# Logic, Language, and Computation — A Historic Journey

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I have never done anything "useful."

— Godfrey Harold Hardy (1877–1947),
*A Mathematician's Apology* (1940)

# Language

Whereof one cannot speak,
thereof one must be silent.
— Ludwig Wittgenstein (1889–1951),
*Tractatus Logico-Philosophicus* (1918)

# Language

- The characteristics of language.
  - Syntax.
  - Semantics.
  - Pronunciation.
  - Evolutionary theories.
  - …

# Examples

All animals are equal but some are more equal than others.

— George Orwell (1903–1950), *Animal Farm* (1945)

- What does it mean?
- What is the context?
- Can’t it be more precise?
- Do you even want it to be precise?

# Examples (cont.)

I'm not a woman you can trust.

— Sharon Stone to Sylvester Stallone on the phone, *The Specialist* (1994)

- What does it convey?
- Self-loops?

# Examples (cont.)

If I am I because you are you, and if you are you because I am I, then I am not I, and you are not you.

— Hassidic rabbi

- It is syntactic.
- What can it possibly mean?
- Is it nonsense?

# Examples (cont.)

"If the king of Taiwan is a man, then he is not a woman."

- True formally?
- This sentence seems meaningful.
- But does it convey any knowledge?
- Do you need to assume such a king exists?
- Must a sentence's components correspond to entities in the physical world?

# Examples (concluded)

It is, I own, not uncommon to be wrong in theory and right in practice[.]

— Edmund Burke (1729–1797), *A Philosophical Enquiry into the Origin of Our Ideas of the Sublime and Beautiful* (1757)

- What does it mean for a statement to be right or wrong?
- How do you verify a sentence?
- What is truth?
- Must truth correspond to facts?
  - What is fact? What is correspondence?

# Ordinary Language vs. the Language of Science

- Precision (no ambiguities?).
- Clarity (no metaphysics?)
- Denotation (existence in the external world?).
- Intention.
- Consistency (freedom from internal contradictions).
- Completeness (no indeterminancy).
- Decidability (finitely computable in principle).

# Symbolic Logic

# Formal Language and Symbolic Logic

- Start with a formal language.
- Equip it with *sound* inference rules.
- Phrase every mathematical (even philosophical and scientific) questions in this language.
- Deduce all valid sentences.
- If your basic axioms are complete and verified by experiences, then you have solved all problems in principle.
- Logical positivism postulates that all other approaches to philosophy are meaningless.

It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature.

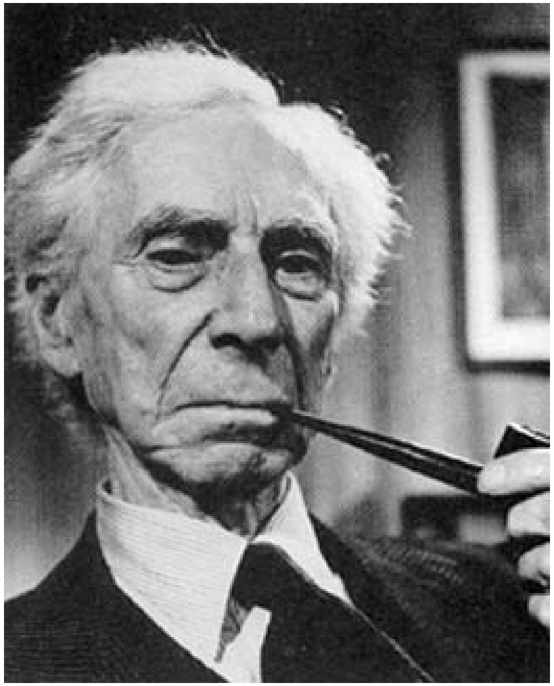
— Niels Bohr (1885–1962)

# Symbolic Logic

- Start with logic for mathematics ("investigates the logical components of mathematics" — Zermelo (1908)).
  - It is easier.
  - Exact sciences have been mathematized for hundreds of years, maybe starting with Galileo (1564–1642).
  - If mathematics does not have a solid foundation, modern science might be shaky.
  - The logicism of Frege (1848–1925) and Russell (1872–1970) posited mathematics can be reduced to logic.

The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself.

— Bertrand Russell, *The Principles of Mathematics* (1903)

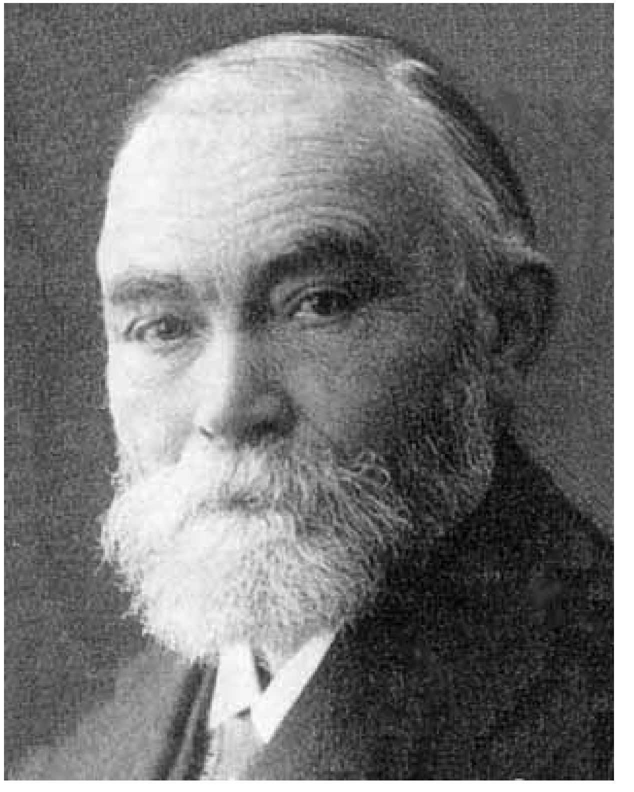


[Russell's] philosophical influence, directly and indirect, over his long period has been unequaled.

— Willard Van Orman Quine (1908–2000), *Theory and Things* (1981)

# Modern Mathematical Logic

- Invented by Gottlob Frege's (1848–1925) *Begriffsschrift* (1879).
- Goal: A system that axiomatizes arithmetic and deduces all theorems of integers.



# Modern Mathematical Logic (concluded)

- Need
  - Constants and variables ($\varnothing$, $x$, $y$, $z$,…).
  - A set of logical connectives ($\forall, \exists, \Rightarrow, \wedge, \vee, \neg$, (, ), $\in, \vdash$).
  - Predicates ($P$, $Q$, …)
  - A precise language syntax to state mathematical results.
  - A mechanism for valid inferences.

# Requirements of a Formal System

- Consistency (Hilbert (1900)).
- Completeness (Hilbert (1918)).
  - For every formula $A$, either $A$ or $\neg A$ is provable.
  - Every provable formula is valid, and conversely (Bernays (1918)).
- Independence of axioms (Huntington (1902) and Veblen (1904)).
- Decidability (Hilbert (1918)).
  - Given a mathematical statement, “decide” in a finite number of steps whether it is true or not from the axiom system.
  - *Entscheidungsproblem* (the decision problem) is the term by Behmann (1921).

# Russell's Paradox (May 1901)

- Consider the set

$$R = \{A : A \notin A\}.$$

  - If $R \in R$, then $R \notin R$ by definition.
  - If $R \notin R$, then $R \in R$ also by definition.
  - In either case, we have a "contradiction."
- Russell sent the paradox to Frege on June 16, 1902.
- Frege's system collapsed.

# Russell’s Angst

In his *Autobiography*, vol. 1 (1967), Russell wrote,

> It seemed unworthy of a grown man to spend his time on such trivialities, but what was I to do?

# Implications of Russell's Paradox

- Perhaps the most fundamental paradox in modern mathematics.
- The "problem" comes from naïve set theory of Georg Cantor (1845–1918).
  - Axiom of abstraction (comprehension) postulates that any proposition $P(x)$ defines a set $\{x: P(x)\}$.

# Implications of Russell's Paradox (concluded)

- An intensive search for the sound foundations of mathematics (thus supposedly anything above it) followed.
  - Three schools: Hilbert's formalism, Russell & Whitehead's logicism, and Brouwer's intuitionism.
- Most attempt to restrict what constitutes a set.
- Whitehead and Russell created a massive 3-volume work, *Principia Mathematica* (1910, 1912, 1913), that attempts to be the new foundation for mathematics, free of contradictions.
  - It proves 1+1=2 on p. 362.
- If a contradiction shatters existing believes that one is unwilling to give up, then it is a paradox (Kuhn (1962)).

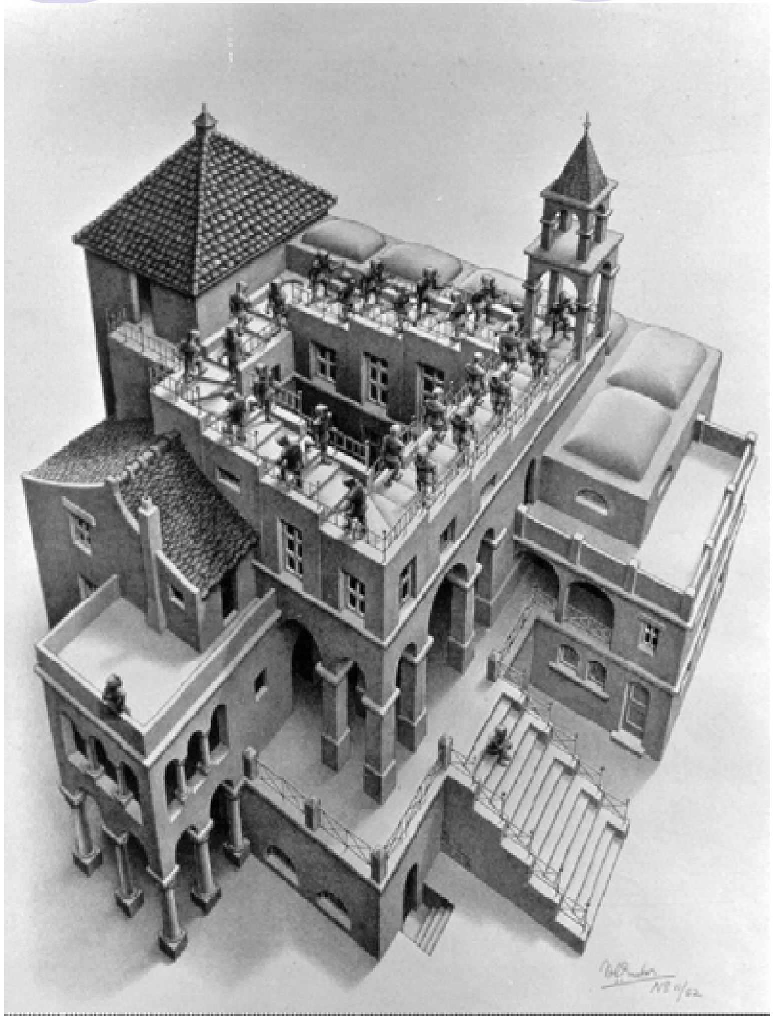
# Self-Loops in Ordinary Languages

- Ordinary languages are “universal,” allowing self-references (Tarski (1933)).
- Eubulides: The Cretan says, “All Cretans are liars.”
- Liar's paradox: “This sentence is false.”
- 我必不背棄我的約，也不改變我口中所出的。
  “My covenant will I not break, nor alter the thing that is gone out of my lips.” (*Psalms* 89:34)
- “Moses was the most humble person in all the world[…]” (*Numbers* 12:3).
- 因為耶和華你們的神他是萬神之神、萬主之主
  “For the LORD your God is God of gods, and Lord of lords[…]” (*Deuteronomy* 10:17)
- 王阿[尼布甲尼撒]、你是諸王之王
  “Thou, O king [Nebuchadnezzar], art a king of kings[…]” (*Daniel* 2:37)

# Self-Loops in Formal Languages

- Russell's paradox contains self-loops: $A \notin A$.
- Naturally, one way to avoid it is to define sets in a hierarchy.
- In the most popular axiomatic set theory Zermelo-Fraenkel-Skolem (ZFS), Russell's $R$ is *not* a set any more.

# Self-Loops in M.C. Escher (1898–1972)



# Hilbert’s Program

- David Hilbert (1862–1943) in 1900 and 1921 proposed to found mathematics on axiomatic systems (Zach (2003)).
  - Geometry, analysis, number theory, etc.
- He asked for “proof” for the consistency of the axioms.
  - No paradoxes or contradictions.
  - Proof is purely symbolic (no direct intuitive meaning).
  - Proof must be “finitary,” to answer intuitionism.
- He believed all meaningful mathematical problems are solvable:

> There is the problem. Seek its solution. You can find it by pure reason, for in mathematics, there is no *ignorabimus*.

# Hilbert’s Program (concluded)

- Hilbert’s proof-theoretical philosophy for the foundation of mathematics is called formalism.
- Since 1914, his school started to be influenced by the logic system of *Principia Mathematica*.
- Some concrete problems for any axiomatic system of mathematics:
  - Are the axioms complete?
  - Are the axioms consistent?
  - Are the validities in first-order logic decidable?

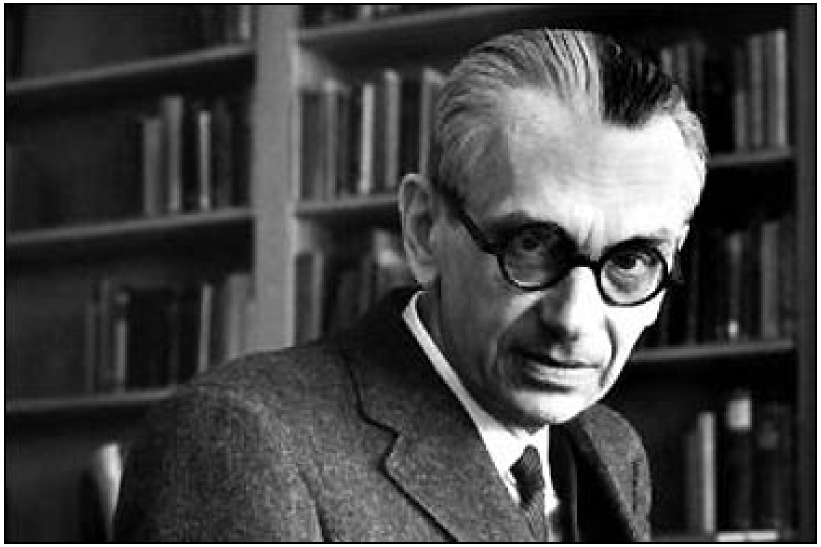
Wir müssen wissen, wir werden wissen.
(We must know, we shall know.)
— David Hilbert (1900)

# Gödel’s Completeness Theorem

- In 1929, Kurt Gödel (1906–1978) proved that the first-order logic is “complete.”
  - His Ph.D. thesis (1930).
  - Every valid formula admits of a finite formal proof.
  - Truth is thus equivalent to provability.

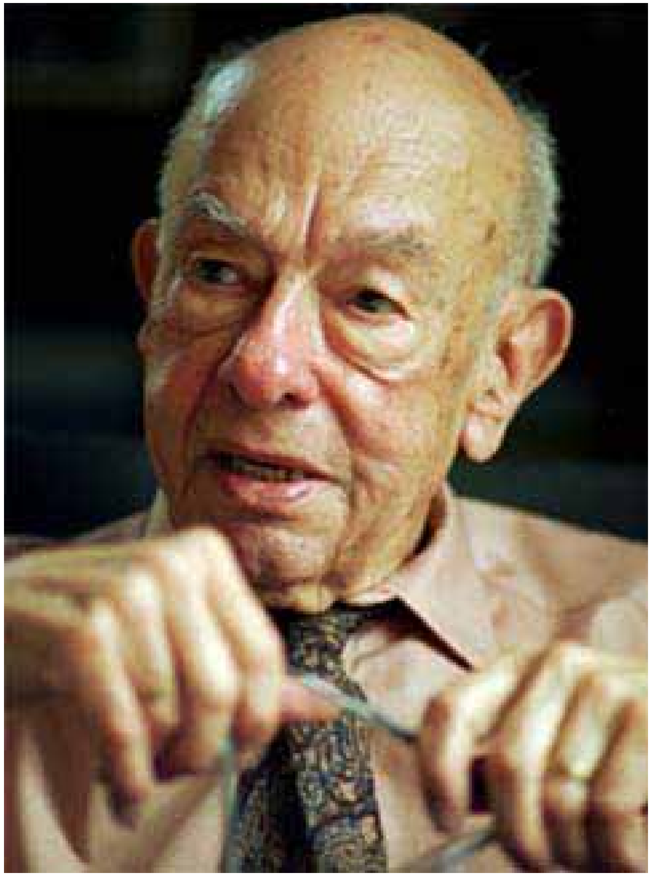
The Completeness Theorem, mathematically, is indeed an almost trivial consequence of Skolem 1923. […] It lies in the widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward non-finitary reasoning.

— Kurt Gödel (1967)



Such completeness was expected. […] But an actual proof of completeness was less expected, and a notable accomplishment.

— Willard Van Orman Quine, *Theory and Things* (1981)



# Gödel's Incompleteness Theorems

- In 1930, Gödel proved two incompleteness theorems.
- Elementary arithmetic is incomplete.
  - There are undecidable propositions in any sufficiently strong formal system involving only + and ×.
- Consistency of number theory cannot be proved *within* the system.
  - Independently by von Neumann, according to Wang (1996).
  - A full 70-page proof was given by Hilbert and Bernays (1939).
- Every sufficiently strong formal system is either inconsistent or incomplete.

# Gödel’s Proof

- The proof uses self-loops: liar's paradox.
- Gödel constructed an arithmetic statement *G* that says

“*G* cannot be proved.”

  - We skip the details that avoid paradoxes.
  - Gödel called it a “parlor trick” (Kreisel (1980)).
- *G* can be proved if and only if it is false.
- Hence *G* must be true but unprovable.
- The incompleteness result sealed Gödel’s *immortality*.

# Why the Proof?

- The proof seemed to "confuse" many people: Zermelo (1931), Wittgenstein (1983), Russell (1963), Chaitin (2005), etc.
- There is an indirect proof, that truth about propositions must be expressed in a metalanguage according to Tarski (1933).
  - To avoid liar's paradox.
- Gödel did not take that shorter route even though he was aware of it as early as 1931 (Kennedy (2007)).
  - The dominant thinking at that time was that truth equals provability.
  - So his paper might have been rejected otherwise (Feferman (1983))?
  - Philosophical inclinations (Krajewski (2004))?

# Far-Reaching Consequences

- Mathematical truth may not be founded upon provability.
  - How about philosophical truths, political truths, legal truths, economic truths, theological truths?
- There is no *absolute* consistency proof of all of mathematics.
  - Relative consistency proof of elementary arithmetic is given by Gentzen in 1936.
- Hilbert’s original goals were dashed.
  - John von Neumann saw it in 1931.
  - Gödel was convinced of it after Turing’s (1937) work on computing machines.

# Impacts on Mathematics

- So Gödel proved that certain formalized systems are incomplete.
- But are there *absolutely* undecidable statements?
- Gödel agreed with “Hilbert’s original rationalistic conception” that there is none.
- Do Gödel’s results impact working mathematicians?
- No famous mathematical statement conjectured to be true has been proved undecidable in axiomatic set theory (Feferman (2006))?

It was anti-Platotic prejudice which prevented people from obtaining my results.

— Gödel to Wang (1976)

I represent this very crude, naïve kind of anti-Platonism[.]

— Alfred Tarski (1902–1983) in 1965

I would have proved Gödel's Theorem in 1921 — had I been Gödel.

— Emil Post (1897–1954) to Gödel (1938)

# Computation

It is unworthy of excellent men to lose hours like slaves in the labor of computation.

— Gottfried Wilhelm von Leibniz (1646-1716)

# What Is Computation?

- *Entscheidungsproblem*: As first-order logic is complete, is there an algorithm that tells you if a statement is true (provable) or not?
- To answer it, we need a rigorous definition of computation.
- Intuitively, an algorithm is a *finite mechanical* procedure that ends with the right answer.
  - Multiplication.
  - Euclid's algorithm for the greatest common divisor (GCD).
  - Polynomial multiplication.
  - Primality testing.
- It is a "mental" proces (with the help of paper and pencil).

# What Is Computation? (concluded)

- But how do you formalize this notion?
- What are the essential elements of computation?
  - What are the primitive operations, for example?
- This is a philosophical question.
- We approach a “mental” phenomenon in a scientific way.
- We will never hope to prove a formalism is correct.
- A counterexample can convince us of its inadequacy.
- But what constitutes a counterexample anyway?

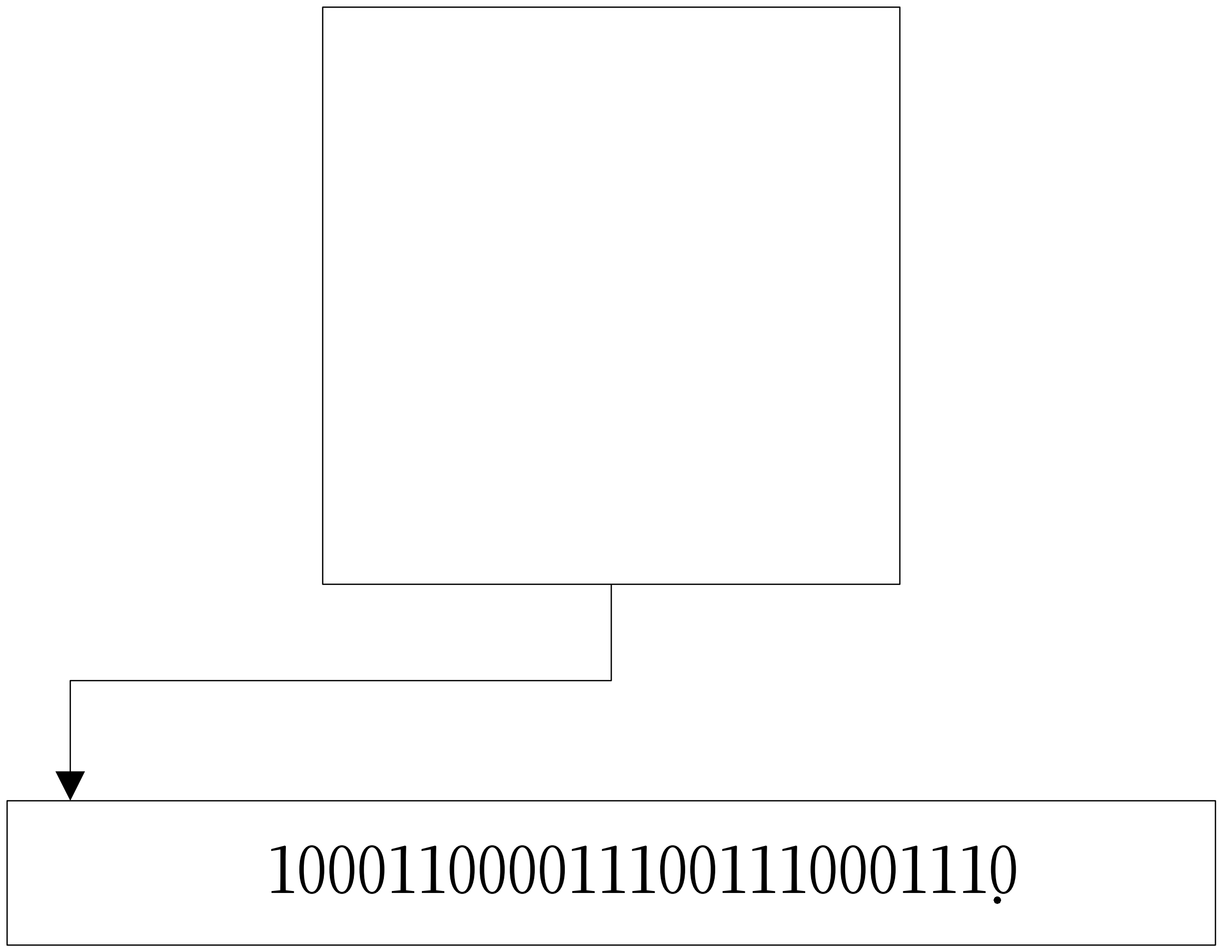
Tarski has stressed in his lecture (and I think justly) the great importance of the concept of general recursiveness (or Turing's computability).

— Kurt Gödel, *Princeton Lectures* (1946)

# Computation Models

- In 1930–6 (and later), various models have been proposed for computation.
  - Church (1903–1995), Gödel in 1934, Kleene (1909–1994), Post (1897–1954), and Turing (1912–1954).
  - The most famous is the Turing machine.
  - They are all shown to be equivalent.
- Nowadays, we call them “computer programs.”

# A Turing Machine



# Church's Thesis

- Church's thesis (1935) states:

  To be computable is to be Turing machine-computable.

  - This term is due to Kleene.
- Once the notion of computation is defined, computability follows.
- This is the birth of computer science.

If you ask who is the greatest logician of this century, no question it's Gödel, but Church was really pre-eminent among American logicians.

— Simon B. Kochen

Alan Turing. Stephen Kleene.

# *Entscheidungsproblem*

- *Entscheidungsproblem* is now shown to be unsolvable.
  - Church (1935), Kleene (1936), Post (1936), and Turing (1936).
  - Turing (1936) is independent of Church (1935).
- In other words, first-order logic is (computably) undecidable.
- This is known as Church's theorem.
- Gödel's completeness theorem is nonconstructive.
- A proof for validity may exist, but there is no program that can verify it in finite time.

The ancient world takes its stand upon the drama of the Universe, the modern world upon the inward drama of the Soul.

— Alfred North Whitehead (1861–1947)

# Minds, Formal Systems, Computers

- According to Quine (1981),

  [Gödel] thought it possible for the mind to transcend formal proof procedures and thus not be bound by his incompleteness theorem.

- Gödel believed mind is separate from matter.
- Mathematicians choose the axioms that are "truths."
- It is a mathematician's intuition that gives the axioms.
- Turing also did not think mental process is purely mechanical (Sieg (2006)).

# Conclusions

- We started with very innocent questions: is mathematics consistent, complete, true, decidable, etc.?
- Gödel destroyed all hopes of positive answers.
- Working mathematicians probably do not care (that much).
- Is truth lost?
- We also ended up with a machine and a revolution.
- We are asking if minds are basically computers.

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