Mar. 22/2000 by LaTeX

National Taiwan University
Computer Science & Information Engineering
Yuh-Daun Lyuu

Derivatives Pricing
Computational Techniques in


Outline

1. Computational complexity

2. Trade against the central bank

3. Derivatives pricing with combinatorics

4. The differential tree approach to model calibration

5. Monte Carlo pricing

6. Path-dependent options pricing

7. Looking into the future
Other published and unpublished papers


References

Yuh-Daun Lyu, Financial Engineering and
—Fischer Black

And I lost the least.

Professor Merton lost the most money.

Instead in warrants,

When Professor Scholes and Merton and I
—Leibniz
in the labor of computation.
to lose hours like slaves
It is unworthy of excellent men

Computational Complexity

Part I:
or path-dependent derivatives

- Maybe an issue for long-dated fixed-income securities

2. Memory

- Simulation, etc.
  - Approximation algorithms, Monte Carlo
  - Approaches: analytical approximations
  - Candidates: Asian options & certain reset options

- Interactable: otherwise

\[
O(n^2) \quad \text{and} \quad O(n)O
\]

- Tractable: "solvable" in polynomial time such as

Measures of Complexity
Part 2: Trade against the Central Bank
assumptions (and therefore more robust) Result is (almost) model-free (no distribution

This adversary trades once, at the highest rate compared against the adversary with complete foresight. Goal: maximize the total JPY amount on day n as

(buy-and-hold only) Converting JPY back to USD is not allowed. The trader acts on each exchange rate. Exchange rates will be revealed. Applicable to any assets with relative prices. The trader wants to trade USD for JPY (say).

Competitive Analysis
- Convert too much and future exchange rates go up
- Trader's Dilemma
The least $c$ that $A$ achieves is called its competitive ratio. 

- $c < 1$: the lower the better
- $c > 1$: the lower the better
- $c = 1$: optimal

\[
\frac{c}{\text{Opt}} \geq \mathbb{E}[A]
\]

A trading algorithm $A$ is $c$-competitive if for any rate sequence, it guarantees a JPY amount at least $1/c$ of the adversary’s amount, i.e.,

\[
A \text{-algorithm } A \text{ is } c \text{-competitive if for any rate}
\]
Brownian motion (used in finance) [Hull 1999]

- Related to the popular lognormal process (geometric)
  - Results available for the general \([r/a, r/b]\) case
  - For the Taipei Stock Exchange \([r, 0.7] \approx \theta\)
  - Geometric upper and lower bounds

The Model
Strategy

Indirect support for the soundness of dollar-averaging

\[ \frac{u-\theta-1}{(1-\theta-1)u} \]

competitive ratio is whose dollar-averaging strategy, whose

Beat the popular dollar-averaging strategy [Chen, Kao, Lyuu, Wong 1999]

can attain: \( \frac{1+\theta}{(z-u)-\theta u} \) achieves the optimal competitive ratio any algorithm

Achieves the optimal competitive ratio any algorithm

\( (z-u)-\theta u \)

Invest dollar on the first and last days

\( z-u \)

Invest dollar on the other days

The optimal strategy per USD:

The Optimal Buy-and-Hold Trading Strategy
—Merton H. Miller

[...] the center of gravity of finance

The shift toward options as

Derivatives Pricing with Combinatorics

Part 3:
<table>
<thead>
<tr>
<th>Name</th>
<th>Futures contracts</th>
<th>Month end open interest</th>
<th>Trading volume</th>
<th>Contracts settled</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dow Jones Industrial Index</td>
<td>CME/IMM, CBT</td>
<td>14,494</td>
<td>3,905,262</td>
<td>31,293</td>
</tr>
<tr>
<td>S&amp;P 500 Index</td>
<td>CME/IMM, CBT</td>
<td>838,403</td>
<td>114,945,293</td>
<td>55,595</td>
</tr>
<tr>
<td>3-month Eurodollar</td>
<td>CME/IMM, CBT</td>
<td>107,386,746</td>
<td>181,051,919</td>
<td>1,556,484</td>
</tr>
<tr>
<td>Treasury bonds</td>
<td>CME/IMM, CBT</td>
<td>2,961,562</td>
<td>8,732,915</td>
<td>500,562,510</td>
</tr>
<tr>
<td>Chicago Mercantile Exchange (CME) and IMM</td>
<td>CME/IMM, CBT</td>
<td>2,602,372</td>
<td>556,213</td>
<td>4,186,906</td>
</tr>
<tr>
<td>Total all exchanges</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dow Jones Industrial Index</td>
<td>CME/IMM, CBT</td>
<td>838,403</td>
<td>114,945,293</td>
<td>55,595</td>
</tr>
<tr>
<td>S&amp;P 500 Index</td>
<td>CME/IMM, CBT</td>
<td>107,386,746</td>
<td>181,051,919</td>
<td>1,556,484</td>
</tr>
<tr>
<td>3-month Eurodollar</td>
<td>CME/IMM, CBT</td>
<td>2,961,562</td>
<td>8,732,915</td>
<td>500,562,510</td>
</tr>
<tr>
<td>Treasury bonds</td>
<td>CME/IMM, CBT</td>
<td>2,602,372</td>
<td>556,213</td>
<td>4,186,906</td>
</tr>
<tr>
<td>Chicago Mercantile Exchange (CME) and IMM</td>
<td>CME/IMM, CBT</td>
<td>2,602,372</td>
<td>556,213</td>
<td>4,186,906</td>
</tr>
<tr>
<td>Total all exchanges</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total CME/IMM</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chicago Board of Trade (CBO)</td>
<td>CME/IMM, CBT</td>
<td>2,398,298</td>
<td>61,369,819</td>
<td>8,073,479</td>
</tr>
<tr>
<td>Dow Jones Industrial Index</td>
<td>CME/IMM, CBT</td>
<td>39,706</td>
<td>3,049,771</td>
<td>31,842,995</td>
</tr>
<tr>
<td>S&amp;P 500 Index</td>
<td>CME/IMM, CBT</td>
<td>274,655</td>
<td>5,049,771</td>
<td>124,107,563</td>
</tr>
<tr>
<td>3-month Eurodollar</td>
<td>CME/IMM, CBT</td>
<td>37,947,756</td>
<td>8,073,479</td>
<td>3,049,771</td>
</tr>
<tr>
<td>Treasury bonds</td>
<td>CME/IMM, CBT</td>
<td>2,398,298</td>
<td>5,049,771</td>
<td>124,107,563</td>
</tr>
<tr>
<td>Chicago Mercantile Exchange (CME) and IMM</td>
<td>CME/IMM, CBT</td>
<td>2,602,372</td>
<td>556,213</td>
<td>4,186,906</td>
</tr>
<tr>
<td>Total all exchanges</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total CME/IMM</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Backward induction

Variations

\[
\max_{S, t} \left( 0, u S_t - X_t \right)
\]

The put option has a terminal payoff given by

\[
\max_{S, t} \left( 0, X_t - u S_t \right)
\]

The call option has a terminal payoff given by

\( S_0, S_1, \ldots, S_T \) denote the prices of the underlying asset

Calls and Puts
\[
\frac{z}{1} = d \\
\n\frac{\nabla^\omega - \nabla (z_{\omega - \nu})^\omega}{\nu^\omega + \nabla (z_{\omega - \nu})^\omega} = p \\
\frac{n}{1} = p \\
\n\rightarrow \text{The Jarow-Rudd (JR) version:} \\
\frac{(p - n)}{(p - \nabla^\omega)} = d \\
\frac{n}{1} = p \\
\n\rightarrow \text{The Cox-Ross-Rubinstein (CRR) version:} \\
\n\text{Stock price can go from } S \text{ to } S_n \text{ with probability } p \text{ or } \}
\]
– Mathematically true.

than pricing American options by an order of magnitude.

A rule of thumb: pricing European options is faster.

– Shortcoming: cannot handle American options.

\[ (u) \mathcal{O} \] time.

\[ (u^2) \mathcal{O} \] time.

Solving the Black-Scholes differential equation takes

\[ (u) \mathcal{O} \] time.

\[ (u^2) \mathcal{O} \] time.

Standard backward induction takes time.

Barrier Option Pricing.
The Reflection Principle for Binomial Random Walk
\[ q + v + u \quad \text{for even } u \quad \left( \frac{q + v + u}{u} \right) \]

Answer: •

How many paths touch the x-axis?

\[ 0 \leq q, v \quad \text{where } \quad q, v \quad \text{are lattice that is to reach } (u, 0), (0, -v) \quad \text{on the integral} \]

Imagine a particle at position (0, 0) on the integral

The Reflection Principle
May introduce fluctuations as well

is the new strike price

is the new barrier

\[
\left[ \frac{\gamma}{u} + \frac{\nabla \sqrt{\omega}}{(S/H) \ln} \right] \equiv \eta \quad \text{and} \quad \left[ \frac{\gamma}{u} + \frac{\nabla \sqrt{\omega}}{(S/X) \ln} \right] \equiv \alpha
\]

Let

Assume without loss of generality \( S > H \) — Assume the barrier is touched — We focus on the down-and-in call with barrier

Single-Barrier Options
The analytical value is 5.6605

Convergence of the Binomial Model
The Combinatorial Formula

\[
\begin{aligned}
&\text{can be summed in } O(n) \text{ steps} \\
&\left( X - \varepsilon_{-u} \mathcal{P}_{\varepsilon} \mathcal{S} \right) \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{The option value equals} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \\
&\text{hits the barrier with probability} \quad \varepsilon_{-u} \mathcal{P}_{\varepsilon} \mathcal{S} \\
&\text{is reached by a path that has} \quad \varepsilon_{-u} \mathcal{P}_{\varepsilon} \mathcal{S} \\
&\text{so the terminal price} \quad \varepsilon_{-u} \mathcal{P}_{\varepsilon} \mathcal{S} \\
&\text{the terminal price is} \quad \varepsilon_{-u} \mathcal{P}_{\varepsilon} \mathcal{S} \\
&\text{there are} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{of them hit} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{there are paths} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{hit them} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{of paths} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{probability} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{probability} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{each path from } \mathcal{S} \text{ to the terminal price} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{Each path has} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{probability} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{probability} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
&\text{probability} \quad \varepsilon_{-u} (d - 1) \varepsilon^d \left( \varepsilon + \varepsilon^2 - u \right) \sum_{\varepsilon} \mathcal{E} \\
\end{aligned}
\]
of DRAM, running Windows NT 4.0

Analytical value 3.66097 100 MHz Intel Pentium processor and 32 MB

<table>
<thead>
<tr>
<th>Value</th>
<th>Time</th>
<th>Value</th>
<th>Time</th>
<th>Value</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>130000.0</td>
<td>5.66093</td>
<td>51.60</td>
<td>3.66097</td>
<td>44.10</td>
<td>4.190</td>
</tr>
<tr>
<td>92000.0</td>
<td>5.66046</td>
<td>43.70</td>
<td>3.66098</td>
<td>36.70</td>
<td>3.613</td>
</tr>
<tr>
<td>67500.0</td>
<td>5.66048</td>
<td>36.70</td>
<td>3.66099</td>
<td>30.78</td>
<td>2.877</td>
</tr>
<tr>
<td>48800.0</td>
<td>5.66049</td>
<td>30.78</td>
<td>3.66092</td>
<td>30.20</td>
<td>2.388</td>
</tr>
<tr>
<td>34800.0</td>
<td>5.66051</td>
<td>24.70</td>
<td>2.3400.0</td>
<td>24.70</td>
<td>2.138</td>
</tr>
<tr>
<td>15400.0</td>
<td>5.66044</td>
<td>19.40</td>
<td>1.731</td>
<td>5.66971</td>
<td>15.00</td>
</tr>
<tr>
<td>9600.0</td>
<td>5.66043</td>
<td>15.00</td>
<td>1.368</td>
<td>5.66038</td>
<td>11.10</td>
</tr>
<tr>
<td>5700.0</td>
<td>5.66033</td>
<td>11.10</td>
<td>1.047</td>
<td>5.65682</td>
<td>7.68</td>
</tr>
<tr>
<td>3080.0</td>
<td>5.66013</td>
<td>8.00</td>
<td>0.768</td>
<td>5.65468</td>
<td>3.33</td>
</tr>
<tr>
<td>1440.0</td>
<td>5.66033</td>
<td>5.60</td>
<td>0.333</td>
<td>5.65223</td>
<td>3.42</td>
</tr>
<tr>
<td>590.0</td>
<td>5.65250</td>
<td>3.60</td>
<td>3.42</td>
<td>5.63541</td>
<td>2.00</td>
</tr>
<tr>
<td>185.0</td>
<td>5.65082</td>
<td>2.00</td>
<td>1.191</td>
<td>5.6354</td>
<td>2.00</td>
</tr>
<tr>
<td>35.0</td>
<td>5.63436</td>
<td>0.30</td>
<td>0.84</td>
<td>5.59759</td>
<td>8.21</td>
</tr>
</tbody>
</table>

Compared with the Trinomial Model (in milliseconds)
No problem if we use an \( O(n) \)-time algorithm.

But then the \( n^2 \) bound becomes too high – we will have to be very large to tackle fluctuations.

Some claimed it makes the binomial model impractical.

When the current stock price is near the barrier
Reflect the path first at \( f \) and then at \( H \).

Must hit both barriers (can \( H \)-hit preceded by an \( f \)-hit).

The Reflection Principle—Iterated
\[ q - \varnothing + \mu \quad \text{for even } \left( \frac{2^{\frac{2}{2}}}{{\mu}^{2^{{q-\varnothing}}+\mu}} \right) \]

\( s - = x \) (appears before a hit of the \( T \)-line is \( 0 = x \))

The number of paths in which a hit of the \( H \)-line appears before a hit of the \( T \)-line

either barrier is hit (knock-in type)

Consider options that come into existence if and only if

\[ H > T \] with

Double barrier options contain two barriers and

Double Barrier Options
The running time is \( O(n) \).

\[
(\|B\| + \|A\|)_{1-\varepsilon} (1 - \varepsilon) \sum_{u=1}^{I} (s', q, \nu) N
\]

The number of paths that hit either barrier equals

\[
\exists \mathcal{Z} \subseteq \underbrace{T + H + I}_{\nu} \text{ a sequence containing } \mathcal{Z}
\]

Let \( B \) denote the set of paths that hit the barriers with

\[
\exists \mathcal{Z} \subseteq \underbrace{H + I + H}_{\nu} \text{ a hit sequence containing } \mathcal{Z}
\]

Let \( A \) denote the set of paths that hit the barriers with

\[ H + \underbrace{I + H}_{\nu} \text{ a sequence of } Hs \text{ and a sequence of } T \]

The Combinatorial Picking Formula
The Combinatorial Picking Formula (continued)
 constitute a standard call; add this to the above
The terminal nodes outside the above-mentioned range

\[ (X - \xi - uP_{\xi} \eta S)_{\xi - u}(d - 1)_{\xi} d ((\eta - \eta)z, 2\eta - \eta, 2\eta - \eta) \bigcap_{\eta} \bigcup_{d} \bigcup_{e} \]

to the option value

together contribute
(The terminal nodes between (inclusive) \( \mathcal{H} \) and \( \mathcal{I} \))

\[ \eta - uP_{\eta} \eta S \equiv \mathcal{I} \quad \text{and} \quad \eta - uP_{\eta} \eta S \equiv \mathcal{H} \]

The barriers are replaced by the barriers

\[ \left[ \frac{z}{u} + \frac{\xi \vee^0 z}{(S/T)z} \right] \equiv \eta \quad \left[ \frac{z}{u} + \frac{\xi \vee^0 z}{(S/H)z} \right] \equiv \eta \]

Define

The Combinatorial Picking Formula (continued)
Figure 1: Comparison of combinatorial and backward methods in running time

Payoff is max\( \text{sum}\left\{ S_i - \text{min}_j S_{i,j} \right\} = 0 \)

Lookback Option
variates approach for pricing arithmetic Asian options

• Needed in some approximation algorithms and control

• Can be priced in time $O(n)$ using backward induction

$$\max(0, X - \frac{1+u}{t}(uS \cdots \cdot S_0S_1 \cdots S_0))$$

The Asian call has a terminal payoff given by

$S_0, S_1, \cdots, S^n$ denote the prices of the underlying asset

Pricing Geometric Asian Options
\[
\left(0, X - \frac{1+u}{t}\left(\frac{u-\varepsilon}{(1+u)v} + \mathcal{P}(n)\right)\right)_{\text{max}} (\mathbf{w}) b_{u-\varepsilon} \sum_{\varepsilon/(1+u)v}^{0=\varepsilon} \text{Value is then} 
\]

\[
x(\mathbf{w}) b_{\varepsilon} \sum_{\varepsilon/(1+u)v}^{0=\varepsilon} = (u x + 1) \cdots (x + 1)(x + 1)(x + 1)
\]

Define with \((1), (0)b, (1)b, (0)b\) each move has identical probability \(1/2\) – use the Jarrow-Rudd binomial model.

The Combinatorial Approach
Comparison with Backward Induction (in seconds)
—Grimm’s Fairy Tales

to slip through it.
to find out if his body was still thin enough
by which they had come in,
The fox often ran to the hole

The Differential Tree Approach to Model Calibration

Part 4:
<table>
<thead>
<tr>
<th>Year</th>
<th>Backed &amp; Asset-Backed MBS</th>
<th>Agency MBS</th>
<th>Agency CMOs</th>
<th>Agency RMBS</th>
<th>Fed Money</th>
<th>Fed Home Loans</th>
<th>Fed Commercial Mortgage Loans</th>
<th>Over-the-Counter ABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985</td>
<td>895.2 895.2</td>
<td>887.9</td>
<td>847.4</td>
<td>833.7</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1986</td>
<td>920.4 920.4</td>
<td>876.3</td>
<td>840.5</td>
<td>819.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1987</td>
<td>1,010.4 1,010.4</td>
<td>949.9</td>
<td>817.2</td>
<td>792.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1988</td>
<td>1,082.3 1,082.3</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1989</td>
<td>1,135.2 1,135.2</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1990</td>
<td>1,184.3 1,184.3</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1991</td>
<td>1,227.2 1,227.2</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1992</td>
<td>1,274.1 1,274.1</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1993</td>
<td>1,372.5 1,372.5</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1994</td>
<td>1,441.7 1,441.7</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1995</td>
<td>1,570.4 1,570.4</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1996</td>
<td>1,715.0 1,715.0</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1997</td>
<td>1,874.3 1,874.3</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1998</td>
<td>1,933.5 1,933.5</td>
<td>979.8</td>
<td>839.6</td>
<td>820.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Outstanding U.S. Debt Market Securities (Trillion)
How to find these derivatives efficiently?

\[
\frac{\partial}{\partial \theta} \left( \cdots \partial \phi \frac{\partial}{\partial \theta} \phi \right) = \frac{\partial}{\partial \theta} \left( \cdots \partial \phi \frac{\partial}{\partial \theta} \phi \right)
\]

Fast root finding usually requires derivatives:

spread, option-adjusted spread, etc.

- Implied volatility, interest rate tree calibration,

\[
\text{for } \phi \quad d = \left( \cdots \partial \phi \frac{\partial}{\partial \theta} \phi \right)
\]

**Calibration** fundamentally is root finding: solve

- Given \( \phi \)

**Picnic** is basically function evaluation: \( \phi(x, \theta) \)

Calibration and Picnic
\[
\frac{xf}{(x)c} + \frac{xq}{(x)q} + \frac{xp}{(x)p} = \frac{xp}{f p}
\]

\[
((x)c \cdot (x)q \cdot (x)p)^f
\]

- Chain rule
- Computation at A is driven by inputs from B and C
- Given a backward induction tree for pruning

The Differential Tree Idea
The zero-coupon bond yield is described by \(0.06 + 0.05 \ln t\). Time partition is one period per year.
<table>
<thead>
<tr>
<th>Year</th>
<th>Time</th>
<th>Efficiency in Calibrating BDT (in seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2015</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2016</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2017</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2018</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2019</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2020</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2021</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2022</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2023</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2024</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2025</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2026</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2027</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2028</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2029</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2030</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2031</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2032</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2033</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2034</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2035</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2036</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2037</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2038</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2039</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>2040</td>
<td>315</td>
<td>39.88</td>
</tr>
<tr>
<td>Iterations</td>
<td>Time</td>
<td>Partitions</td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
<td>------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1850</td>
<td>10617.370</td>
</tr>
<tr>
<td>5</td>
<td>9533.900</td>
<td>17500</td>
</tr>
<tr>
<td>5</td>
<td>8502.950</td>
<td>16500</td>
</tr>
<tr>
<td>5</td>
<td>15500</td>
<td>7548.760</td>
</tr>
<tr>
<td>5</td>
<td>6589.360</td>
<td>14500</td>
</tr>
<tr>
<td>5</td>
<td>5714.440</td>
<td>13500</td>
</tr>
<tr>
<td>5</td>
<td>4912.680</td>
<td>12500</td>
</tr>
<tr>
<td>5</td>
<td>4169.570</td>
<td>11500</td>
</tr>
<tr>
<td>5</td>
<td>3503.410</td>
<td>10500</td>
</tr>
</tbody>
</table>

Efficiency in Calculating Spread (in seconds)
Efficiency in Calculating Implied Volatility (in seconds)
Part 5: Monte Carlo Pricing
• Quasi-Monte Carlo: jury still out

• Efficiency remains an issue

• Promising applicability to American-style options

• Mortgage-backed securities and multivariate options

• Maybe the only viable method for complex securities

• Bounded is only probabilistic (no guarantee)

• Average the replications

• Simulation of the underlying asset price

Ideas and Facts
to have the lowest variance

For many path-dependent options, control variates seem

Arithmetically, control variates, conditioning

often necessary to improve convergence

Variance reduction (efficiency improving) schemes are

\( O(1/\sqrt{N}) \)

of Crude Monte Carlo, converges relatively slowly, at a rate

Variance Reduction Schemes
Part 6: Path-Dependent Options Pricing
Monte Carlo simulation, etc.
• Analytical approximations, approximate algorithms,
  • No natural options have been identified as such yet
  hard to price
  • Theory says there are derivatives which are provably
  • Arithmetic Asian options, e.g.
  • Other path-dependent derivatives seem hard to price
  geometric Asian options, etc.
  • Barrier-type options, (simple) reset options,
  • Some path-dependent derivatives are easy to price

Issues
Want to calculate the expected payoff

\[
(0, S^{\frac{0=t}{0=t}} S^{\frac{I + u}{I}} - X) \max
\]

- Arithmetic Asian put's terminal payoff:

\[
(0, X - S^{\frac{0=t}{0=t}} S^{\frac{I + u}{I}}) \max
\]

- Arithmetic Asian call's terminal payoff:

\[
S^{\frac{0}{0}} S^{\frac{1}{1}} \cdots S^{u} \text{ denote the prices of the underlying asset}
\]

Asian Option Defined
That leaves us exact and approximate algorithms.

Analytical approximations fail under some circumstances.

Quasi-Monte Carlo is not well understood.

Mostly to European options.

Monte Carlo: no control over the error and limited.

It seems almost every path has to be explored.

Due to the non-combinining of the tree, the time is in the order of $O(2^n)$.

The binomial model converges to the analytic value.

Issues
[Dai and Lyuu 1999]

A converging general-purpose quasi-polynomial-time

[000] Huang and Lyuu 2000

An unpublished result lowers the error bound to

— Similar to Hull-White, but analyzable

[000] Angworth, Motwani, and Oldham 2000

A recent \( O(n^2) \) algorithm for the binomial tree algorithm by at most

— Overpricing

— Interpolation on the price tree (see [Hull 1999])

The popular Hull-White algorithm of 1993

Approximation and Exact Algorithms
Note that $2^{141} \approx 3 \times 10^{42}$.

- DL can be executed comfortably at $u = 141$ if reduced.
- The extent of the exponential explosion is dramatically.

- Contrast this with Hull-White.
- Backward induction is carried out exactly.
- Convergence to the continuous-time model.
- Finite-precision binary numbers.
- A trinomial tree that guarantees all the asset prices to

There are memory optimization issues

(approximations) such as interpolation in Hull-White

– This key property relates backward induction of

at each node, the running subtotal, \( S_t \), is an integer

Since a sum of integers is an integer, the state variable

make sure every asset price on the tree is another integer

Multiply the stock price and the exercise price by \( 2^n \) to

Option value is homogeneous of degree one in the stock

More Details of the Dai-Lyuu Algorithm
A Grand Comparison

Hull-White
AMO
Dai-Lyu
Upper
Lower
year, and the option has a life of 0.5 year. The risk-free rate is 10% per year, the volatility is 0.3 per

The initial underlying asset value is $50, the exercise price is $70.

<table>
<thead>
<tr>
<th>Time</th>
<th>Value</th>
<th>Upper</th>
<th>Lower</th>
<th>Monte Carlo</th>
<th>Hull-White</th>
<th>AMO</th>
<th>D&amp;I-Yun</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.326170</td>
<td>0.324930</td>
<td>0.320767</td>
<td>0.32861144</td>
<td>141</td>
<td>0.32463966</td>
<td>130</td>
</tr>
<tr>
<td>2</td>
<td>0.326839</td>
<td>0.325458</td>
<td>0.320263</td>
<td>0.3276216</td>
<td>119</td>
<td>0.32621672</td>
<td>108</td>
</tr>
<tr>
<td>3</td>
<td>0.327743</td>
<td>0.324427</td>
<td>0.329614</td>
<td>0.3220728</td>
<td>97</td>
<td>0.32205736</td>
<td>96</td>
</tr>
<tr>
<td>4</td>
<td>0.328661</td>
<td>0.323828</td>
<td>0.3228873</td>
<td>0.3239254</td>
<td>86</td>
<td>0.32347476</td>
<td>85</td>
</tr>
<tr>
<td>5</td>
<td>0.324195</td>
<td>0.323094</td>
<td>0.323787</td>
<td>0.3223322</td>
<td>75</td>
<td>0.32264844</td>
<td>74</td>
</tr>
<tr>
<td>6</td>
<td>0.324167</td>
<td>0.323176</td>
<td>0.3224244</td>
<td>0.3220507</td>
<td>64</td>
<td>0.3222054</td>
<td>63</td>
</tr>
<tr>
<td>7</td>
<td>0.322646</td>
<td>0.3220507</td>
<td>0.3224244</td>
<td>0.3220507</td>
<td>53</td>
<td>0.3222054</td>
<td>52</td>
</tr>
<tr>
<td>8</td>
<td>0.322185</td>
<td>0.321940</td>
<td>0.3224244</td>
<td>0.3220507</td>
<td>42</td>
<td>0.3220542</td>
<td>41</td>
</tr>
</tbody>
</table>

Comparison with Monte Carlo and Hull-White.
<table>
<thead>
<tr>
<th>Years</th>
<th>Levy</th>
<th>DL</th>
<th>MC</th>
<th>HW</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.174</td>
<td>2.234</td>
<td>4.527</td>
<td>5.627</td>
</tr>
<tr>
<td>1.0</td>
<td>1.054</td>
<td>2.956</td>
<td>6.359</td>
<td>9.759</td>
</tr>
<tr>
<td>0.317</td>
<td>0.342</td>
<td>0.324</td>
<td>0.332</td>
<td>0.321</td>
</tr>
</tbody>
</table>

More Comparisons
Levy denotes Levy’s approach. Dai-Lyuu method with the number of periods equal to 30. Simulations (MC) are based on 100,000 trials. DL is the HW denotes the Hull-White algorithm. Monte Carlo 10% per year, and the volatility is 0.3 per year. The initial underlying asset value is 50, the risk-free rate is
The post-Widder method, and TW is the Thurnbull-Wakeman underlying asset, GE is the German-Eydeland method, PW is the exercise price is 2.0, S is the initial price of the

<table>
<thead>
<tr>
<th></th>
<th>0.055</th>
<th>0.172</th>
<th>0.219</th>
<th>0.055</th>
<th>0.036</th>
<th>0.172</th>
<th>0.219</th>
<th>0.355</th>
<th>0.350</th>
<th>0.351</th>
<th>0.350</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.351</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.172</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.219</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.055</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.036</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.172</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.219</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.055</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
<tr>
<td>0.036</td>
<td>0.48</td>
<td>0.479</td>
<td>0.365</td>
<td>0.364</td>
<td>0.364</td>
<td>0.172</td>
<td>0.219</td>
<td>0.355</td>
<td>0.350</td>
<td>0.351</td>
<td>0.350</td>
</tr>
</tbody>
</table>

Extreme case comparisons with many methods
Part 7:
Looking into the Future
How many data points are needed?

- Will this work for complex options?

Issues:

- Afterward, we only need to interpolate from the surface.
- May invest few rights‘ work in approximating the surface.
- If the surface of the function is reasonably smooth, we
  think of the option values as the range of a function.

Do we really have to compute it?
Conclusions

1. Guarded optimism: Inherent complexity is probably not
2. Much work remains to be done
3. Efficient algorithms allow more strategies to be explored
4. Derivative pricing draws ideas from many fields