Sample solution for midterm

(1) Consider the following automaton $A$.

(i) Is $A$ deterministic or non-deterministic?
   Ans: Non-deterministic.

(ii) Is $aaa$ accepted by $A$?
    Ans: No.

(iii) Is $ababababa$ accepted by $A$?
    Ans: Yes.

(iv) Construct the deterministic automaton for $A$.
    Ans:

(2) Construct a DFA for the following language over the alphabet $\{0, 1\}$:

$$L_0 := \{w \mid w \text{ represents an integer divisible by 3}\}.$$

Ans: First, we calculate the following:

- $2 \cdot 0 + 0 \equiv 0 \pmod{3}$
- $2 \cdot 1 + 0 \equiv 2 \pmod{3}$
- $2 \cdot 2 + 0 \equiv 1 \pmod{3}$
- $2 \cdot 0 + 1 \equiv 1 \pmod{3}$
- $2 \cdot 1 + 1 \equiv 0 \pmod{3}$
- $2 \cdot 2 + 1 \equiv 2 \pmod{3}$

Then, we can construct a DFA with three states $q_0, q_1, q_2$ corresponding to 0, 1, 2, respectively.
(3) Construct the CFG for each of the following languages.

- \( L_1 := \{ w \# w^R \# \mid w \in \{0,1\}^* \} \)
  
  **Ans:** \( L_1 \) can be generated by the following grammar with \( S \) being the start variable.

  \[
  S \rightarrow X\#
  \]

  \[
  X \rightarrow \# \mid 0X0 \mid 1X1
  \]

- \( L_2 := \{ w_1 \# w_1^R \# w_2 \# w_2^R \# \cdots \# w_k \# w_k^R \# \mid \text{each } w_i \in \{0,1\}^* \text{ for some } k \geq 1 \} \)
  
  **Ans:** \( L_2 \) can be generated by the following grammar with \( T \) being the start variable.

  \[
  T \rightarrow ST \mid S
  \]

  \[
  S \rightarrow X\#
  \]

  \[
  X \rightarrow \# \mid 0X0 \mid 1X1
  \]

(4) Prove or disprove the following.

- If \( L \) is regular and \( K \) is CFL, then \( L \cap K \) is regular.
  
  **Ans:** The statement is wrong. Consider the following languages \( L \) and \( K \).

  \[
  L := \{ a^n b^n \mid n \geq 0 \}
  \]

  \[
  K := \Sigma^*
  \]

  We have learned that \( L \) is CFL, but not regular, while \( K \) is obviously regular. Thus, \( L \cap K = L \) is not regular.

- If \( L \) is regular and \( K \) is CFL, then \( L \cup K \) is regular.
  
  **Ans:** The statement is wrong. Consider the following languages \( L \) and \( K \).

  \[
  L := \{ a^n b^n \mid n \geq 0 \}
  \]

  \[
  K := \emptyset
  \]

  We have learned that \( L \) is CFL, but not regular, while \( K \) is obviously regular. Thus, \( L \cup K = L \) is not regular.

(5) Prove that if \( L \) is regular, then \( \text{half}(L) \) is also regular.

**Ans:** Suppose \( L \) is regular and is accepted by a DFA \( A = (\Sigma, Q, q_0, F, \delta) \).

Consider the following \( \epsilon \)-NFA \( A' = (\Sigma, Q', q'_0, F', \delta') \).

- \( Q' = Q \times Q \cup \{ p \} \), where \( p \notin Q \).
- \( q'_0 = p \).
- \( F = \{(q, q) \mid q \in Q \} \).
- \( \delta' \) is the following set of transitions.

  \[
  \delta' = \{(p, \epsilon, (q_0, q_f)) \mid q_f \in F \}
  \cup \{(q_1, q_2), a, (q'_1, q'_2) \mid (q_1, a, q_2) \in \delta \text{ and } (q'_2, b, q_2) \in \delta \text{ for some } b \in \Sigma \}
  \]

  The idea is that on input word \( c_1 \cdots c_n \), \( A' \) simulates \( A \) both “going forward” from the initial state \( q_0 \) and “going backward” from one of the final states \( q_f \in F \).
We will prove that \( L(\mathcal{A}') = \text{half}(L) \). If a word \( c_1c_2 \cdots c_n d_1 \cdots d_n \) is accepted by \( \mathcal{A} \), where each \( c_i, d_i \in \Sigma \), with an accepting run:

\[
q_0 \ c_1 \ q_1 \ \cdots \ q_{n-1} \ c_n \ q_n \ d_n \ q_{n+1} \ \cdots \ q_{2n-1} \ d_n \ q_{2n}, \quad \text{where } q_{2n} \in F;
\]

then the following is a run of \( \mathcal{A}' \) on \( c_1c_2 \cdots c_n \) by definition of \( \delta' \):

\[
p \in (q_0, q_{2n}) \ c_1 (q_1, q_{2n-1}) \ \cdots \ (q_{n-1}, q_{n+1}) \ c_n (q_n, q_n).
\]

Since \( (q_n, q_n) \in F \), the word \( c_1 \cdots c_n \) is accepted by \( \mathcal{A}' \).

Vice versa, if \( c_1c_2 \cdots c_n \) is accepted by \( \mathcal{A}' \), the accepting run must be of the form:

\[
p \in (q_0, q_{2n}) \ c_1 (q_1, q_{2n-1}) \ \cdots \ (q_{n-1}, q_{n+1}) \ c_n (q_n, q_n), \quad \text{where } q_{2n} \in F.
\]

By definition of \( \delta' \), we have the following run on some \( d_1 \cdots d_n \):

\[
q_n \ d_1 \ q_{n+1} \ \cdots \ q_{2n-1} \ d_n \ q_{2n}
\]

This means there is a run of \( \mathcal{A} \) on \( c_1 \cdots c_n d_1 \cdots d_n \):

\[
q_0 \ c_1 \ q_1 \ \cdots \ q_{n-1} \ c_n \ q_n \ d_n \ q_{n+1} \ \cdots \ q_{2n-1} \ d_n \ q_{2n}
\]

Since \( q_{2n} \in F \), \( c_1 \cdots c_n d_1 \cdots d_n \) is accepted by \( \mathcal{A} \). Therefore, \( \mathcal{A}' \) accepts \( \text{half}(L) \).