Sample solution to HW 3

(1) (i) \( q_0 \vdash q_{\text{rej}} \).
(ii) \( q_0011 \vdash \langle p_011 \vdash \langle 0p_1 \vdash \langle 01s1 \vdash \langle 0s10 \vdash \langle s000 \vdash t < 100 \vdash q_{\text{acc}} < 100 \). \\
(iii) \( q_0100 \vdash \langle p_100 \vdash \langle 1p_0 \vdash \langle 10p_0 \vdash \langle 10q_{\text{rej}} \). \\
(iv) \( q_0111 \vdash \langle p_111 \vdash \langle 1p_1 \vdash \langle 11s1 \vdash \langle 1s10 \vdash \langle s100 \vdash s < 000 \vdash r_1000 \vdash \langle r_1000 \vdash \langle 10r_00 \vdash \langle 100r_0 \vdash \langle 1t1000 \vdash t < 1000 \vdash q_{\text{acc}} < 100 \).

(2) A configuration is a string over the alphabet \( Q \cup \Gamma \) in which every symbol from \( Q \) appears exactly once and does not contain the blank symbol \( \square \). Then, we can easily construct an DFA that accepts such string. Note that an DFA can be viewed as a DTM that always moves right and does not change the input symbol.

(3) In the following algorithm, let \( w[i] \) denote the symbol in position \( i \) of the string \( w \).

On input string \( w \), do the following. We assume that we have already checked that \( w \) is a configuration of \( M_1 \).

- Store the transition \( \delta \) as a list.
- Scan the string \( w \) from left to right to find out the position in \( w \) that contains the symbol from \( Q \).
- Say, \( i \) is the position of \( w \) that contains the symbol from \( Q \).
- If the state is either \( q_{\text{acc}} \) or \( q_{\text{rej}} \), output NULL.
- Otherwise, consider the symbols \( w[i - 1], w[i], w[i + 1] \) and \( w[i + 2] \).
- Scan the list \( \delta \) to find out the appropriate transition to use on state \( w[i] \) with input symbol \( w[i + 1] \), say, \( (w[i], w[i + 1]) \rightarrow (q, a, \alpha) \).
- Let \( v \) the string defined as follows.
  - If \( \alpha = \text{Stay} \), then for every position \( j \),
    \[
    v[j] := \begin{cases} 
    q & \text{if } j = i \\
    a & \text{if } j = i + 1 \\
    w[j] & \text{for all the other } j
    \end{cases}
    
  - If \( \alpha = \text{Left} \), then for every position \( j \),
    \[
    v[j] := \begin{cases} 
    q & \text{if } j = i - 1 \\
    w[i - 1] & \text{if } j = i \\
    a & \text{if } j = i + 1 \\
    w[j] & \text{for all the other } j
    \end{cases}
    
  - If \( \alpha = \text{Right} \), then for every position \( j \),
    \[
    v[j] := \begin{cases} 
    a & \text{if } j = i \\
    q & \text{if } j = i + 1 \\
    w[j] & \text{for all the other } j
    \end{cases}
    
- Output \( v \).

*It is also OK, if you assume a configuration may contain a blank symbol.

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(4) We will show that $\text{HALT} \leq_m L_{\text{fin}}$.

On input $[M]w$, do the following.

- Construct a TM $M'$ that works as follows.
  
  On input $v$:
  
  - Simulate $M$ on input $w$.
  - Accepts if and only if $M$ accepts $w$.

- Decide whether $[M'] \in L_{\text{fin}}$.

It is obvious that if $M$ accepts $w$, then $L(M') = \Sigma^*$. Otherwise, $L(M') = \emptyset$. In other words, $[M]w \in \text{HALT}$ if and only if $[M'] \in L_{\text{fin}}$.

(5) We show a reduction from CFL-Universality.

On input CFG $G$, do the following.

- Construct a DFA $A$ that accepts $\Sigma^*$.
- Decide whether $L(G) = L(A)$.

Obviously, $L(G) = \Sigma^*$ if and only if $L(G) = L(A)$.

(6) Recall the language $L_5 := \{[M] \mid L(M) \text{ is regular}\}$ defined in Lesson 11. We prove that it is undecidable by reduction from HALT.

The problem in question (6) can be proved similarly by reduction from the language HALT.

On input $[M]w$, do the following.

- Construct a TM $M'$ that works as follows.
  
  On input $v$:
  
  - Scan the string $v$ from left to right to determine whether $v$ is of the form $a^k b^l c^m$, for some integers $k, l, m$.
    
    If it is not, it rejects immediately.
  - If it is, it counts the number of $a$'s, the number of $b$'s and the number of $c$'s.
    
    If they are not all the same, it rejects immediately.
    
    If they are the same, simulate $M$ on input $w$.
  - Accepts if and only if $M$ accepts $w$.

- Decide whether $L(M')$ is a CFL.

It is obvious that if $M$ accepts $w$, then $L(M') = \{a^n b^n c^n \mid n \geq 0\}$, which is not CFL. Otherwise, $L(M') = \emptyset$, which is CFL. In other words, $[M]w \in \text{HALT}$ if and only if $L(M')$ is not CFL.