# Pricing of Moving-Average-Type Options with Applications

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**Abstract**

Moving-average-type options are complex path-dependent derivatives whose payoff depends on the moving average of stock prices. This paper concentrates on two such options traded in practice: the moving-average-lookback option and the moving-average-reset option. Both options were issued in Taiwan in 1999, for example. The moving-average-lookback option is an option struck at the minimum moving average of the underlying asset's prices. This paper presents efficient algorithms for pricing geometric and arithmetic moving-average-lookback options. Monte Carlo simulation confirms that our algorithms converge quickly to the option value. The price difference between geometric averaging and arithmetic averaging is found to be small. As it takes much less time to price the geometric-moving-average version, it serves as a practical approximation to the arithmetic-moving-average version. When applied to the moving-average-lookback options traded on Taiwan's stock exchange, our algorithm gives almost exactly the issue prices. The numerical delta and gamma of the options reveal subtle behavior and have implications for hedging. The moving-average-reset option is struck at a series of decreasing contract-specified prices based on moving averages. Similar results are obtained for such options using the same methodology.

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# 1 Introduction

Path-dependent derivatives are derivative securities whose payoff depends nontrivially on the price history of the underlying asset. Some path-dependent derivatives such as lookback and reset options can be efficiently priced. Others, however, are known to be difficult to price as surveyed in Lyuu (2002). Options based on the arithmetic average of the underlying asset's prices—such as the (arithmetic) Asian option—are prominent examples. As these options typically lack simple closed-form solutions, the development of numerical algorithms is essential.

This paper focuses on moving-average-type options. They are path-dependent derivatives whose payoff depends on the moving average of the stock prices. For many investors, moving average is a popular technical measure for a short-term trend in or a fair value of the stock price. This paper concentrates on two such options found in practice: the moving-average-lookback option and the moving-average-reset option. The moving-average-lookback option is struck at the minimum moving average of the underlying stock prices over a contractual period. The moving-average-reset option is similar. It is struck at a series of decreasing contract-specified prices over a contractual period based again on the moving averages. These two options were traded on Taiwan's exchange, for example. In practice, both options use arithmetic average and are American-style. Our algorithmic methodology is the same for both options and is applicable to a wide variety of moving-average-type options.

The reset feature makes the option useful in portfolio insurance as it protects the investors amidst stock price declines. The advantages of using a moving average instead of the stock price alone, as in ordinary reset options, are (1) to mitigate the possibility of stock price manipulation, especially for thin or shallow markets, (2) to provide a strike price correlated with a perceived price trend or fair value, and (3) to lower the option price compared with ordinary reset options. These advantages make the products appealing to some investors.

Taiwan's listed options market was born in 1997. Table 1 documents its trading activities over the 1997–2001 period. Options whose strike prices are related to the arithmetic moving average of the underlying stock prices were first issued in 1997. The issuance of options with similar features was particularly active in 1999 and involved all the major players in the capital market. Among them, the most prominent examples are the moving-average-lookback options and the moving-average-reset options, the focus of the paper. Moving-average-type options made up a significant portion of the listed options market in 1999. In that year, a total of 13 moving-average-type options were issued. Their combined premium stood at 3.07 billion TWD, which was nearly 23% of the market (see Table 2).[1]

The price that comes with the above-mentioned advantages is complexity. The moving-average-lookback option contains the features of Asian, lookback, and reset options, while the moving-average-reset option contains the features of Asian and reset options. It is the combination of features and the adoption of a moving average that make the options so difficult to price. This paper will concentrate on the moving-average-lookback option (MAL, hereafter). The slightly simpler moving-average-reset option (MAR, hereafter) can be handled by the same methodology and hence will receive less coverage. At any rate, the general conclusions for MARs are similar to those for MALs.

Pricing moving-average-type options is an intricate problem. Throughout the paper $S_t$ denotes the stock price at date $t$. Let $m_t$ be the 6-day moving average at date $t$, where $t \geq 5$; thus

$$m_t = \frac{S_{t-5} + S_{t-4} + S_{t-3} + S_{t-2} + S_{t-1} + S_t}{6}.$$

Even given the previous day's moving average,

$$m_{t-1} = \frac{S_{t-6} + S_{t-5} + S_{t-4} + S_{t-3} + S_{t-2} + S_{t-1}}{6},$$

$m_t$ remains dependent on the earlier price $S_{t-6}$ because

$$m_t = m_{t-1} - \frac{S_{t-6} - S_t}{6}.$$

This shows that process $(m_t, S_t)$ is not Markovian. In contrast, the averaging process $(A_t, S_t)$ most relevant in pricing Asian options, where

$$A_t \equiv \frac{\sum_{i=0}^{t} S_i}{t+1},$$

is Markovian because

$$A_t = \frac{tA_{t-1} + S_t}{t+1}.$$

Recall that the payoff of the Asian option is a function of $A_t$. Moving average hence introduces complexity not present in Asian options.

Although process $(m_t, S_t)$ is not Markovian, the 6-day moving window $(S_{t-5}, S_{t-4}, S_{t-3}, S_{t-2}, S_{t-1}, S_t)$ clearly is. It is a moving window instead of the

[1]The average exchange rate in 1999 was 32.27 TWD/USD.

fixed window utilized by $A_t$ because the starting date of the 6-day averaging window moves with the passage of time. The 6-day stock prices certainly suffice to calculate the 6-day moving average $m_t$. The main reason the moving averaging process $(m_t, S_t)$ is non-Markovian is, therefore, that the *order* of the 6 stock prices comprising $m_t$ counts. The requirement to keep track of the order of individual stock prices complicates the design of numerical algorithms. This paper develops practical resolution-reduction methods to lower the amount of information needed to encode the states.

Absent analytic formulas, the Monte Carlo method is always available (see Boyle, Broadie, and Glasserman (1997)). But it suffers from the inability to handle early exercise. Recently, Longstaff and Schwartz (2001) have developed a least-squares Monte Carlo approach to tackle the early-exercise problem. A common disadvantage of both methods is that the resulting option value is probabilistic. The alternative deterministic tree method can handle early exercise. The challenge lies in designing efficient tree-based algorithms. This paper will develop algorithms based on the CRR binomial model. Monte Carlo simulation is used to verify the calculated prices. Besides being accurate, the algorithms converge quickly to the option value. Interestingly, the price difference between geometric and arithmetic MALs is very small. As it takes much less time to price the geometric version, it is a practical approximation to the arithmetic version.

When the algorithms are applied to the actual contracts traded in Taiwan, their issue prices are essentially obtained exactly. Such excellent matches in market prices independently confirm the accuracy and practicality of the methodology. Subsequent data after the issue date show patterns of volatility qualitatively typical of Taiwan's traded securities. Sensitivity measures are needed for hedging purposes. Numerical data will reveal subtle behavior in deltas and gammas for moving-average-type options. They have implications for traders engaged in risk management and hedging.

There is scarcely any prior work on the pricing of moving-average-type options. Cheng and Zhang (2000) discuss an option similar to the MAL. However, it is based on geometric, continuous averaging over a fixed window instead of arithmetic, discrete averaging over a moving window found in practice and treated in this paper. Zvan, Vetzal, and Forsyth (1999) discuss the discrete Asian barrier option, but again it is based on a fixed window. Babbs (2000) and Cheuk and Vorst (1997) discuss lookback options, Gray and Whaley (1997) analyze the S&P 500 bear market warrant with a single reset, and Hull and White (1993) and Klassen (2001) propose several approximation algorithms to price Asian options. These works do not translate directly into solutions to moving-average-type options because they address only one aspect of the complexities; furthermore, none of them addresses moving averages. Chang, Chung, and Shackleton (2000) use the Hull-White paradigm to price an MAR-like option. But they use fixed windows, and the state-reducing interpolation scheme lacks convergence guarantees as mentioned in Forsyth, Vetzal, and Zvan (2001).

The remainder of this paper is organized as follows. The models are succinctly reviewed in Section 2. Section 3 covers the pricing of geometric MALs. Section 4 covers the pricing of arithmetic MALs. Section 5 discusses the numerical delta and gamma of MALs. Section 6 conducts empirical studies of two arithmetic MALs and two arithmetic MARs traded on the Taipei Stock Exchange. Section 7 concludes the paper.

# 2 A Quick Review of the Models

The stock price is assumed to follow the lognormal diffusion,

$$dS/S = (r - q)\, dt + \sigma\, dW,$$

in a risk-neutral economy with a dividend yield of $q$. The binomial model in Cox, Ross, and Rubinstein (1979) will be adopted to approximate the stock's price dynamics. That this binomial model, called the CRR model, converges to the continuous-time model is well-known; see, for example, Heston and Zhou (2000).

The binomial model has parameters $u$ and $d$ with $u > d$. From any node with stock price $S$, price $Su$ follows with probability $p$ (the up move), and price $Sd$ follows with probability $1 - p$ (the down move). If the binomial model has $n$ periods, covering $T$ years, then $u = e^{\sigma\sqrt{T/n}}$, $d = 1/u$, and $p = [\, e^{(r-q)\,T/n} - d\,]/(u - d)$.

Let node $(i, j)$, $0 \leq j \leq i \leq n$, stand for the node at time $i$ with $j$ cumulative up moves and, by implication, $i - j$ cumulative down moves. Node $(i, j)$ is reached with probability $\binom{i}{j} p^j (1 - p)^{i-j}$. The resulting topology has $\sim n^2/2$ nodes and will be called the binomial tree.

This paper focuses on calls as puts can be treated similarly. MAL and MAR have the same payoff function as the vanilla call, but their strike prices may be lowered in the future. As a consequence, MAL and MAR cannot cost less than an otherwise identical vanilla call. Recall that the American-style vanilla call costs more than its exercise payoff if the underlying stock pays no dividends or is dividend-protected. As American-style MAL and MAR cost as much as an otherwise identical vanilla call, they cost more than the exercise payoff, too. Hence they will not be exercised early under the same conditions.

# 3 Pricing Geometric MALs

This section studies the geometric-moving-average-lookback option (GMAL, hereafter). The purpose is two-fold. First, a methodology is developed to price the geometric-averaging version. Second, after the methodology's accuracy is verified, it is applied to the arithmetic case. Later the arithmetic and the geometric cases will be found to produce very close prices.

## 3.1 Defining GMALs

Daily closing prices are used in calculating moving averages. Let $0, 1, \ldots, n$ be the $n+1$ trading dates on or before the reset date $T_{\mathrm{s}}$ (in years)—or date $n$. When calculating an $a$-day moving average, $a$ stock prices are involved. The $a$-day moving product at date $i$ is defined as the product of the $a$ prices up to date $i$:

$$S_{i-a+1} S_{i-a+2} \cdots S_i = \prod_{j=i-a+1}^{i} S_j, \quad a-1 \leq i \leq n.$$

The $a$-day geometric moving average at date $i$ is then $(\prod_{j=i-a+1}^{i} S_j)^{1/a}$. The minimum $a$-day geometric moving average as of the reset date is therefore

$$m_a \equiv \min \left[ \left( \prod_{i=0}^{a-1} S_i \right)^{1/a}, \left( \prod_{i=1}^{a} S_i \right)^{1/a}, \ldots, \left( \prod_{i=n-a+1}^{n} S_i \right)^{1/a} \right].$$

The payoff of the European-style GMAL (EGMAL, hereafter) at expiration date $m > n$ is $\max(S_m - X, 0)$ with the strike price

$$X = \max(\min(m_a, \mathrm{UB}), \mathrm{LB}).$$

The seemingly complex strike price is easy to explain. It is struck at the minimum $a$-day moving average but must be banded between LB and UB. LB sets the lower bound on the strike price and UB the upper bound. Both LB and UB are written in the contract. In Taiwan, for example, UB is set to $S_0$ and LB is some proportion (e.g., 90%) of UB. The American-style GMAL is identical except that in determining the strike price $X$ used at date $i$, the moving average $m_a$ should be the minimum $a$-day geometric moving average up to date $i$.

## 3.2 The Setup of the Binomial Tree

A key feature of the $a$-day geometric average on the binomial tree is that it must be $S_0 u^{k/a}$ for some integer $k$. Let $k_{\text{UB}}$ be the smallest integer $k$ such that $S_0 u^{k/a} \geq \text{UB}$ and $k_{\text{LB}}$ be the largest integer $k$ such that $S_0 u^{k/a} \leq \text{LB}$. The set of potential moving averages for strike prices is then

$$\{S_0 u^{k/a} : k \text{ is an integer}, k_{\text{LB}} \leq k \leq k_{\text{UB}}\},$$

which contains $k_{\text{UB}} - k_{\text{LB}} + 1$ numbers.

Assume the binomial tree partitions each day into $L$ periods. The number of periods before the reset date is therefore $nL$. The GMAL is identical to the vanilla call at expiration with the strike price determined earlier at the reset date. Therefore, to price European-style GMALs, the tree is built up to the reset date and then the Black-Scholes formula is plugged in at the terminal nodes. The tree contains $\sim (nL)^2/2 = O(n^2L^2)$ nodes.

The number of nodes can be reduced. Observe that moving averages involve only daily closing prices. So although the tree covers $nL$ periods, only nodes falling on dates $0, 1, 2, \ldots, n$ are essential. In fact, the $(nL)$-period tree can be made $n$-period without losing information, with one period per day instead of $L$. The idea is to keep only those nodes on the dates. Each node is then linked to those nodes that can be reached in $L$ periods on the original binomial tree. A node now has $L + 1$ successor nodes. This creates an $n$-period $(L+1)$-ary tree with $O(Ln^2)$ nodes instead of $O(L^2n^2)$. For example, when $L = 2$, a trinomial tree with half as many periods is created, as shown in Fig. 1. It is important to note that every branch on the trinomial tree corresponds to 2 branches on the binomial tree. As another example, when $L = 6$, a heptanomial tree with one sixth as many periods results, as shown in Fig. 2.

Let $N(i, j)$ denote the node on the $(L+1)$-ary tree at date $i$ with $j$ cumulative up moves—hence $i - j$ down moves by implication—on the binomial tree. Let $S(i, j)$ denote the stock's price at node $N(i, j)$. Node $N(i, j)$ has $L + 1$ successor nodes

$$N(i + 1, j), N(i + 1, j + 1), \ldots, N(i + 1, j + L).$$

In particular, the $\ell$th branch takes node $N(i, j)$ to node $N(i + 1, j + \ell)$. The probability that this branch is followed is

$$p(\ell) \equiv \binom{L}{\ell} p^{\ell} (1 - p)^{L-\ell}$$

because it corresponds to $\ell$ up moves and $L - \ell$ down moves on the original binomial tree. Their stock prices are hence related by

$$S(i+1, j+\ell) = S(i,j)\, u^{\ell} d^{L-\ell} = S(i,j)\, u^{2\ell - L}. \tag{1}$$

By induction, the stock price at node $N(i,j)$ is

$$S(i,j) = S_0 u^{2j-iL}, \quad 0 \leq i \leq n, \quad 0 \leq j \leq iL.$$

Nodes on the $(L+1)$-ary tree must be augmented with states besides the stock price. Each state at time $t$ records the $a$ daily stock prices involved in the average at time $t$, i.e.,

$$(S_{t-a+1}, S_{t-a+2}, \ldots, S_t).$$

When the next stock price $S_{t+1}$ appears, the new state becomes

$$(S_{t-a+2}, S_{t-a+3}, \ldots, S_{t+1}),$$

a simple left-shifting operation. Because $S_{t+1}$ is the result of taking one of the $L+1$ branches from $S_t$, a number between 0 and $L$ suffices to represent that transition. In general, any state $(S_{t-a+1}, S_{t-a+2}, \ldots, S_t)$ contains $a-1$ transitions; hence it can be represented by a tuple $\overline{s_1 s_2 \cdots s_{a-1}}$, where $s_i \in \{0, 1, \ldots, L\}$. The number $s_i$ signifies that stock price $S_{t-a+i}$ takes the $s_i$th branch to reach price $S_{t-a+i+1}$. By Eq. (1),

$$S_{t-a+i+1} = S_{t-a+i} u^{2s_i - L}. \tag{2}$$

Take $L = 2$ and $a = 3$ for example. The tree is trinomial. Suppose 0, 1, and 2 mean down, flat, and up movements, respectively (recall Fig. 1). The state $\overline{12}$ means a flat move and then an up move have been taken to reach the current state. Table 3 tabulates the $(2+1)^2 = 9$ states and their interpretations.

For completeness, state $(S_{t-a+1}, S_{t-a+2}, \ldots, S_t)$ should be encoded by

$$(\overline{s_1 s_2 \cdots s_{a-1}}, S_t)$$

instead of merely $\overline{s_1 s_2 \cdots s_{a-1}}$. $S_t$ is often dropped because stock price $S_t$ is often implicit and because ambiguity is seldom an issue. Given state $(\overline{s_1 s_2 \cdots s_{a-1}}, S_t)$, the $a-1$ prices preceding $S_t$ are, in reverse chronological order,

$$S_t u^{L-2s_{a-1}}, S_t u^{2L-2(s_{a-1}+s_{a-2})}, \ldots, S_t u^{(a-1)\,L-2(s_{a-1}+s_{a-2}+\cdots+s_1)}$$

according to relation (2). The $a$-day geometric average for the state is therefore

$$\begin{aligned} & S_t \left[ u^{L-2s_{a-1}} u^{2L-2(s_{a-1}+s_{a-2})} \cdots u^{(a-1)\,L-2(s_{a-1}+s_{a-2}+\cdots+s_1)} \right]^{1/a} \\ = \; & S_t u^{(a-1)\,L/2-2[\,s_{a-1}(a-1)+s_{a-2}(a-2)+\cdots+s_1\,]/a}. \end{aligned}$$

Recall that state transition is accomplished by left-shifting of the state:

$$\overline{s_1 s_2 \cdots s_{a-1}} \to \overline{s_2 s_3 \cdots s_a}.$$

The leftmost number $s_1$ is dropped as the new number $s_a$ is shifted in from the right. If $b$ is the current state and the transition takes the $\ell$th branch, then the next state is denoted by $\textsc{Next}(b, \ell)$. See Table 4 for the complete tabulation for the case of 3-day moving average with $L = 2$.

By making the stock price component of a state explicit, state transition

$$(\overline{s_1 s_2 \cdots s_{a-1}}, S_t) \to (\overline{s_2 s_3 \cdots s_a}, S_{t+1})$$

can be used to calculate the new $a$-day moving average. Here, $S_{t+1}$ is the result of $S_t$ taking the $s_a$th branch. Suppose the moving average for the left state is $A$. Then the new moving average equals

$$\begin{aligned} \left[ \frac{A^a S_{t+1}}{S_t u^{(a-1)\,L-2(s_{a-1}+s_{a-2}+\cdots+s_1)}} \right]^{1/a} &= \left[ \frac{A^a}{u^{aL-2(s_a+s_{a-1}+s_{a-2}+\cdots+s_1)}} \right]^{1/a} \\ &= \frac{A}{u^{L-2(s_a+s_{a-1}+s_{a-2}+\cdots+s_1)/a}} \qquad (3) \end{aligned}$$

by relation (2). Define function

$$\textsc{ma}((\overline{s_1 s_2 \cdots s_{a-1}}, S_t), s_a) = k$$

if the new average is $S_0 u^{k/a}$. The computation of $\textsc{ma}()$ can be based on formula (3).

Take $L = 2$ and $a = 3$ for example. Consider node $N(i, j)$ with stock price $S_0 u^{2j-iL}$ and state $\overline{12}$ from Table 3. The state is the result of a flat move followed by an up move. After backtracking those moves, node $N(i, j)$ is reached from node $N(i-1, j-2)$ at date $i-1$ and node $N(i-2, j-3)$ at date $i-2$. The associated prices are $S_0 u^{2(j-2)-(i-1)L}$ and $S_0 u^{2(j-3)-(i-2)L}$, respectively. The 3-day moving average for the state is therefore

$$\left\{ S_0^3 u^{[\,6j-3L(i-1)-10\,]} \right\}^{1/3} = S_0 u^{[\,6j-3L(i-1)-10\,]/3}.$$

### 3.3 Pricing European-Style GMALs

Let $C(i, j; b, k)$ denote the option price on node $N(i, j)$ with path-dependent state $(b, k)$. The parameter $b$ encodes the state. But $b$ alone is not sufficient for pricing purposes; the prevailing strike price is also needed. The integer $k$ signifies that the minimum $a$-day moving product of the stock prices up to date $i$ is $S_0^a u^k$ (equivalently, the moving average is $S_0 u^{k/a}$). By contract, this number is bounded: $k_{\text{LB}} \leq k \leq k_{\text{UB}}$. This feature helps reduce the state space. This integer $k$ will be called the power index. Forward induction can determine the minimum and maximum $k$ precisely, denoted by $\min k_{i,j}$ and $\max k_{i,j}$, respectively, for each node $N(i, j)$.

The minimum geometric moving average may change after each transition. Consider the transition from node $N(i, j)$ with state $b$ and minimum moving average $S_0 u^{k/a}$ to node $N(i+1, j+\ell)$, by taking the $\ell$th branch. Denote the power index of the new minimum $a$-day moving average by $x(b, \ell, k)$. By considering how the minimum moving average is affected if (1) the new moving average is not smaller than the current minimum moving average, (2) the new moving average is between lower bound LB and the current minimum moving average, and (3) the new moving average is smaller than LB, the following formula obtains:

$$x(b, \ell, k) \equiv \begin{cases} k, & \text{if } k \leq \text{MA}(b, \ell) \\ \text{MA}(b, \ell), & \text{if } k_{\text{LB}} \leq \text{MA}(b, \ell) < k \\ k_{\text{LB}}, & \text{if } \text{MA}(b, \ell) < k_{\text{LB}} \end{cases}.$$

The function $x(b, \ell, k)$ that updates the prevailing strike price is straightforward to calculate.

Ground has been laid for the pricing algorithm. The terminal payoffs are given by the Black-Scholes formula for all combinations of stock price $S(n, j)$ and strike price $S_0 u^{k/a}$, where $\min k_{n,j} \leq k \leq \max k_{n,j}$. Inductively, the backward-induction formula is

$$C(i, j; b, k) = \frac{\sum_{\ell=0}^{L} p(\ell)\, C(i+1, j+\ell; \text{NEXT}(b, \ell), x(b, \ell, k))}{R}, \tag{4}$$

where $\min k_{i,j} \leq k \leq \max k_{i,j}$ and $R$ represents the gross riskless return per day. The option value appears in $C(0, 0; 0, 0)$.

The algorithm for the EGMAL runs in $O(n^2 L^{a+3})$ time. Here is the analysis. There is a total of $O(n^2 L)$ nodes. Each node contains $(L+1)^{a-1}$ possible $b$'s and $O(L^2)$ possible $k$'s. Formula (4) takes $O(L)$ time to compute. Multiply these four bounds to obtain the time bound. As the time bound is exponential in $a$, both $a$ and $L$ cannot be large. This is typically the case in practice.

### 3.4 Pricing American-Style GMALs

For American-style GMALs, two changes must be made. First, the option holder can exercise the option on or before the reset date. Early exercise is addressed by taking the maximum of the continuation value $C(i, j; b, k)$ and the exercise payoff $S_0 u^{2j-iL} - S_0 u^{k/a}$. Second, after the reset date when the strike price is set, the option is simply a vanilla American-style call. The Black-Scholes formula from each state on a terminal node should be replaced with the familiar binomial tree algorithm to price an American-style option.

### 3.5 Numerical Results

The binomial tree and Monte Carlo simulation are used to price the 3-day and 5-day EGMALs. Assume $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$ (year), and $T_s = 1/12$ (year). Suppose there are 22 trading days in a month; hence $n = 22$. LB, $\sigma$, and the moving-average length $a$ will be varied. Fix $L = 8$ for the 3-day cases ($a = 3$) and $L = 3$ for the 5-day cases ($a = 5$). The pricing results by Monte Carlo simulation are based on 1,000,000 paths: 500,000 plus 500,000 antithetic.

The numerical results are tabulated in Table 5. Two observations will be made. First, the prices calculated by the algorithm are within two times the standard error generated by Monte Carlo simulation. The accuracy of the algorithm is therefore assured. Second, empirically, the option value decreases with the moving-average length $a$ and LB, but it increases with $\sigma$.

The convergence behavior is shown in Fig. 3. The price can be seen to converge quickly, with accuracy up to two decimal places when the number of periods per day, $L$, is at least 3. The pattern of convergence oscillates in a regular manner. This suggests Richardson's extrapolation as a heuristic to improve the accuracy, and it indeed leads to tighter option prices.

## 4 Pricing Arithmetic MALs

This section applies the same methodology to price the arithmetic-moving-average-lookback option (AMAL, hereafter). AMAL is similar to GMAL except that the geometric moving average is replaced with the arithmetic average:

$$m_a \equiv \min_{a-1 \leq t \leq n} \frac{\sum_{i=t-a+1}^{t} S_i}{a}.$$

Because of the nonnormality of the logarithm of the arithmetic sum of stock prices when modeled by the geometric Browning motion, the same complexity issue facing Asian options reappears.

### 4.1 Pricing AMALs on the Binomial Tree

A strike price must be some moving average. In preparation for the algorithm, all the possible strike prices between LB and UB are enumerated and put in table $M$, sorted from the smallest to the largest:

$$\mathrm{LB} \leq M(1) \leq \cdots \leq M(k_{\max}) \leq \mathrm{UB},$$

where $k_{\max}$ is the size of $M$. The possible strike prices in the arithmetic version therefore form the set

$$\{M(k) : k \text{ is an integer}, 1 \leq k \leq k_{\max}\}.$$

The basic computational structure remains the same as in the geometric version. Let $C(i, j; b, k)$ denote the option price on node $N(i, j)$ with path-dependent state $(b, k)$. The number $b$ is the same as in the GMAL case. Integer $k$ is an index into the table $M$ such that $M(k)$ is the prevailing strike price. As $M$ grows exponentially in $a$, its size should be reduced further. The algorithm simply rounds the strike prices in $M$ to 3 decimal places.[2] The reduced resolution has the effect of equating strike prices in $M$ that are close to each other such as 20.0001 and 20.0004. Take LB $= 45$ and UB $= 50$ for example. The set of possible strike prices is a subset of $\{45.000, 45.001, \ldots, 50.000\}$, about 5001 in number. The size of $M$ can thus be treated as a constant.

Let $\mathrm{MA}(b, \ell)$ denote the new moving average after taking the $\ell$th branch at node $N(i, j)$ with state $b$. For any moving average $y$, let function $f(y)$ give the index $k$ of (rounded) $y$ in table $M$, i.e., $M(f(y)) = y$. Such a function can be easily computed and put in a table once and for all for fast lookup later. The backward-induction formula for the AMAL is identical to that for the GMAL in formula (4) except for the definition of $x(b, \ell, k)$. This function now becomes

$$x(b, \ell, k) = \begin{cases} k, & \text{if } M(k) \leq \mathrm{MA}(b, \ell) \\ f(\mathrm{MA}(b, \ell)), & \text{if } \mathrm{LB} \leq \mathrm{MA}(b, \ell) < M(k) \\ 1, & \text{if } \mathrm{MA}(b, \ell) < \mathrm{LB} \end{cases}.$$

The explanation is the same as before. The overall pricing structure thus mirrors that in the geometric case.

The algorithm runs in $O(n^2 L^{a+1})$ time. Although the time bound seems lower than that for the GMAL, the opposite is typically true in practice. The reason is the big constant factor in the running time.

[2]The idea comes from the fact that quoted prices are rounded to 2 decimal places in the markets. It turns out that 3 decimal places already demonstrate excellent accuracy even though some information is lost. See Dai and Lyuu (2002) for a more elaborate approach.

## 4.2 Numerical Results

With the same parameters as in the geometric version, Table 6 demonstrates that the prices of European-style AMALs (EAMALs, hereafter) are close to the earlier geometric counterparts in Table 5. The convergence is quite fast as shown in Fig. 4. The price of a GMAL is greater than that of an otherwise identical AMAL because of the smaller mean of the geometric average. This claim is proved rigorously in Fang (2002). Although their price difference increases with the moving-average period $a$ and volatility $\sigma$, it remains small. Figure 5 shows that the difference is quite stable when $L \geq 3$. As it takes much less time to price the GMAL, it is a good approximation to the AMAL in practice.

# 5 Some Greeks

Delta and gamma are key to risk management and hedging. Because of the price similarity between GMAL and AMAL, EGMAL will be the focus. The general observations made below apply to all MALs, however. The sensitivity of MAR will also be studied. The formulas to compute the sensitive measures are from Pelsser and Vorst (1994).

## 5.1 Delta

Figure 6 shows option prices with different combinations of LB and UB. In the extreme case when UB = LB, the EGMAL reduces to a vanilla call. The price difference between the vanilla call and the EGMAL diminishes as $S_0$ increases. When $S_0$ decreases, on the other hand, the reset feature makes the EGMAL's price decrease more slowly than the vanilla call's. An interesting feature of the figure is the concavity of the EGMAL price when the stock price falls between LB and UB. It suggests nonmonotonicity in the value of delta, unlike the vanilla call. This phenomenon reflects the two forces determining the option value. As the stock price decreases, the downward stock price tends to drag down the option value, whereas at the same time the prospect of a lower strike price tends to raise the option value. When the stock price is between LB and UB, the strike-price effect is at its strongest and is able to counteract the decrease in the option value. The option holder is therefore protected by the downward-reset feature. When the stock price penetrates below LB, however, the stock-price effect dominates, and the protection ceases to exist.

Figure 7 explores this issue with delta information. Indeed, the delta of EGMAL is not a monotonically increasing function of $S_0$. Looked at more closely, the delta is a decreasing function of the stock price when the stock price is roughly between LB and UB.

## 5.2 Gamma

Delta's complex behavior suggests a similar case for gamma, and Figure 8 confirms that. Gamma is negative roughly between LB and UB; for vanilla European-style calls, in contrast, gamma is always positive. When a securities firm writes a derivative security with a positive gamma, the firm's position has a negative gamma. In that case, the delta hedge strategy will result in hedging losses—the so-called gamma risk. A delta-gamma hedge strategy may be followed instead. This is especially relevant when the underlying stock price is volatile or when the hedging frequency is low. The opposite is true when the derivative has a negative gamma. Apparently, issuing a derivative with negative gamma benefits the issuer.

MAL without an LB is an extreme case. Gamma is essentially zero for $S_0 <$ UB, which implies that the option value decreases with the underlying stock price in roughly a linear manner. This reflects the fact that the strike price can be continuously adjusted down to zero.

## 5.3 Delta Jumps of MARS

One more parameter $N_s$ is added in the specification of MARs. The MAR has $N_s$ reset strike prices set linearly between LB and UB:

$$\text{UB} - h, \text{UB} - 2h, \ldots, \text{LB},$$

where $h \equiv (\text{UB} - \text{LB})/N_s$. The strike price is changed to the lowest reset strike price that the prevailing moving average touches or penetrates.

Like most reset options, the feature of discrete reset strike prices makes the MAR witness delta jump at the stock price such that the strike price is lowered. Hence MAR can be very sensitive to the stock price. Consider an arithmetic MAR (AMAR, hereafter) at the reset date $n$. The prevailing strike price is 50, $a = 3$, $S_{n-2} = S_{n-1} = 50$, LB = 48, UB = 50, and $N_s = 2$. The reset strike prices are thus 49 and 48. Clearly, the strike price will be revised down to 49 if $44 < S_n \leq 47$, and down to 48 if $S_n \leq 44$. This results in two kinks for the option value and two delta jumps at $S_n = 47$ and $S_n = 44$ as shown in Fig. 9.

Because the delta jump depends on past $a - 1$ stock prices, it makes the stock price at the reset date, $S_n$, play a less critical role in MAR than for ordinary reset options, whose strike price depends solely on the stock price.

The probability of a delta jump at the reset date is also smaller. This is yet another advantage of MARs.

# 6 Empirical Studies

Now that the performance of the algorithms has been assured and sensitivity measures studied, the paper turns to empirical studies using them. The focus is on pricing and delta-hedge effectiveness. Two American-style AMALs, `PL06` and `PL07` issued by Polaris Securities, and two American-style arithmetic MARs, `GC06` and `NS02` issued by Grand Cathay Securities and National Securities, respectively, will be the targets of study.

The contracts are specified in Table 7, and the ensuing parameter setups for the algorithms are tabulated in Table 8. The prices calculated by the algorithms, in Table 9, are essentially identical to the issue prices: $26.81 \approx 26.98$, $16.67 \approx 16.76$, $19.88 \approx 20$, and $19.89 \approx 20.25$. The relative differences between the implied volatilities calculated by the algorithms and those released by the securities firms are less than 1%. The GMAL approximations are also very tight for `PL06` and `PL07`. Prices calculated by both AMAL and GMAL algorithms are within one standard error of the Monte Carlo simulation based on 2,000,000 paths (1,000,000 plus 1,000,000 antithetic). The tightness of the results gives us confidence in their respective correctness.

The implied volatilities of `PL06` and `PL07` in Figs. 10 and 11 are calculated with the AMAL algorithm up to September 18, 1999.[3] The two plots show that the volatilities lie within the range of 55% and 65%. This level of volatility is typical of stocks in Taiwan. Interestingly, there is obvious divergence of movements between implied volatilities and the underlying stock prices in both figures. Because these options are issued in fixed quantities, they may carry a fixed-supply premium. This feature also makes short selling somewhat constrained.

Delta hedge for a short option position will be investigated for the two American-style AMALs: `PL06` and `PL07`. Daily hedging is adopted. The experiments show that, on September 20, 1999, a hedge loss of 2.24 TWD ($-8.30\%$) is incurred on `PL06`, and a hedge loss of 0.06 TWD ($-0.03\%$) is incurred on `PL07`. This is probably because of the increase in volatilities during this period: The issue volatilities are 54.80% and 54.95% for `PL06` and `PL07`, respectively, whereas the implied volatilities on September 20, 1999,

[3]Because of a major earthquake on September 21, 1999, the Taipei Stock Exchange was shut down from September 21 to September 26. Therefore, the implied volatilities in Fig. 11 stop on September 20, 1999. After September 27, all the options were no longer resettable (review Table 7) and became vanilla calls.

are 63.82% and 58.24%. Observe also that during the period, the prices of the underlying asset of `PL06` exceed UB, whereas those of `PL07` move between LB and UB. The previous section implies that these price behaviors will disadvantage the hedging of `PL06` but benefit the hedging of `PL07` because of negative gamma. These inferences are consistent with the above findings.

# 7 Conclusions

This paper develops a general methodology for pricing moving-average-type options. In particular, it gives practical algorithms for the moving-average-lookback option and the moving-average-reset option. Both options were traded on Taiwan's exchange, for example. The algorithms converge quickly to the correct value as verified by simulation. Interestingly, the price difference between the geometric and arithmetic moving-average-lookback options is very small. As it takes much less time to price the geometric version, it is a good approximation to the arithmetic version. The delta and gamma of these options are investigated and found to possess complex behavior. This phenomenon has implications for hedging. The algorithms are tested empirically against those moving-average-type options traded on Taiwan's stock exchange. The results show surprisingly tight matches in the issue prices and implied volatilities supplied by the issuers. When applied to delta hedge, the algorithms show conclusions consistent with those suggested by the numerical results.

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Table 1:

Listed Options Market in Taiwan: 1997–2001

| Year | Issue Premium (Millions TWD) | Issue Volume (Millions of Shares) | Trading Volume per Year (Millions TWD) | (Millions of Shares) |
|---|---|---|---|---|
| 1997 | 4,032 | 204 | 1,960 | 165 |
| 1998 | 4,743 | 270 | 13,069 | 1,492 |
| 1999 | 13,381 | 866 | 64,782 | 3,807 |
| 2000 | 12,668 | 1,110 | 162,262 | 11,588 |
| 2001 | 4,772 | 893 | 28,440 | 7,784 |

Note. These options were called warrants because they were issued in fixed quantities. They were listed by the Taiwan Stock Exchange Corporation (TSEC). Options in Taiwan are dividend-protected. TWD: New Taiwan dollar.

Table 2:

Moving-Average-Type Options Issued in 1999 on the Taipei Stock Exchange

| Contract | Type | Issue Date | Issue Volume (Thousands of Shares) | Issue Premium (Millions TWD) |
|---|---|---|---|---|
| GC06 | Reset | 28-Apr | 10,700 | 216.68 |
| GC07 | Reset | 27-May | 17,500 | 239.40 |
| GC08 | Reset | 09-Jun | 10,000 | 248.30 |
| GC09 | Reset | 14-Jun | 13,600 | 206.86 |
| GC10 | Reset | 20-Oct | 12,000 | 262.44 |
| YT07 | Reset | 23-Nov | 22,000 | 226.64 |
| NS02 | Reset | 16-Jun | 10,000 | 200.00 |
| NS03 | Reset | 13-Sep | 10,000 | 205.00 |
| FB01 | Reset | 08-Jul | 20,000 | 220.00 |
| FB02 | Reset | 18-Aug | 18,000 | 306.00 |
| CS04 | Reset | 04-Sep | 11,700 | 216.74 |
| PL06 | Lookback | 21-Aug | 10,000 | 269.80 |
| PL07 | Lookback | 27-Aug | 15,000 | 251.40 |

Note. All the options have a maturity of one year. To put the numbers in perspective, the total dollar amount of options issued in 1999 was 13.4 million TWD. GC: Grand Cathay Securities (capital 10.5 billion TWD); YT: Yuanta Securities (capital 11.7 billion TWD); NS: National Securities (capital 8 billion TWD); FB: Fubon Securities (capital 10.5 billion TWD); CS: Capital Securities (capital 9.2 billion TWD); PL: Polaris Securities (capital 6 billion TWD). The capitalization figures are based on 1999 filings.

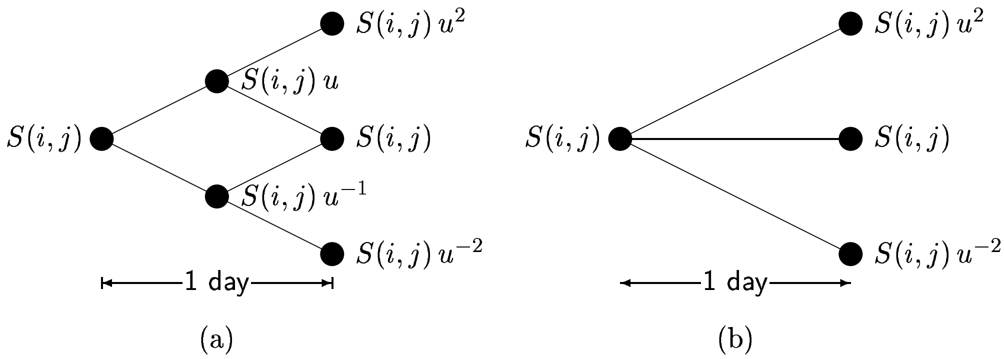


Figure 1:
Turning a Binomial Tree into an $(L+1)$-ary Tree: $L = 2$. (a) A 2-period binomial tree covering one day. (b) A 1-period trinomial tree covering one day after the intermediate nodes are removed from the binomial tree of (a). A period for the trinomial tree lasts twice as long as that for the binomial tree. A down move on the trinomial tree (b) corresponds to 2 down moves on the binomial tree (a). A flat move on the trinomial tree corresponds to a total of 1 up move and 1 down move on the binomial tree. An up move on the trinomial tree corresponds to 2 up moves on the binomial tree.

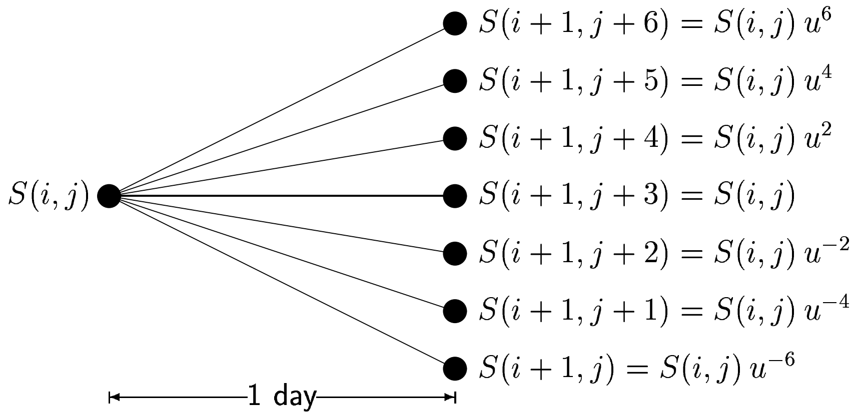


Figure 2:
Turning a Binomial Tree into an $(L+1)$-ary Tree: $L = 6$. Because a 6-period binomial tree has 7 terminal prices, a heptanomial tree obtains. For example, $S(i+1, j+4)$ is reached by taking 4 up moves (hence 2 down moves) on the original binomial tree.

Table 3:

Encoding 2-Day Movements

| State | Moves on Binomial Tree | |
|---|---|---|
| $\overline{00}$ | 0 up move | 4 down moves |
| $\overline{01}$ | 1 up move | 3 down moves |
| $\overline{02}$ | 2 up move | 2 down moves |
| $\overline{10}$ | 1 up moves | 3 down moves |
| $\overline{11}$ | 2 up move | 2 down moves |
| $\overline{12}$ | 3 up moves | 1 down moves |
| $\overline{20}$ | 2 up move | 2 down moves |
| $\overline{21}$ | 3 up moves | 1 down move |
| $\overline{22}$ | 4 up moves | 0 down move |

Note. Graphically from Fig. 1(b), 0, 1, and 2 mean down, flat, and up movements, respectively. Figure 1(a) gives a more complete picture: 0 means 2 down moves, 1 means a total of 1 down move and 1 up move, and 2 means 2 up moves, all referring to the original binomial tree.

Table 4:

State Transition for 2-Day Movements

| Current State $b$ | One-Day Forward $\textsc{Next}(b, \ell)$ |
|---|---|
| $\overline{00}$ | $\overline{0\,\ell}$ |
| $\overline{01}$ | $\overline{1\,\ell}$ |
| $\overline{02}$ | $\overline{2\,\ell}$ |
| $\overline{10}$ | $\overline{0\,\ell}$ |
| $\overline{11}$ | $\overline{1\,\ell}$ |
| $\overline{12}$ | $\overline{2\,\ell}$ |
| $\overline{20}$ | $\overline{0\,\ell}$ |
| $\overline{21}$ | $\overline{1\,\ell}$ |
| $\overline{22}$ | $\overline{2\,\ell}$ |

Note. The branch taken is $\ell$. The new state results from dropping the leftmost number and appending $\ell$ from the right. Assume $a = 3$ and $L = 2$.

Table 5:

Pricing EGMALs

| | | $a = 3,\ L = 8$ | | | $a = 5,\ L = 3$ | | |
|---|---|---|---|---|---|---|---|
| LB | $\sigma$ | CRR | MC | SE | CRR | MC | SE |
| 45 | 0.3 | 6.1689 | 6.1712 | (0.0019) | 6.0769 | 6.0745 | (0.0019) |
| | 0.4 | 8.1916 | 8.1942 | (0.0029) | 8.0924 | 8.0871 | (0.0028) |
| | 0.5 | 10.1367 | 10.1392 | (0.0039) | 10.0360 | 10.0339 | (0.0038) |
| 40 | 0.3 | 6.2694 | 6.2723 | (0.0018) | 6.1566 | 6.1521 | (0.0018) |
| | 0.4 | 8.4219 | 8.4242 | (0.0026) | 8.2832 | 8.2797 | (0.0027) |
| | 0.5 | 10.4953 | 10.4992 | (0.0036) | 10.3402 | 10.3332 | (0.0036) |
| 35 | 0.3 | 6.2714 | 6.2731 | (0.0018) | 6.1579 | 6.1551 | (0.0018) |
| | 0.4 | 8.4414 | 8.4460 | (0.0026) | 8.2970 | 8.2922 | (0.0026) |
| | 0.5 | 10.5581 | 10.5604 | (0.0035) | 10.3882 | 10.3836 | (0.0035) |

Note. The parameters are $S_0 = \mathrm{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, and $T_{\mathrm{s}} = 1/12$ ($n = 22$). SE is the standard error of Monte Carlo simulation (MC) based on 1,000,000 paths: 500,000 plus 500,000 antithetic.

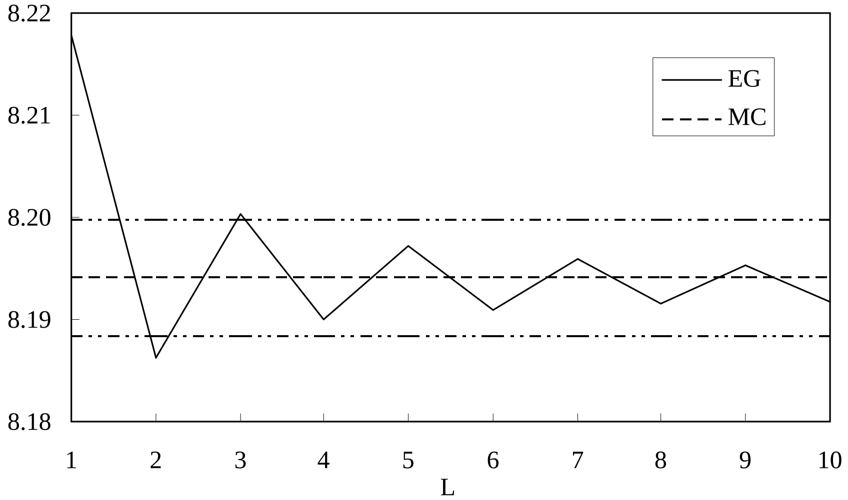


Figure 3:
Convergence of the EGMAL Algoerithm. The parameters are $S_0 = \mathrm{UB} = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $n = 22$, and $a = 3$. EG* and MC represent the prices obtained by Richardson's extrapolation and the Monte Carlo simulation (MC), respectively. Monte Carlo simulation (MC) is based on 1,000,000 paths: 500,000 plus 500,000 antithetic. Here, MC gives a vlaue of 8.1942 with a standard error of 0.0029. A band with a width of 2 standard errors above and below MC is plotted for reference.

Table 6:

Pricing EAMALs

| | | $a = 3,\ L = 8$ | | | $a = 5,\ L = 3$ | | |
|---|---|---|---|---|---|---|---|
| LB | $\sigma$ | CRR | MC | SE | CRR | MC | SE |
| | 0.3 | 6.1684 | 6.1706 | (0.0019) | 6.0757 | 6.0726 | (0.0019) |
| 45 | 0.4 | 8.1909 | 8.1933 | (0.0029) | 8.0907 | 8.0864 | (0.0028) |
| | 0.5 | 10.1358 | 10.1380 | (0.0039) | 10.0340 | 10.0314 | (0.0038) |
| | 0.3 | 6.2688 | 6.2715 | (0.0018) | 6.1552 | 6.1507 | (0.0018) |
| 40 | 0.4 | 8.4209 | 8.4225 | (0.0026) | 8.2809 | 8.2775 | (0.0027) |
| | 0.5 | 10.4937 | 10.4987 | (0.0036) | 10.3371 | 10.3299 | (0.0036) |
| | 0.3 | 6.2708 | 6.2724 | (0.0018) | 6.1564 | 6.1542 | (0.0018) |
| 35 | 0.4 | 8.4404 | 8.4449 | (0.0026) | 8.2946 | 8.2881 | (0.0026) |
| | 0.5 | 10.5563 | 10.5573 | (0.0035) | 10.3847 | 10.3812 | (0.0035) |

Note. The parameters are $S_0 = \text{UB} = 50$, $r = 2\%$, $q = 4\%$, $T = 1$, and $T_{\mathrm{s}} = 1/12$ ($n = 22$). SE is the standard error of Monte Carlo simulation (MC) based on 1,000,000 paths: 500,000 plus 500,000 antithetic.

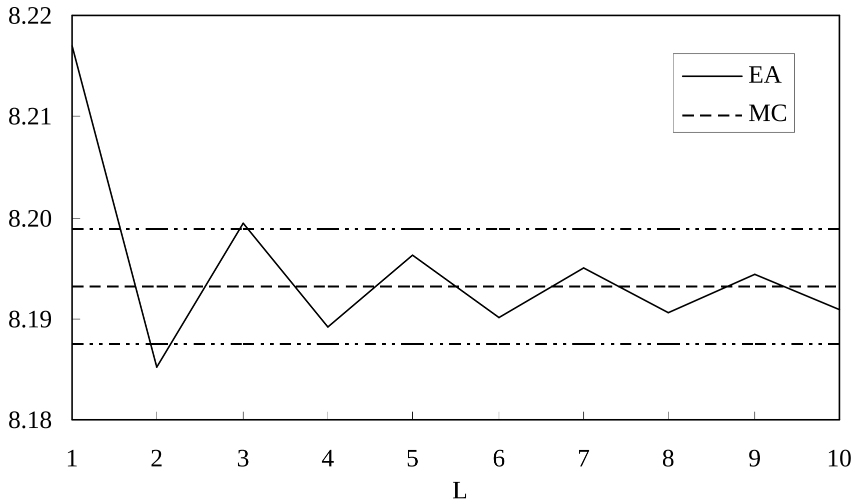


Figure 4:
Convergence of the EAMAL Algorithm. The parameters are $S_0 = \text{UB} = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), and $a = 3$. EA* and MC represent the prices obtained by Richardson's extrapolation and the Monte Carlo simulation, respectively. The MC result is based on 1,000,000 paths: 500,000 plus 500,000 antithetic. It gives a value of 8.1933 with a standard error of 0.0029. A band with a width of 2 standard errors above and below MC is plotted for reference.

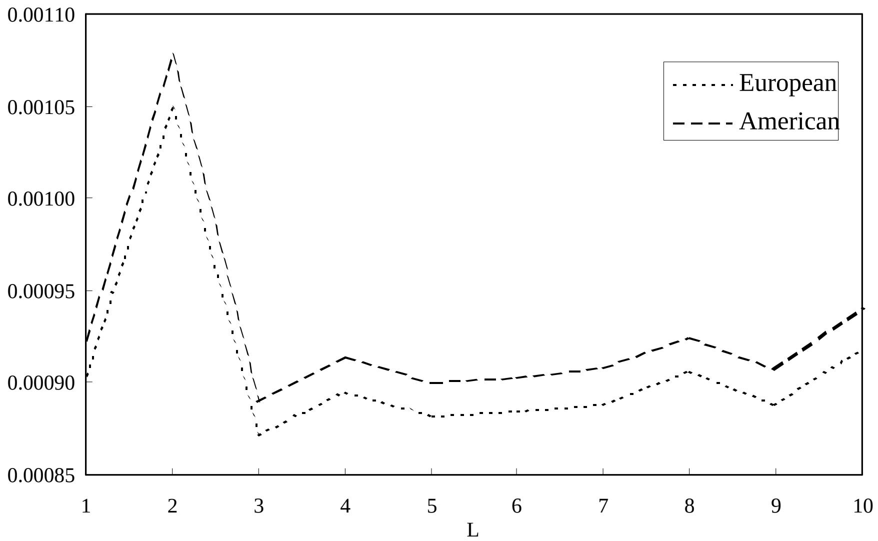


Figure 5:
Price Difference of GMAL and AMAL. Both European style and American style are considered. The parameters are $S_0 = \mathrm{UB} = 50$, $\sigma = 40\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_{\mathrm{s}} = 1/12$ ($n = 22$), and $a = 3$.

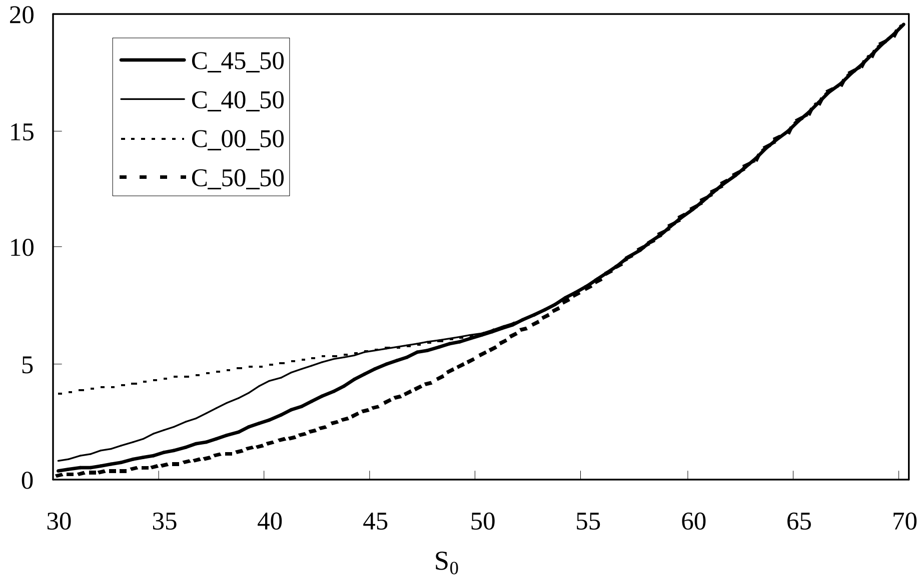


Figure 6:
Dependency of EGMAL's Price on Lower and Upper Bounds. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ $(n = 22)$, $L = 3$, and $a = 3$. C_x_y means the EGMAL comes with LB = x and UB = y.

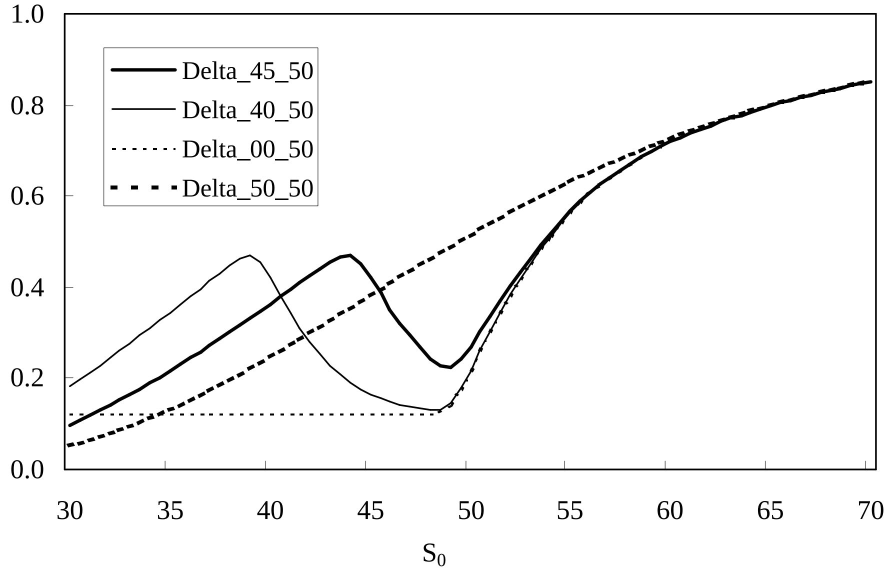


Figure 7:
Delta of EGMAL. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ ($n = 22$), $L = 3$, and $a = 3$. Delta_x_y means LB = x and UB = y.

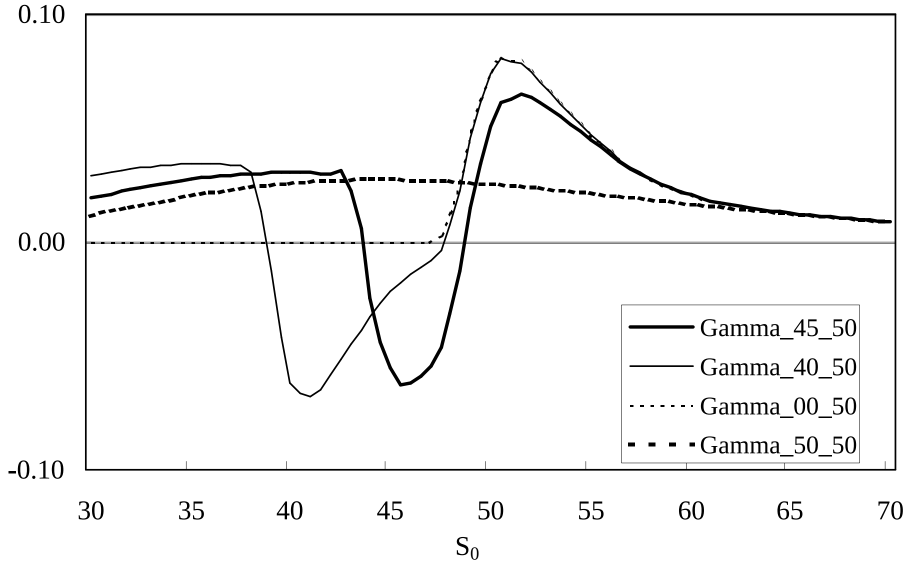


Figure 8:
Gamma of EGMAL. The parameters are $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 1$, $T_s = 1/12$ $(n = 22)$, $L = 1$, and $a = 3$. Gamma_x_y means LB = x and UB = y.

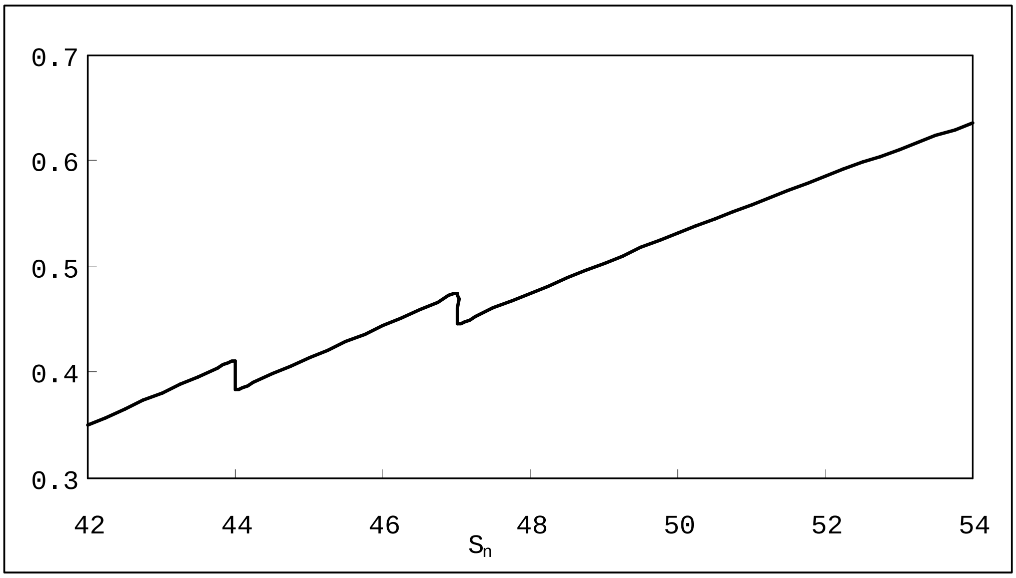


Figure 9:
Delta of AMAR at Reset Date. The parameters are $S_{n-2} = S_{n-1} = 50$, $\sigma = 30\%$, $r = 2\%$, $q = 4\%$, $T = 11/12$, $n = 0$, $a = 3$, LB = 48, UB = 50, and $N_s = 2$ (thus, reset strike prices are 49 and 48). The prevailing strike price is 50.

Table 7:

Contract Specifications of PL06, PL07, GC06, and NS02

| | PL06 | PL07 | GC06 | NS02 |
|---|---|---|---|---|
| Type | Lookback | Lookback | Reset | Reset |
| Issue date | 08/21/'99 | 08/27/'99 | 04/28/'99 | 06/16/'99 |
| First trading date | 09/02/'99 | 09/06/'99 | 05/12/'99 | 06/30/'99 |
| Reset date | 09/20/'99 | 09/27/'99 | 08/11/'99 | 07/15/'99 |
| Expiration date | 09/01/'00 | 09/05/'00 | 05/11/'00 | 06/29/'00 |
| Moving-average period (days) | 6 | 6 | 6 | 3 |
| UB | 103.75 | 64.45 | 81.00 | 81.30 |
| LB | 0.9×UB | 0.9×UB | 0.9×UB | 0.9×UB |
| Number of reset strike prices | — | — | 5 | 5 |
| Issue volatility | 54.38% | 54.58% | 49.10% | 50.43% |
| Issue price (TWD) | 26.98 | 16.76 | 20.25 | 20.00 |

Note. Information is provided by the issuing firms.

Table 8:

Parameter Setups for `PL06`, `PL07`, `GC06`, and `NS02` for the Tree Algorithm

| | `PL06` | `PL07` | `GC06` | `NS02` |
|---|---|---|---|---|
| $S_0$ | 103.75 | 64.45 | 81.00 | 81.30 |
| UB | 103.75 | 64.45 | 81.00 | 81.30 |
| LB | 93.38 | 58.01 | 72.90 | 73.17 |
| $\sigma$ | 54.38% | 54.58% | 49.10% | 50.43% |
| $r$ | 5.00% | 5.00% | 5.00% | 5.00% |
| $q$ | 0.00% | 0.00% | 0.00% | 0.00% |
| $T$ | 378/365 | 376/365 | 380/365 | 380/365 |
| $T_{\mathrm{s}}$ | 31/365 | 32/365 | 105/365 | 30/365 |
| $n$ | 24 | 24 | 81 | 21 |
| $a$ | 6 | 6 | 6 | 3 |
| $L$ | 2 | 2 | 2 | 11 |
| $N_{\mathrm{s}}$ | — | — | 5 | 5 |

Table 9:

Prices and Volatilities of PL06, PL07, GC06 and NS02

| | AMAL | | AMAR | |
|---|---|---|---|---|
| | PL06 | PL07 | GC06 | NS02 |
| CRR | 26.8125 | 16.6689 | 19.8866 | 19.8841 |
| GMAL approximation (CRR) | 26.8181 | 16.6725 | — | — |
| MC | 26.8160 | 16.6714 | 19.9003 | 19.8786 |
| Standard error | 0.0071 | 0.0045 | 0.0104 | 0.0050 |
| Implied volatility by CRR | 54.80% | 54.95% | 49.50% | 50.78% |

Note. The parameters are based on Table 8. Recall the earlier proof that says MAL and MAR will not be exercised early when options are dividend-protected, as is the case here. Monte Carlo simulation thus gives valid results even if the MAL and MAR above are American-style.


Figure 10:

Implied Volatilities of PL06.

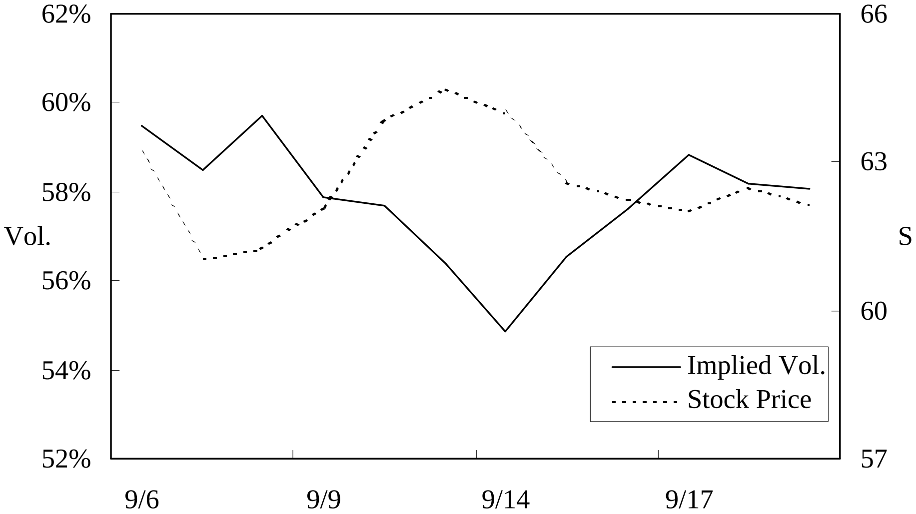


Figure 11:

Implied Volatilities of PL07.