Lesson 12: Basic time and space complexity classes

Theme: Classification of languages/problems according to number of steps (time) and cells (space) needed by Turing machines to decide them.

In the following let \( \mathbb{N} \) denote the set of natural numbers \( \{0, 1, 2, \ldots\} \), and \( f, g \) are functions from \( \mathbb{N} \) to \( \mathbb{N} \). Recall the big ‘oh’ notation \( f(n) = O(g(n)) \), which means that there is \( c > 0 \) and \( n_0 \) such that for every \( n \geq n_0 \),

\[
f(n) \leq c \cdot g(n).
\]

1 Time complexity

Deterministic Turing machines. We say that a DTM \( M \) accepts \( w \) in \( N \) steps, if \( M \) accepts \( w \) and the length of the accepting run is \( N \):

\[
C_0 \vdash C_1 \vdash \cdots \vdash C_N \quad \text{where } C_N \text{ is an accepting configuration.}
\]

Likewise, we can say that \( M \) rejects \( w \) in \( N \) steps, if \( M \) rejects \( w \) and the length of the rejecting run is \( N \):

\[
C_0 \vdash C_1 \vdash \cdots \vdash C_N \quad \text{where } C_N \text{ is a rejecting configuration.}
\]

We say that \( M \) decides \( w \) in \( N \) steps, if it either accepts or rejects \( w \) in \( N \) steps. Equivalently, we may also say that on input \( w \), \( M \) runs in \( N \) steps.

**Definition 12.1** Let \( f : \mathbb{N} \to \mathbb{N} \). We say that a DTM \( M \) decides a language \( L \) in time \( O(f(n)) \), if there is a constant \( c > 0 \) and a constant \( n_0 \) such that for every word \( w \) with length \( \geq n_0 \), \( M \) decides \( w \) in \( \leq c \cdot f(|w|) \) steps.

**Definition 12.2** For a function \( f : \mathbb{N} \to \mathbb{N} \), we define the class \( \text{DTIME}[f(n)] \) as follows.

\[
\text{DTIME}[f(n)] := \{L \mid \text{there is a DTM } M \text{ that decides } L \text{ in time } O(f(n))\}
\]

Of particular interest are the classes \( \text{DTIME}[n], \text{DTIME}[n^2], \text{DTIME}[n^3], \ldots \). The class \( \mathbf{P} \) is defined as follows.

\[
\mathbf{P} := \bigcup_{k \geq 1} \text{DTIME}[n^k]
\]

Note that the class \( \mathbf{P} \) is closed under complement, union and intersection.

Non-deterministic Turing machines. We say that an NTM \( M \) accepts \( w \) in \( N \) steps, if there is an accepting run of \( M \) on \( w \) with length \( N \):

\[
C_0 \vdash C_1 \vdash \cdots \vdash C_N \quad \text{where } C_N \text{ is an accepting configuration.}
\]

We say that \( M \) decides \( w \) in \( N \) steps, if **every** run of \( M \) on \( w \) is of length \( \leq N \).
Definition 12.3 Let \( f : \mathbb{N} \to \mathbb{N} \). We say that an NTM \( M \) decides a language \( L \) in time \( O(f(n)) \), if there is a constant \( c > 0 \) and a constant \( n_0 \) such that for every word \( w \) with length \( \geq n_0 \), \( M \) decides \( w \) in \( \leq c \cdot f(|w|) \) steps.

Definition 12.4 For a function \( f : \mathbb{N} \to \mathbb{N} \), we define the class \( \text{Ntime}[f(n)] \) as follows.

\[
\text{Ntime}[f(n)] := \{ L \mid \text{there is an NTM } M \text{ that decides } L \text{ in time } O(f(n)) \}
\]

Of particular interest are the classes \( \text{Ntime}[n], \text{Ntime}[n^2], \ldots \). The class \( \text{NP} \) is defined as follows.

\[
\text{NP} := \bigcup_{k \geq 1} \text{Ntime}[n^k]
\]

Of particular interest is also the following class.

\[
\text{coNP} := \{ L \mid \Sigma^* - L \in \text{NP} \}
\]

2 Space complexity

Definition 12.5 Let \( f : \mathbb{N} \to \mathbb{N} \).

- A DTM \( M \) decides a language \( L \) in space \( O(f(n)) \), if there is a constant \( c > 0 \) and a constant \( n_0 \) such that for every word \( w \) with length \( \geq n_0 \), the run of \( M \) on \( w \) uses/visits \( \leq c \cdot f(|w|) \) cells of its tape.
- An NTM \( M \) decides a language \( L \) in space \( O(f(n)) \), if there is a constant \( c > 0 \) and a constant \( n_0 \) such that for every word \( w \) with length \( \geq n_0 \), every run of \( M \) on \( w \) uses/visits \( \leq c \cdot f(|w|) \) cells of its tape.

For a function \( f : \mathbb{N} \to \mathbb{N} \), we can define the class \( \text{Dspace}[f(n)] \) and \( \text{Nspace}[f(n)] \) as follows.

\[
\text{Dspace}[f(n)] := \{ L \mid \text{there is a DTM } M \text{ that decides } L \text{ in space } O(f(n)) \}
\]

\[
\text{Nspace}[f(n)] := \{ L \mid \text{there is an NTM } M \text{ that decides } L \text{ in space } O(f(n)) \}
\]

\[
\text{coNspace}[f(n)] := \{ L \mid \Sigma^* - L \in \text{Nspace}[f(n)] \}
\]

Polynomial space. Of particular interest are the classes \( \text{Dspace}[n^k] \) and \( \text{Nspace}[n^k] \). The classes \( \text{PSPACE} \) and \( \text{NPSPACE} \) are defined as follows.

\[
\text{PSPACE} := \bigcup_{k \geq 1} \text{Dspace}[n^k]
\]

\[
\text{NPSPACE} := \bigcup_{k \geq 1} \text{Nspace}[n^k]
\]

\[
\text{coNPSPACE} := \bigcup_{k \geq 1} \text{coNspace}[n^k] = \{ L \mid \Sigma^* - L \in \text{NPSPACE} \}
\]
Logarithmic space. Another interesting classes are $L$ and $NL$. We say that a language $L$ is in $L$, if there is a 2-tape DTM $M$ that decides $L$ such that on input word $w$:

- The first tape always contains only the input word $w$, i.e., $M$ never changes the content of the first tape.
- $M$ uses/visits $\leq c \cdot \log(|w|)$ cells of its second tape.

Likewise, we say that a language $L$ is in $NL$, if there is a 2-tape NTM $M$ that decides $L$ such that the above two conditions are satisfied.

3 Some classic complexity results

Obviously, we have $L \subseteq NL, P \subseteq NP$, and $PSPACE \subseteq NPSPACE$.

Proposition 12.6

- $L \subseteq P$.
- $NP \subseteq PSPACE$.

The following are classic results in complexity theory. (We will not prove them in the class.)

- $NL \subseteq P$.
- If $L \in NSPACE[n^k]$, then $\Sigma^* - L \in NSPACE[n^k]$.
- $NSPACE[n^k] \subseteq DSPACE[n^{2k}]$.

The second bullet implies that $coNSPACE[n^k] = NSPACE[n^k]$, and hence, $NPSPACE = coNPSPACE$. The third bullet implies that $NPSPACE = PSPACE$.

Combining all these inclusions together, we obtain:

$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE$

It is also known that $L \nsubseteq PSPACE$ (which we will not prove in the class). In fact, we also know that $NL \nsubseteq PSPACE$. So, we know that at least one of the inclusions must be strict, but we don’t know which one.