Homework 1
(Due on Friday, 10:30 am, 13-April-2017)

(1) Define the following function $\text{GT}(x, y) = \{ 1, \text{ if } x > y \}
\{ 0, \text{ otherwise } \}$

Prove that $D(\text{GT}) = n + 1$. (Note that here you have to prove both lower and upper bounds.)

(2) Recall the function $\text{DISJ}$: For every $x, y \subseteq \{1, \ldots, n\}$,
\[ \text{DISJ}(x, y) = \{ 1, \text{ if } x \cap y = \emptyset \}
\{ 0, \text{ otherwise } \} \]

Prove that the size of any 1-monochromatic rectangle of $\text{DISJ}$ is at most $2^n$. Then, deduce that $D(\text{DISJ}) \geq n$.

(3) Let $X, Y \subseteq 2^{\{1,\ldots,n\}}$ such that for every $x \in X$ and $y \in Y$, $|x \cap y| \leq 1$.

Define the function $f : X \times Y \to \{0, 1\} as follows: f(x, y) = |x \cap y|$. Prove that $D(f) = O(\log^2 n)$.

(4) Let $n$ and $k$ be positive integers such that $k$ divides $n$. Define the following function $\text{DISJ}_k$: For every $x, y \subseteq \{1, \ldots, n\}$ such that $|x| = |y| = k$,
\[ \text{DISJ}_k(x, y) = \{ 1, \text{ if } x \cap y = \emptyset \}
\{ 0, \text{ otherwise } \} \]

Prove that $D(\text{DISJ}_k) \geq \log(n/k)$.

(5) In the following, let $[n] = \{1, \ldots, n\}$ and $\vec{z} = (z_1, \ldots, z_n)$ be a vector of $n$ variables. For integers $0 \leq k \leq n$, we define $N_{n,k} = \sum_{i=0}^{k} \binom{n}{i}$.

(a) For $x \subseteq [n]$, we define the (multi-variate) monomial $q_x(\vec{z}) := \prod_{i \in x} z_i$. For $y \subseteq [n]$, define the vector $\vec{c}_y = (c_1, \ldots, c_n)$ where each $c_i$ is as follows.
\[ c_i = \begin{cases} 1, & \text{if } i \notin y \\ 0, & \text{if } i \in y \end{cases} \]

Note that $q_x(\vec{c}_y) = 1$ if and only if $x \cap y = \emptyset$.

Consider the following (multi-linear) polynomial $p(\vec{z})$.
\[ p(\vec{z}) := \alpha_0 + \alpha_1 q_{x_1}(\vec{z}) + \cdots + \alpha_m q_{x_m}(\vec{z}), \]
where $x_1 \subseteq x_2 \subseteq \cdots \subseteq x_m$ and $|x_m| \leq k$, and each $\alpha_i$ is non-zero real number. Prove that there is $y \subseteq [n]$ such that $|y| \leq k$ and $p(\vec{c}_y) \neq 0$.

(b) Consider the following polynomial $p(\vec{z})$.
\[ p(\vec{z}) := \alpha_1 q_{x_1}(\vec{z}) + \cdots + \alpha_m q_{x_m}(\vec{z}), \]
where $x_1, \ldots, x_m$ are all the subsets of $[n]$ with cardinality $\leq k$ and not all of $\alpha_i$ is zero. Prove that there is $y \subseteq [n]$ such that $|y| \leq k$ and $p(\vec{c}_y) \neq 0$.

(c) Define the following function $\text{DISJ}_{\leq k}$: For every $x, y \subseteq [n]$ such that $|x|, |y| \leq k$,
\[ \text{DISJ}_{\leq k}(x, y) = \{ 1, \text{ if } x \cap y = \emptyset \}
\{ 0, \text{ otherwise } \} \]

Prove that $\text{rank}(\text{DISJ}_{\leq k}) = N_{n,k}$, and hence, $D(\text{DISJ}_{\leq k}) \geq \log N_{n,k}$.

Hint: Consider the matrix of $\text{DISJ}_{\leq k}$, and also what the polynomial $p(\vec{z})$ in (b) represents.