Lesson 3: Rectangles and fooling sets

Theme: Rectangles as special partitions of the inputs induced by deterministic protocols.

1 Rectangles

A set $R \subseteq X \times Y$ is called a (combinatorial) rectangle in $X \times Y$, if $R = A \times B$, for some $A \subseteq X$ and $B \subseteq Y$.

**Proposition 3.1** $R$ is a rectangle in $X \times Y$ if and only if for every two tuples $(x_1, y_1), (x_2, y_2) \in R$, $(x_1, y_2) \in R$.

Let $\mathcal{P}$ be a protocol that computes a function $f : X \times Y \rightarrow Z$. To determine the value of $\mathcal{P}$ on input $(x, y)$, we traverse the binary tree $\mathcal{P}$ from the root node to a leaf node according to the rules set out in Definition 2.1:

- For a node $u$ in $\mathcal{P}$, let $R_u$ be the set of inputs that reach $u$.
- A subset $R \subseteq X \times Y$ is called $f$-monochromatic, or monochromatic, for short, if $f$ is fixed on $R$, i.e., there is $z \in Z$ such that for every $(x, y) \in R$, $f(x, y) = z$.

**Theorem 3.2** Let $\mathcal{P}$ be a protocol that computes a function $f : X \times Y \rightarrow Z$.

- For every node $u$ in $\mathcal{P}$, $R_u$ is a rectangle.
- The input set $X \times Y$ is partitioned into $R_{\ell_1} \cup \cdots \cup R_{\ell_m}$, where $\ell_1, \ell_2, \ldots, \ell_m$ are the leaf nodes in $\mathcal{P}$.
- For every leaf node $\ell$ in $\mathcal{P}$, each $R_\ell$ is $f$-monochromatic.
- The number of monochromatic rectangles in $X \times Y$ is $\leq$ the number of leaf nodes in $\mathcal{P}$.

**Corollary 3.3** Let $f : X \times Y \rightarrow Z$ be a function. If any partition of $X \times Y$ into $f$-monochromatic rectangles requires at least $t$ triangles, then $D(f) \geq \log t$.

2 Fooling sets

Let $f : X \times Y \rightarrow Z$. A set $S \subseteq X \times Y$ is called a fooling set for $f$, if there exists $z \in Z$ such that:

- For every $(x, y) \in S$, $f(x, y) = z$.
- For every two distinct pairs $(x_1, y_1), (x_2, y_2) \in S$, either $f(x_1, y_2) \neq z$ or $f(x_2, y_1) \neq z$.

**Theorem 3.4** Let $f : X \times Y \rightarrow Z$ be a function and let $\mu$ be a probability distribution on $X \times Y$.

- If $f$ has a fooling set $S$ of cardinality $t$, then $D(f) \geq \log t$.
- If every $f$-monochromatic rectangle $R$ has measure $\mu(R) \leq \delta$, then $D(f) \geq \log(1/\delta)$.

We will consider the following functions:

- For $x, y \in \{0, 1\}^n$, $\text{EQ}(x, y) = 1$, if $x = y$ and $\text{EQ}(x, y) = 0$, if $x \neq y$.
- For $x, y \subseteq \{1, \ldots, n\}$, $\text{DISJ}(x, y) = 1$, if $x \cap y = \emptyset$ and $\text{DISJ}(x, y) = 0$, if $x \cap y \neq \emptyset$.
- For $x, y \in \{0, 1\}^n$, $\text{IP}(x, y) = \sum x_i y_i \pmod 2$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

By counting monochromatic rectangles, we will show that $D(\text{EQ}) = n + 1$, $D(\text{DISJ}) = n + 1$ and $D(\text{IP}) \geq n - 1$. 

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