Lesson 1: Preliminaries

Theme: Review of some basic facts from discrete mathematics.

1 Sets

- A set is a collection of things, which are called its members or elements.
  
  \( a \in X \) (read: \( a \) is in \( X \), or \( a \) belongs to \( X \)) means \( a \) is a member or an element of \( X \), whereas \( a \notin X \) means \( a \) is not a member of \( X \).

- An empty set is denoted by \( \emptyset \).

- \( X \) is a subset of \( Y \), denoted by \( X \subseteq Y \), if every element of \( X \) is also an element of \( Y \).

- \( X \) is a proper subset of \( Y \), denoted by \( X \subset Y \), if \( X \neq Y \) and \( X \subseteq Y \).

- For two sets \( X \) and \( Y \), we write \( X \cap Y \) and \( X \cup Y \) to denote their intersection and union, respectively.

- The cartesian product between two sets \( X \) and \( Y \) is the following.
  
  \[ X \times Y := \{(a, b) \mid a \in X \text{ and } b \in Y\} \]

  We write \( X^n \) to denote \( X \times \cdots \times X \), where \( X \) appears \( n \) time.

2 Relations

- A relation \( R \) over two sets \( X, Y \) is a subset of \( X \times Y \).

- A binary relation \( R \) over \( X \) is a subset of \( X \times X \).

- An \( n \)-ary relation \( R \) over \( X \) is a subset of \( X^n \).

3 Functions

- A relation \( R \) over \( X, Y \) is a function or a mapping, if for every \( x \in X \), there is exactly one \( y \in Y \) such that \((x, y) \in R\).
  
  In this case, we will say \( R \) is a function from \( X \) to \( Y \), or \( R \) maps \( X \) to \( Y \). We denote it by \( R : X \rightarrow Y \).

- We will usually use the letters \( f, g, h, \ldots \) to represent functions. As usual, we write \( f(x) \) to denote the element \( y \) in which \((x, y) \in f\).

- A function \( f : X \rightarrow Y \) is an injective function, if for every \( y \in Y \), there is at most one \( x \in X \) such that \( f(x) = y \). An injective functions is also called an injection.

- A function \( f : X \rightarrow Y \) is a surjective function, if for every \( y \in Y \), there is at least one \( x \in X \) such that \( f(x) = y \).

- A function \( f : X \rightarrow Y \) is a bijection, if it is both injective and surjective.
4 Equivalence relations

The symbol $\sim$ is reserved to denote a special relation, called equivalence relation. Using the standard notation, we write $x \sim y$ to mean that the pair $(x, y)$ belongs to the relation $\sim$.

Recall that $\sim$ being an equivalence relation (over some set, say, $X$) means it satisfies the following conditions.

- Reflexive: $x \sim x$, for every $x \in X$.
- Symmetric: $x \sim y$ if and only if $y \sim x$, for every $x, y \in X$.
- Transitive: for every $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

For $x \in X$, the equivalence class of $x$ in $\sim$ is defined as:

$$[x]_\sim := \{ y \mid x \sim y \}$$

**Theorem 1.1** If $\sim$ is an equivalence relation over $X$, then the following holds.

- $[x]_\sim = [y]_\sim$ if and only if $x \sim y$.
- If $[x]_\sim \neq [y]_\sim$, then $[x]_\sim \cap [y]_\sim = \emptyset$.
- The equivalence classes of $\sim$ partition $X$, i.e., every member of $X$ belongs to exactly one equivalence class of $\sim$.

5 Countable and uncountable sets

Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, \ldots\}$. A set $X$ is countable, if there is an injective function from $X$ to $\mathbb{N}$. Otherwise, it is called an uncountable set.

**Theorem 1.2** The following sets are all countable.

- The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers.
- The set $\mathbb{N}^k$, for every integer $k \geq 1$.
- The set $\mathbb{N}^* := \bigcup_{k \geq 1} \mathbb{N}^k$.

**Theorem 1.3** The set $2^\mathbb{N}$ is uncountable.

6 The notion of alphabets and languages

- An alphabet is a finite set of symbols. We usually use the symbol $\Sigma$ to denote an alphabet.
- A (finite) string/word over $\Sigma$ is a finite sequence of symbols from $\Sigma$.
- We will usually write $w = a_1 \ldots a_n$ to denote a word whose label in position $i$ is $a_i$. The length of $w$ is denoted by $|w|$.
- We write $\varepsilon$ to denote the empty string/word, i.e., the word of length 0.
- For an integer $n \geq 0$, $\Sigma^n$ denotes all the words over $\Sigma$ of length $n$.
- $\Sigma^*$ denotes the set of all finite words over $\Sigma$, i.e., $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$.
- A language $L$ over $\Sigma$ is a subset of $\Sigma^*$.

**Theorem 1.4** For every alphabet $\Sigma$, the set $\Sigma^*$ is countable.
Appendix

In this course it is important to be able to read mathematical/formal statements. It will take a while to get used to them. One important aspect of a formal statement is its use of “quantifiers.” Consider the following statement.

Every student stays in a dormitory room. \hspace{2cm} (1)

If we want to write in strict logical form, we will have to write it in the following way.

For every student \(x\), there is a dormitory room \(y\) such that \(x\) stays in \(y\).

“For every” and “there exists” in the above sentence are called quantifiers.

The negation of statement (1) is:

There is student \(x\), for every dormitory room \(y\) such that \(x\) does not stays in \(y\). \hspace{2cm} (2)

Note also that neither (1) nor (2) are equivalent to the following sentence:

There is student \(x\), for every dormitory room \(y\) such that \(x\) stays in \(y\). \hspace{2cm} (3)