Lesson 11: Discrepancy

**Theme:** The notion of discrepancy for establishing lower bounds on distributional complexity.

Let $f : X \times Y \to \{0, 1\}$, and let $\mu$ be a probability distribution on $X \times Y$. Let $R$ be a rectangle (not necessarily monochromatic) in $X \times Y$. The discrepancy of $f$ on $R$ according to $\mu$ is defined as follows.

$$\text{Disc}_\mu(R, f) := \left| \Pr_\mu[f(x, y) = 0 \text{ and } (x, y) \in R] - \Pr_\mu[f(x, y) = 1 \text{ and } (x, y) \in R] \right|$$

The discrepancy of $f$ according to $\mu$ is defined as follows.

$$\text{Disc}_\mu(f) := \max_{R \text{ is a rectangle}} \text{Disc}_\mu(R, f)$$

**Theorem 11.1** For every function $f : X \times Y \to \{0, 1\}$, for every probability distribution $\mu$ on $X \times Y$, for every $0 \leq \epsilon \leq 1/2$,

$$D_{1/2-\epsilon}^\mu(f) \geq \log_2 \left( \frac{2\epsilon}{\text{Disc}_\mu(f)} \right)$$

Recall the function $\text{ip} : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$, where $\text{ip}(x, y)$ is the inner product of $x$ and $y$ modulo 2.

**Theorem 11.2** $\text{Disc}_{\mu_0}(\text{ip}) = 2^{-n/2}$, where $\mu_0$ is the uniform distribution on $\{0, 1\}^n \times \{0, 1\}^n$. Thus, $D_{1/2-\epsilon}^\mu(\text{ip}) \geq n/2 - \log(1/\epsilon)$.

**Corollary 11.3** $R_{1/2-\epsilon}^{\text{pub}}(\text{ip}) \geq n/2 - \log(1/\epsilon)$. 