\text{Row}(T) \leq 2^{-\delta n}$ or $\text{Row}(T) > 2^{-\delta n}$.

First consider the case that $\text{Row}(T) \leq 2^{-\delta n}$. By Lemma 12.5, $\text{Row}_0(T) \leq 2 \cdot \text{Row}(T) \leq 2 \cdot 2^{-\delta n}$. If $T$ is $x$-bad,

\[
\text{Row}(T) < \frac{\text{Row}_0(T)}{3} - 2^{-\delta n} < \frac{2}{3} \cdot 2^{-\delta n} - 2^{-\delta n} < 0
\]

A contradiction. Thus, for this particular $T$, a legal $T = (T_1, T_2, \{i\})$ is never $x$-bad whenever $T_2 = T$. That is, $\Pr[\text{Bad}_x(T) \mid e_2(T)] = 0$, which is $< 1/5$.

Now consider the case $\text{Row}(T) > 2^{-\delta n}$. Let $[n] - T = \{ k_1, \ldots, k_{2\ell} \} = K$. Note that each $i \in K$ determines a legal partition $([n] - (T \cup \{i\}), T, \{i\})$. Let $K'$ be the following set.

\[
K' := \{ i \in K \mid ([n] - (T \cup \{i\}), T, i) \text{ is } x\text{-bad} \}.
\]
Let \( Z = (Z_1, \ldots, Z_{2\ell}) \) be the indicator random variable for the set \( s \) chosen uniformly from \( S \). Assume to the contrary that \( \Pr[T] \) is \( x \)-bad \( |e_2(T)| \geq 1/5 \). This means that \( |K'| \geq 2\ell/5 \).

Now, for each \( i \in K' \), the partition \( T = ([n] - (T \cup \{i\}), T, \{i\}) \) is \( x \)-bad. By Lemma 12.8 when the set \( s \) is chosen uniformly from \( S \), \( \Pr[i \in s] < 1/4 \). Or, equivalently, \( \Pr[Z_i = 1] < 1/4 \).

We calculate an upper bound of the entropy of \( Z \).

\[
H(Z) \leq \sum_{i=1}^{2\ell} H(Z_i) = \sum_{i \in K'} H(Z_i) + \sum_{i \not\in K'} H(Z_i) \\
\leq \sum_{i \in K'} H(1/4) + \sum_{i \not\in K'} H(1/2) \\
< \frac{2\ell}{5} H(1/4) + \frac{2\ell \cdot 4}{5} H(1/2) \\
< 1.928 \cdot \ell.
\]

On the other hand, since \( s \) is chosen uniformly from \( S \),

\[
H(Z) = \log_2 |S| = \log_2 \left( \frac{2\ell}{\ell} \cdot \frac{\text{Row}(T)}{\text{Row}(T)} \right) = \log_2 \left( \frac{2\ell}{\ell} \cdot 2^{-\delta n} \right)
\]

The second equality is from Lemma 12.8. The last inequality is the assumption \( \text{Row}(T) > 2^{-\delta n} \).

Using an estimate on binomial coefficient, there is \( \lambda \) such that:

\[
\log_2 \left( \frac{2\ell}{\ell} \cdot 2^{-\delta n} \right) \geq \log_2 \left( \frac{2^{2\ell}}{\lambda \sqrt{\ell}} \cdot 2^{-\delta n} \right)
\]

Combining all those inequalities:

\[
\log_2 \left( \frac{2^{2\ell}}{\lambda \sqrt{\ell}} \cdot 2^{-\delta n} \right) \leq H(Z) < 1.928 \cdot \ell.
\]

If \( \delta \) is fixed to be small enough \( \delta \), the inequality above will yield a contradiction.

**Step 5: The measure on a rectangle \( R \).**

Let \( R = C \times D \) be a rectangle that is fixed as before. All the random variables \( \text{Row}(T) \), \( \text{Row}_0(T) \), \( \text{Row}_1(T) \), \( \text{Col}(T) \), \( \text{Col}_0(T) \), \( \text{Col}_1(T) \) are defined with respect to \( R \).

**Lemma 12.10** For every set \( T \subseteq [n] \) such that \( |T| = 2\ell - 1 \),

\[
E[\text{Row}_0(T) \text{Col}_0(T) \text{Bad}_x(T) | e_2(T)] \leq (2/5) \cdot E[\text{Row}_0(T) \text{Col}_0(T) | e_2(T)]
\]

\[
E[\text{Row}_0(T) \text{Col}_0(T) \text{Bad}_y(T) | e_1(T)] \leq (2/5) \cdot E[\text{Row}_0(T) \text{Col}_0(T) | e_1(T)]
\]

**Proof.** Let \( T \) be a subset of \([n]\) such that \( |T| = 2\ell - 1 \). By Lemma 12.6, \( \text{Row}(T) \) and \( \text{Col}_0(T) \) are fixed, say to \( c_1 \) and \( c_2 \), respectively, for every \( T = (T_1, T_2, \{i\}) \) such that \( T_2 = T \).

The inequality comes from the following.

\[
E[\text{Row}_0(T) \cdot \text{Col}_0(T) \cdot \text{Bad}_x(T) | e_2(T)] = c_2 \cdot E[\text{Row}_0(T) \cdot \text{Bad}_x(T) | e_2(T)] \quad (1)
\]

\[
\leq c_2 \cdot E[2 \cdot \text{Row}(T) \cdot \text{Bad}_x(T) | e_2(T)] \quad (2)
\]

\[
= 2 \cdot c_2 \cdot c_1 \cdot E[\text{Bad}_x(T) | e_2(T)] \quad (3)
\]

\[
\leq 2 \cdot c_2 \cdot c_1 \cdot (1/5) \quad (4)
\]

\[
= (2/5) \cdot c_2 \cdot E[\text{Row}(T) | e_2(T)] \quad (5)
\]

\[
= (2/5) \cdot c_2 \cdot E[\text{Row}_0(T) | e_2(T)] \quad (6)
\]

\[
= (2/5) \cdot E[\text{Col}_0(T) \text{Row}_0(T) | e_2(T)] \quad (7)
\]
Lemma 12.11
\[
\begin{align*}
\mathbb{E}[\text{Row}_0(T)\text{Col}_0(T)\text{Bad}_x(T)] & \leq \frac{2}{5} \cdot \mathbb{E}[\text{Row}_0(T)\text{Col}_0(T)] \\
\mathbb{E}[\text{Row}_0(T)\text{Col}_0(T)\text{Bad}_y(T)] & \leq \frac{2}{5} \cdot \mathbb{E}[\text{Row}_0(T)\text{Col}_0(T)].
\end{align*}
\]

**Proof.** We only prove the first inequality. The second can be proved in a similar manner. Let \( L \) be the set of all legal partitions. For \( T \subseteq \{n\} \) such that \( |T| = 2\ell - 1 \), let \( L(T) \) be the set of legal \( T = (T_1, T_2, \{i\}) \) such that \( T_2 = T \). Note that \( |L(T)| = |L(T')|, \) whenever \( |T| = |T'| = 2\ell - 1 \). Let \( N = |L(T)| \).

\[
\begin{align*}
\mathbb{E}[\text{Row}_0(T)\text{Col}_0(T)\text{Bad}_x(T)] & = \frac{1}{|L|} \sum_{T \in L} \text{Row}_0(T) \cdot \text{Col}_0(T) \cdot \text{Bad}_x(T) \\
& = \frac{N}{|L|} \sum_{T \text{ s.t. } |T| = 2\ell - 1} \frac{1}{N} \sum_{T \in L(T)} \text{Row}_0(T) \cdot \text{Col}_0(T) \cdot \text{Bad}_x(T) \\
& = \frac{N}{|L|} \sum_{T \text{ s.t. } |T| = 2\ell - 1} \mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T) \cdot \text{Bad}_x(T) | e_2(T)] \\
& \leq \frac{N}{|L|} \sum_{T \text{ s.t. } |T| = 2\ell - 1} \left( \frac{2}{5} \right) \cdot \mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T) | e_2(T)] \\
& = \frac{2N}{5|L|} \sum_{T \text{ s.t. } |T| = 2\ell - 1} \frac{1}{N} \sum_{T \in L(T)} \text{Row}_0(T) \cdot \text{Col}_0(T) \\
& = \frac{2}{5|L|} \sum_{T \in L} \text{Row}_0(T) \cdot \text{Col}_0(T) \\
& = \left( \frac{2}{5} \right) \cdot \mathbb{E}[\text{Row}_0(T) \text{Col}_0(T)]
\end{align*}
\]

The inequality comes from Lemma [12.10] \( \blacksquare \)

Lemma 12.12
\[
\mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T) \cdot \text{Bad}(T)] \leq \frac{4}{5} \mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T)].
\]

**Hence,**
\[
\mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T) \cdot (1 - \text{Bad}(T))] > \frac{1}{5} \mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T)].
\]

**Proof.** Note that \( \text{Bad}(T) \leq \text{Bad}_x(T) + \text{Bad}_y(T) \). Thus, the inequality follows immediately from Lemma [12.11] \( \blacksquare \)

Lemma 12.13
- \( \mu(B \cap R) = \frac{3}{4} \mathbb{E}[\text{Row}_1(T) \cdot \text{Col}_1(T)]. \)
- \( \mu(A \cap R) = \frac{3}{4} \mathbb{E}[\text{Row}_0(T) \cdot \text{Col}_0(T)]. \)
Proof. We compute $\mu(R \mid B)$. Let $L$ be the set of all legal partitions.

$$
\mu(R \mid B) = \frac{1}{|L|} \sum_{T=(T_1,T_2,\{i\})\in L} \Pr_\mu[x \in C \land y \in D \mid x \cap y \neq \emptyset]
$$

$$
= \frac{1}{|L|} \sum_{T=(T_1,T_2,\{i\})\in L} \Pr_\mu[x \in C \land y \in D \mid e(T) \land i \in x \land i \in y]
$$

$$
= \frac{1}{|L|} \sum_{T=(T_1,T_2,\{i\})\in L} \left( \Pr_\mu[x \in C \mid e(T) \land i \in x \land i \in y] \cdot \Pr_\mu[y \in D \mid e(T) \land i \in x \land i \in y] \right)
$$

$$
= \frac{1}{|L|} \sum_{T=(T_1,T_2,\{i\})\in L} \Pr_\mu[x \in C \mid e(T) \land i \in x] \cdot \Pr_\mu[y \in D \mid e(T) \land i \in y]
$$

$$
= \frac{1}{|L|} \sum_{T=(T_1,T_2,\{i\})\in L} \operatorname{Row}_1(T) \cdot \operatorname{Col}_1(T)
$$

$$
= \operatorname{E}[\operatorname{Row}_1(T) \cdot \operatorname{Col}_1(T)]
$$

The equality follows from $\mu(R \cap B) = \mu(R \mid B)\mu(B)$ and by Lemma 12.4 $\mu(B) = 1/4$. The second equality can be proved in a similar manner.

Step 6: Wrapping all up.

Lemma 12.14 Let $\mu$, $A$, $B$ be as defined in Steps 1 and 2 above. Let $R = C \times D$ be a rectangle. Then, there are positive real numbers $\alpha$ and $\delta$ such that $\mu(B \cap R) \geq \alpha \cdot \mu(A \cap R) - 2^{-\delta n}$.

Proof. We let $\delta$ to be the value obtained in Lemma 12.9 As before, let $L$ denote the set of all legal partitions. Let $L_{\text{bad}}$ denote the set of all bad partitions. By Lemma 12.13 $\mu(B \cap R) = (1/4)\operatorname{E}[\operatorname{Row}_1(T)\operatorname{Col}_1(T)]$. Since Bad$(T)$ is 0-1 random variable and Row$_1(T), \operatorname{Col}_1(T)$ take only non-negative real numbers,

$$
\mu(B \cap R) \geq (1/4) \cdot \operatorname{E}[\operatorname{Row}_0(T)\operatorname{Col}_0(T)(1 - \text{Bad}(T))]
$$

$$
= \frac{1}{4|L|} \sum_{T \in L} \operatorname{Row}_1(T)\operatorname{Col}_1(T)(1 - \text{Bad}(T))
$$

$$
= \frac{1}{4|L|} \sum_{T \notin L_{\text{bad}}} \operatorname{Row}_1(T)\operatorname{Col}_1(T)(1 - \text{Bad}(T))
$$

$$
\geq \frac{1}{4|L|} \sum_{T \notin L_{\text{bad}}} \left( \frac{\operatorname{Row}_0(T)}{3} - 2^{-\delta n} \right) \left( \frac{\operatorname{Col}_0(T)}{3} - 2^{-\delta n} \right) (1 - \text{Bad}(T))
$$

$$
\geq \left( \frac{1}{4 \cdot 9} \cdot \operatorname{E}[\operatorname{Row}_0(T)\operatorname{Col}_0(T)(1 - \text{Bad}(T)))] \right) - 2^{-\delta n}
$$

$$
\geq \left( \frac{1}{4 \cdot 9 \cdot 5} \cdot \operatorname{E}[\operatorname{Row}_0(T)\operatorname{Col}_0(T)] \right) - 2^{-\delta n}
$$

$$
= \left( \frac{1}{4 \cdot 9 \cdot 5} \cdot \frac{4}{3} \cdot \mu(A \cap R) \right) - 2^{-\delta n}
$$

(8) to (9) is the definition of Bad$(T)$. (10) to (11) is by Lemma 12.12 (11) to (12) is by Lemma 12.13
Appendix

The entropy $H(X)$ of a random variable $X$ is defined as:

$$H(X) := - \sum_{x \in \text{range}(X)} \Pr[X = x] \cdot \log(\Pr[X = x]),$$

where the logarithm is of base 2.

In this lesson, we are only interested in the entropy of 0-1 random variable $X$. For $0 \leq p \leq 1$, $H(p)$ denotes the entropy of a 0-1 random variable $X$, where $\Pr[X = 1] = p$.

The following facts can be easily verified.

- If $X$ is a uniform distribution on $\text{range}(X)$,
  $$H(X) = \log |\text{range}(X)|$$

- For $0 \leq p_1 \leq p_2 \leq 1/2$,
  $$H(p_1) \leq H(p_2).$$

- For a vector $X = (X_1, \ldots, X_m)$, where each $X_i$ is a random variable,
  $$H(X) \leq \sum_{i=1}^{m} H(X_i).$$