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利用具相關性之二元樹模型做信用組合違約模擬

Credit Portfolio Simulation Using Correlated Binomial Lattices

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Abstract

We revisit the models developed in Das and Sundaram (2004) and Bandreddi, et al. (2007). Bandreddi, et al. (2007) use a simplified version of the model developed by Das and Sundaram for correlated default simulation. We find that in their setting, problematic probabilities may arise which may cause biased results for the purpose of default simulation and the pricing of derivative products. We suggest an alternative model — the D-CEV model, as an alternative to address this problem. The new model is an extension of a popular binomial model and is easy to implement. We further explore the natural characteristics of our alternative method with several numerical experiments. Our proposed model is found to resolve the unpleasant flaws in the model of Bandreddi, et al. (2007) while preserving its desirable properties. We also show how this framework accounts for several empirical features.

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I. Introduction

The market in credit derivative products has experienced a tremendous growth in the past years and plays an important role in today's financial market by facilitating the transfer and trading of credit risk. The International Swaps and Derivatives Association (ISDA) reported in April 2007 that the total notional amount on outstanding credit derivatives was $35.1 trillion with a gross market value of $948 billion. Portfolio credit derivatives such as collateralized debt obligations (CDOs) and basket default swaps account for a significant portion of the market, which have drawn much attention in recent years.[1] A CDO is an asset-backed securitized structure that distributes credit risk to investors by creating "tranches." Tranches are each responsible, in a sequential order, for credit losses in the reference portfolio backing the CDO. Thus its valuation highly depends on the correlated default behavior of the underlying assets.

Two main quantitative approaches for valuing credit risk and default correlation are the structural-form credit models and the reduced-form credit models. Structural-form models are based on the original framework developed by Merton (1974) and its extension by Black and Cox (1976), using the principles of option pricing. In this model a default occurs if the value of a company falls below a default barrier before the maturity of its debt liabilities. Zhou (2001b), Hull and White (2001), and Hull, Predescu, and White (2005) propose structural-form models for multi-issuer cases. Their models are dynamic in the sense that the credit qualities of companies evolve through time. Structural-form models possess economic rationale in which they provide a link between the credit quality of a firm and the firm's financial situation. Still, due to the assumption of complete information about the firm's assets and liabilities, the default event is not a total surprise. This is often referred to as the "predictability" of structural-form models (see Giesecke (2004) and Jarrow and Protter (2004)). Since default can be anticipated, the model price of a credit sensitive security converges continuously to its recovery value, in conflict with empirical observation where prices abruptly drop to its recovery value upon default. Moreover, the model implied credit spread for the firm's debt tends to zero for short time-to-maturity, at odds with positive short-term spreads seen in practice (see

[1] According to the Securities Industry and Financial Markets Association, aggregate global CDO issuance totaled US$ 157 billion in 2004, US$ 272 billion in 2005, US$ 552 billion in 2006 and US$ 486 billion in 2007. Research firm Celent estimates the size of the CDO global market to close to $2 trillion by the end of 2006.

Giesecke (2004)). Hence, such predictability of default times in this type of models is regarded as a major drawback. This consideration brings us to reduced-form models.

Reduced-form models, first studied by Jarrow and Turnbull (1995) and Duffie and Singleton (1999), overcome this deficiency of predictability by assuming that default occurs without warning at an exogenous default rate (or intensity), characterized by jump processes. The intensity is extracted from the market prices of a firm's defaultable instruments such as corporate bonds or credit default swaps, which contain a default risk premium demanded from investors of uncertainty about default events. The main shortcoming with reduced-form models is that the arrival of default is not based on any characteristic of the firm's underlying credit quality, but on market prospects. Nevertheless, they are more commonly used by practitioners for pricing, hedging, and trading purposes (see Jarrow and Protter (2004)). Among reduced-form approaches for multi-issuer cases, copula methods have become popular for pricing correlated default. These types of models were first introduced by Li (2000) and later extended by Gregory and Laurent (2005), which has become the standard market model for portfolio credit derivatives. However, common copula models are static models; they assume constant hazard rates through the whole term of a contract. But a changing default environment is more realistic.

Nowadays, researchers and practitioners seek to find innovative methods for the valuation of credit portfolio defaults. For example, Longstaff and Rajan (2006) develop a so called top-down approach to model credit portfolio losses using a direct method that does not require modeling individual correlations, whereas Carayannopoulos and Kalimipalli (2003), and Das and Sundaram (2004) suggest another method where equity correlations may be used to drive intensity correlations.

Das and Sundaram (2004) introduce a simple model for pricing securities with equity, interest-rate, and default risk. Their model is a reduced-form model. Default probabilities in their framework are derived endogenously on a binomial lattice calibrated to credit default swap markets. It is claimed that the model captures default information from both equity- and debt-market information rather than just from equity-market information (as in structural-form credit models) or just from debt-market information (as in reduced-form credit models). Bandreddi et al. (2007) use a simplified version of their model to simulate correlated defaults for credit portfolios. We will call it the defaultable CRR model (D-CRR for short). Their framework contains three main components. First, one develops for all reference issuers in a credit portfolio their equity binomial lattices with default risk considered. Second, one calibrates the lattices to the credit default swap market. Third, one simulates default with the correlated lattices to examine default risk distributions and default correlations.

Unfortunately, the model developed in Bandreddi et al. (2007) gives rise to problematic probabilities on the lattices. In particular, probabilities outside the range of 0 to 1 can arise, leading to biased results when applied to default modeling. Furthermore, our numerical analyses show other deficiencies of their model in practice, such as its conflict with traditional structural-form models in the relationship between credit risk and equity volatility. To remedy them, we suggest an alternative method to build the lattice. Our method is a generalized extension of the well-known CEV model that takes into account the “leverage effect” and is easy to implement. We call it the defaultable-CEV model (D-CEV for short). The D-CEV model is a simpler version of the model first seen in Das and Sundaram (2007), which was used for the pricing of convertible bonds with equity, interest rate, and credit risk. This thesis will analyze how our proposed model addresses the drawbacks of the D-CRR model without sacrificing their desirable properties. We compare the D-CEV model with the D-CRR model in the pricing of credit derivative products, and find that our model produces higher mean levels of default, which was neglected by the previous D-CRR approach. An examination on how different input parameters affect the results for default simulation in our framework is also performed.

In summary, the framework discussed in this thesis is a new approach that is easy to understand, in which observable equity prices are used along with an intensity-base model to simulate default in an arbitrage-free setting. The model is dynamic in that hazard rates evolve through time, which is an appealing feature for the hedging of credit positions and the valuation of new-generation credit products. The flexibility of the model enables accommodation of several known empirical phenomena. First, empirical research has shown evidence of the joint movements between credit spreads and stock option implied volatilities (see Hull, Nelken and White (2004) and Carr and Wu (2006)). The D-CEV model proposed in our thesis can be simultaneously calibrated to both the term structure of credit default swap spreads and the equity options market. We will also show how in the D-CRR model one may encounter undesirable outcomes when we calibrate to equity volatility. Second, there is evidence indicating that corporate defaults “cluster” in time (see Das, Duffie, Kapadia, and Saita (2005)). The framework discussed in this thesis allows for the accommodation of this fact. We can switch between the assumption of independence of default events, where we consider only intensity correlations, and the assumption of dependence of default events conditioned on intensities, where the default clustering effect is matched.

The remainder of this thesis is organized as follows. In Chapter II we review the previous framework for default modeling. Chapter III points out their disadvantages and describes our improved model. In Chapter IV we perform numerical experiments to explore the characteristics of our model. Chapter V makes a comparison between the D-CRR model and D-CEV model in the valuation of portfolio credit derivatives. Chapter VI concludes the thesis.

II. Literature Review

The equity binomial lattice introduced in Das and Sundaram (2004) extends the Cox, Ross, Rubinstein (CRR) model by including a jump-to-default branch for each node. Bandreddi et al. (2007) apply this model to simulate correlated default. Their framework is summarized in this Chapter.

2.1 The Defaultable CRR Model

In a discrete-time setting with time intervals of length h, the evolution of the equity price S_t to a stochastic value S_{t+h} is assumed to be of the following pattern:

$$S_{t+h} = \begin{cases} uS_t & \text{(up move)} \quad \text{w/prob } q(1-\pi_t) \\ dS_t & \text{(down move)} \quad \text{w/prob } (1-q)(1-\pi_t) \\ 0 & \text{(default)} \quad \text{w/prob } \pi_t \end{cases} \tag{1}$$

The respective "up-move" and "down-move" parameters u and d are based on CRR's settings, where $u = \mathrm{e}^{\sigma\sqrt{h}}$ and $d = 1/u$. Intuitively, equity is a security that is assumed to receive zero recovery upon the occurrence of a credit event, thus a third branch to 0 is incorporated. Given the risk-neutral default intensity from time t to $t+h$ as ξ_t, the default arrival follows a Poisson process, where the default probability π_t over the period t to $t+h$ should be

$$\pi_t = 1 - \mathrm{e}^{-\xi_t h} .$$

In a risk-neutral world, the equity price should fulfill the martingale condition

$$S_t = e^{-rh} E[S_{t+h}].$$

Hence,

$$e^{rh} S_t = uS_t \times q \times (1-\pi_t) + dS_t \times (1-q) \times (1-\pi_t) + 0 \times \pi_t ,$$

leading to the risk-neutral probability

$$q = \frac{e^{(r+\xi)h} - d}{u - d}. \tag{2}$$

Notice that the probability of an up-move becomes higher compared to that of CRR's since the jump to zero of the stock price must be compensated.

The most critical step for building the lattice in this model is to give a characterization of the default intensity. The intensity is endogenously determined with the following equation:

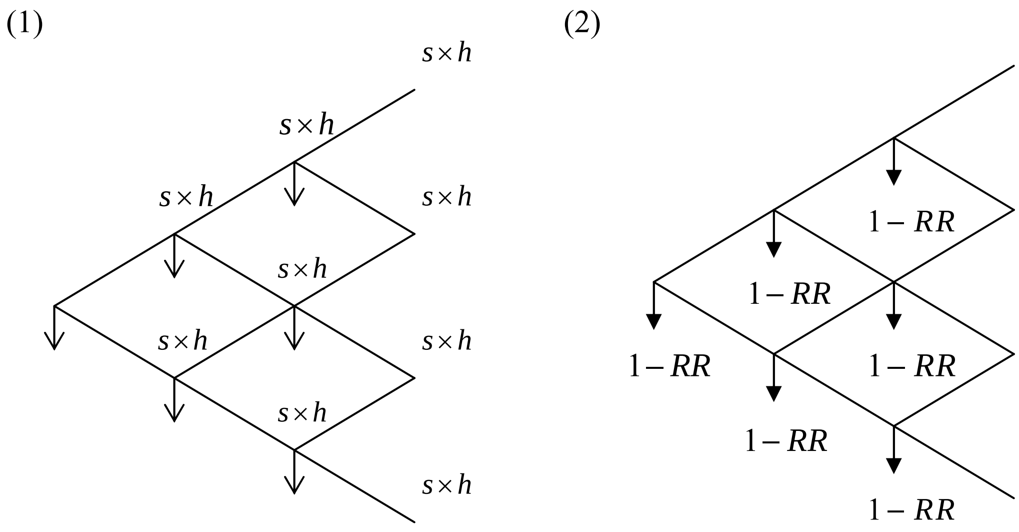
$$\xi_t = \frac{\exp(\alpha + \gamma t)}{S_t^{\beta}}, \tag{3}$$

where α, β, γ are entitled the intensity-function parameters. The idea of this function comes from the fact that equity prices tend to reflect the credit risk of a firm. It is easy to see that in this inverse relationship, as stock prices tend to be low, the default intensity moves higher. We can visualize how such a feature is analogous to structural-form models, where lower equity value leads to lower firm value and higher tendency of hitting the default barrier. The zero equity value upon default is used as a response to the barrier condition. When a firm's value is below its liabilities, equity holders get nothing. This link to equity values is economically attractive since stock prices in this model are observable whereas firm values in traditional structural models are unobservable. In the D-CRR model, the actual default event is not determined by the absolute equity value, however, but by the intensity in a Poisson arrival, as in reduced-form models. Therefore this model differs from typical structural-form models in the degree of predictability of default. The intensity in this model is an endogenous process characterized by the parameters α, β, and γ. We will show in chapter IV that the three intensity function parameters α, β, γ capture the level, slope, and curvature of the term structure of credit default swap spreads, respectively.

2.2 Calibrating Parameters

The parameters α, β, γ are calibrated to the credit default swap market. A credit default swap is an insurance-like instrument to transfer the credit risk of fixed-income products. It is a contract between two counterparties in which a protection buyer pays a fixed fee periodically to a protection seller that guarantees a contingent payment

Figure 1
Illustration of the Pricing of Credit Default Swaps



upon a credit event (such as default or failure to pay) happening in the reference entity agreed upon in the contract. The market of credit default swaps represents one of the fastest-growing derivatives markets.[2] Its high liquidity provides efficient default information, where the availability of term structures of credit default swap spreads allows one to fit the intensity-function parameters mentioned above.

Credit default swaps are quoted as the spread payment per annum made by the protection buyer. How they are priced is much discussed in the literature (for example, see Duffie (1999) or Hull and White (2000) for their no-arbitrage approaches). The main concept behind pricing a credit default swap is that the periodic payment should be the spread that equates the present value of payments made by the protection buyer with the expected loss on default over the life of the default swap contract. This is easy to implement on the lattice since the probabilities of moving on to all states and defaulted nodes are already determined when the lattice is built. As shown in Figure 1, one may begin by putting a spread payment $s \times h$ on each non-defaulted node. Then, by calculating the expected present value of these payments through backward recursion, one may get $s \times h \times PV_1$, where PV_1 is the present value of \$1 paid at all non-defaulted nodes. Second, by putting a loss-given-default of $1 - RR$ per dollar on each defaulted node, where RR is the recovery rate, and zero elsewhere, one can

[2] The Bank for International Settlements reported the notional amount on outstanding OTC credit default swaps to be \$42.6 trillion in June 2007, up from \$28.9 trillion in December 2006 (\$13.9 trillion in December 2005).

compute the present value of expected losses PV_L . Finally, equate $s \times h \times PV_1$ with PV_L , and the annual spread payment made from the protection buyer should be:

$$s = \frac{PV_L}{PV_1} \times \frac{10000}{h} \text{ (in bps)} \tag{4}$$

The illustration in Figure 1 is a general treatment that does not specify the timing of defaults and the notional value of the credit default swap. There is much flexibility in considering these details. For example, the notional value of the default swap can be assumed to be a fixed amount for convenience. In our thesis we will assume that (a) in any period in which default occurs, recovery payoffs are realized at the end of the period, (b) default is based on the default intensity at the beginning of the period. In addition, we will adopt the recovery of market value (RMV) condition in our research. Basically, the RMV condition stipulates that at default of one counterparty, the other counterparty's claim is a fraction, $1-RR$, of the market value of a non-defaulted, but otherwise equivalent, security (if the value of this security is positive). This condition is briefly described in Appendix A and can be seen in detail in Schönbucher (2003).

As mentioned before, the main purpose of this pricing procedure is to calibrate the parameters α, β, γ. This can be done by extracting different credit default swap spreads of the reference entity from its term structure of credit default swap spreads, and then fitting the parameters to this market data. The three parameters can be solved directly given three market spreads, or a sum-of-least-squares fit can be used for more data provided.

2.3 Simulating Correlated Default

Now, to simulate correlated default, one starts by building the above binomial lattices for all reference entities in a credit portfolio. Assuming N entities (issuers) in a portfolio, one may get N sets of parameters by calibrating to market data, leading to N intensity functions of the form

$$\xi_{it} = \frac{\exp(\alpha_i + \gamma_i t)}{S_{it}^{\beta_i}}, \quad i = 1, \ldots, N.$$

This equation implies that if one simulates a path of stock prices S_{it} for an issuer i, it is equivalent to simulating a path of default intensities ξ_{it} for the issuer. Therefore, with the given equity correlation, one can simulate a joint process of stock prices for all reference issuers in the portfolio, which at the same time produces the joint process of ξ_{it}, i.e., correlated intensity paths. This is how equity correlations drive intensity correlations.

The simulation starts from the root node of each issuer. At the root nodes, the initial stock prices S_{i0}, $i = 1,...,N$, are known, which implies realizing the initial values of the intensities ξ_{i0} for all firms. The simulation then proceeds in two steps:

1. One first checks if any of the reference entities has defaulted or not. When the intensity ξ_{it} at a time t is realized, the default probability π_{it} at that time is also determined — which is $\pi_{it} = 1 - \exp(-\xi_{it} h)$. Then, draw a set of uniform random numbers between 0 and 1, denoted by $u = \{u_1, u_2, \cdots, u_N\}$. If $u_i \le \pi_{it}$, then issuer i defaults at time t. The random numbers u can be independent uniform random numbers between 0 and 1 drawn with correlation using copula methods (or other techniques).

2. For the firms that have not defaulted in the first step, one further determines whether the stock prices move up or down on the binomial lattices for all the non-defaulted issuers. The probabilities of an up move q_{it} and a down move $1 - q_{it}$ are known at each time period. Given the correlation matrix Σ of stock returns, one can decompose Σ using a Cholesky factorization and sample a set of correlated standard normal random variables $\mathrm{x} = \{x_1, x_2, \cdots, x_N\}$. Given $\Phi(\cdot)$ the standard normal cumulative distribution function, if $\Phi(x_i) \le q_{it}$, then the stock price goes up for issuer i. On the other hand, if $\Phi(x_i) > q_{it}$, the stock price goes down for i.

By repeating these two steps, starting from the root node until maturity, N sample paths of stock movements (equivalently, intensity movements) are obtained. The timing and number of defaults can also be found as a result. Therefore, we are able to model correlated defaults to evaluate portfolio credit derivatives.

Drawing independent random numbers in the first step assumes independence of default events, where we consider only intensity correlations. This way a "doubly-stochastic assumption" is invoked, similar to most intensity-based models. The doubly-stochastic assumption means that, conditional on the path of the underlying state process determining default intensities, the respective default times are the first event times of independent Poisson arrivals. Comparatively, drawing random numbers with correlation assumes "conditional dependence," where additional correlation between default events is injected besides intensity correlation. We present a brief introduction of the one-factor Gaussian copula model in Appendix B, which is the method we use to draw correlated random variables in our thesis. These assumptions will be investigated in Chapter V to understand their relative strengths.

In addition to the original framework, we further suggest a more particular criterion in step 2 to grasp a general concept. Since stock values reduce to zero and remain there after default events, there should actually be no equity correlation between the defaulted firms and those that have survived. Therefore, given the outcomes after the first step in each time period, the portion of those firms that have defaulted should be excluded from the equity correlation matrix. In this way we are ensuring that the random variable draws are not interfered with by irrelevant draws in the simulation algorithm.

III. An Alternative Model: the Defaultable CEV Model

3.1 Problems with the D-CRR Model

The D-CRR model gives rise to problems of negative probabilities. Recall in the model for stock price movements discussed in the previous chapter, the probability of an up move is given by Eq. (2), which is a modification of the transition probabilities in the original CRR model. Furthermore, the stock-intensity relationship is given by Eq. (3). We now examine a set of base parameters as selected in Bandreddi et al. (2007):

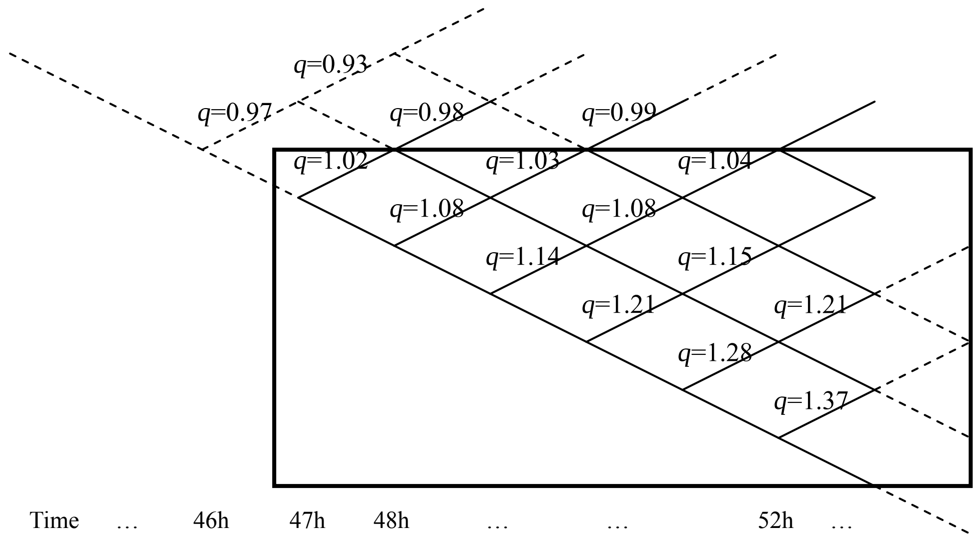
$$S = 50\,,\ \sigma = 0.3\,,\ \alpha = -0.5\,,\ \beta = 1\,,\ \gamma = 0.1\,,\ r_f = 0.03\,,$$

and a term of 5 years with monthly time intervals $h = 1/12$. When implementing this model, we encounter problematic probabilities in the lower part of the lattice. For example, when the stock price follows the lowest nodes for 47 time steps, it will become $Sd^{47} = 50*(e^{-0.3\sqrt{1/12}})^{47} = 0.854$. The intensity is then computed as $\xi_t = \mathrm{e}^{(-0.5+0.1*47/12)}/0.854 = 1.05$, and the probability of an up move will be $q = \left(e^{(0.03+1.05)/12} - e^{-0.3\sqrt{1/12}}\right)/\left(e^{0.3\sqrt{1/12}} - e^{-0.3\sqrt{1/12}}\right) = 1.022$, a dubious number. In fact, the probabilities on the nodes with stock price lower than 0.854 are all out of their valid range in this example. Figure 2 shows that part of the lattice.

One can easily examine how this happens for different parameter values. This is due to the intensity-function setting of Eq. (3) where the relationship between stock and intensity is actually a convex function. When stock prices are low, a slight decrease in stock price will lead to an immense increase in default intensity. In this case where intensity is not bounded, probabilities move out of their valid range. Although in practice this can be relieved for some cases by choosing an appropriate time interval *h*, how large the number of time steps should be chosen varies widely for different cases. Therefore, implementation of the model is unstable and computationally inefficient for simulation purposes when a large number of time steps is needed.

The existence of these probabilities also affects the results when we simulate correlated defaults. Recall the simulation method described in section 2.3. The stock price movements are determined by comparing the risk-neutral probability, *q*, with the cumulative distribution function values of the sample draws. Since the cumulative distribution function values are always in the range of 0 to 1, the stock price will

Figure 2



always move up according to the algorithm when $q > 1$. This implies that the stock price can never reach below the stock price that starts yielding invalid probabilities (0.854 in the above example). This portion of the lattice, which contains higher default intensities (and higher default probabilities) than any other parts of the lattice, will never be reached inside during the simulation process. It is hence unused, despite its significant contribution to the calibration of the intensity-function parameters. Moreover, pricing other derivatives on the lattice through backward induction seems unreasonable with these dubious probabilities.

One main motivation of our thesis is to address this issue. What we need is an alternative model that mitigates the problem and still preserves the desirable properties of the D-CRR model. We come up with the defaultable CEV model (D-CEV for short) in the next section.

3.2 The Defautable CEV Model

A similar version of this model was first seen in Das and Sundaram (2007) to price convertible bonds with default risk. In their paper, stochastic interest rates are also considered, yet it is switched off here in our work for credit modeling objectives. Prior to introducing our model, we first take a glance of what the "leverage effect" is and how taking account of it benefits the solution of the probability issue.

The "leverage effect" is a phenomenon suggesting that stock price and the volatility of its return are negatively correlated. Previous research shows that the log-normality of stock prices assumed in the Black-Scholes framework does not hold empirically due to the leverage effect in actual stock price behavior. Therefore, to account for this evidence, different diffusion processes are studied in the literature. Cox (1975) and Cox and Ross (1976) focused on a general class of stochastic processes known as the constant elasticity of variance (CEV) diffusion class:

$$dS = \mu S dt + \sigma S^c dZ \quad , \quad 0 < c \le 1 .$$

Here the instantaneous variance of dS/S (percentage price change) is equal to σ^2 / S^{2-2c} and hence an inverse function of the stock price. When the stock price is high, volatility is smaller than when the stock price is low. This matches the leverage effect qualitatively. In Eq. (2), the denominator $u-d$ will become larger with higher volatility when stock prices are low, in accordance with the leverage effect, forming a force to pull back the probabilities to their appropriate range. Therefore, the probability issue can be mitigated with this alternative setting. Actually, taking account of both the leverage effect and the inverse relationship between stock prices and default intensities describes empirical phenomena more faithfully. It is a fact that a firm usually becomes more volatile when its default intensity is high, connecting both our intensity-function setting and the leverage effect in the equity prices.

Nelson and Ramaswamy (1990) developed a recombining binomial lattice that converges weakly to the CEV process, which proves to be computationally simple. We apply their method for our lattice construction, summarized as follows:

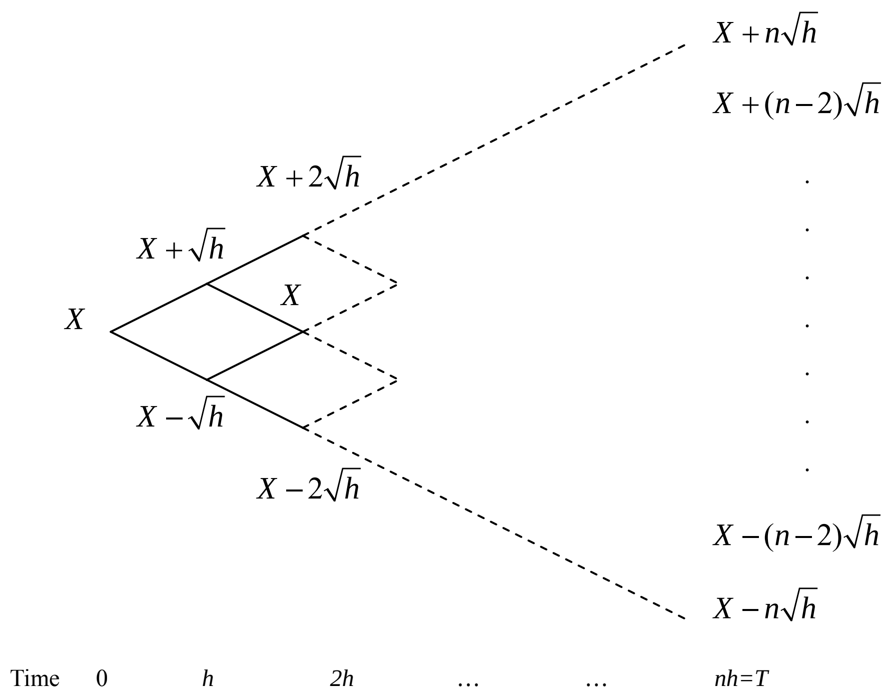
1. First, since the instantaneous volatility of the stock price is stochastic in the CEV process, building a discrete model for the stock process will result in a non-recombining lattice. Therefore, we need to make a transformation from the process of S into another process X, that produces constant instantaneous volatility. Nelson and Ramaswamy (1990) show that the transformation can be computed as

$$X(S) = \sigma^{-1} \int^{S} Z^{-c} dZ = \frac{S^{1-c}}{\sigma(1-c)} .$$

   The function $X(S)$ is the said mapping from S to X. Notice that the inverse function of the $X(S)$ is

$$S(X) = \left[\sigma(1-c)X\right]^{1/(1-c)} . \tag{5}$$

Figure 3
The *X*-Lattice in the Nelson and Ramaswamy (1990) Framework



After building a lattice for the process with constant volatility, *X*, we are able to transform from *X* back to *S* using this function. Nelson and Ramaswamy (1990) show that the process of *S*(*X*) is monotonically increasing in *X*; therefore a lattice built for *S* with this transformation inherits the computational simplicity that the lattice built for *X* demonstrates.

2. The next step is to form the *X*-lattice correspondent to the stock price, defined as

$$X_{t+h} = \begin{cases} X_t + \sqrt{h} & \text{(up move)} \quad \text{w/prob } q(1-\pi_t) \\ X_t - \sqrt{h} & \text{(down move) w/prob } (1-q)(1-\pi_t) \\ 0 & \text{(default)} \quad \text{w/prob } \pi_t \end{cases},$$

with the initial condition $X_0 = X(S_0)$. Hence by applying the relationship shown in Eq. (5), this lattice can be mapped to the stock price process

$$S_{t+h} = \begin{cases} S_t^+(X) = S_t(X_t + \sqrt{h}) & \text{(up move)} \quad \text{w/prob } q(1-\pi_t) \\ S_t^-(X) = S_t(X_t - \sqrt{h}) & \text{(down move) w/prob } (1-q)(1-\pi_t) \\ 0 & \text{(default)} \quad \text{w/prob } \pi_t \end{cases}$$

with the same set of probabilities. Here, the probability of an up-move is

$$q = \frac{e^{(r+\xi)h} S(X) - S^-(X)}{S^+(X) - S^-(X)} = \frac{e^{(r+\xi)h} - d(X)}{u(X) - d(X)}$$

by risk-neutral arguments, where

$$u(X) = S^+(X)/S(X) \quad \text{and} \quad d(X) = S^-(X)/S(X).$$

3. Finally, to build a D-CEV lattice that effectively alleviates the probability issue described in Section 3.1, we need to select an appropriate choice for the CEV leverage coefficient, c, in the CEV diffusion process. We have seen that the instantaneous variance of dS/S is equal to σ^2 / S^{2-2c} from the CEV diffusion process. Thus when $c = 1$ the instantaneous variance is simply σ^2, which corresponds to the constant volatility assumption in the Black-Scholes framework. As c is varied from 1 to 0, the negative effect of the stock price on the instantaneous variance becomes more profound, where $c = 0$ yields an inverse relationship with quadratic order, σ^2 / S^2. Consequently, c is the parameter that determines the degree of the leverage effect, and can be chosen to build a D-CEV lattice with appropriate leverage effect that relieves the probability issue.

   To choose the appropriate c the following inequality should hold:

$$0 \le q = \frac{e^{(r+\xi)h} S(X) - S^-(X)}{S^+(X) - S^-(X)} \le 1, \tag{6}$$

This does not need to be checked for all probabilities q on the lattice. Since we claimed that problematic probabilities happen in the lower part of the lattice due to the convex relationship between the stock price and the intensity in Eq. (3), all we need to do is to assure c is chosen such that the risk-neutral probability for the lowest node satisfies the inequality, for example, the node with value $X - n\sqrt{h}$ in Figure 3. As long as the node with the lowest equity value, and hence highest intensity, is free of the probability issue, then all the other risk-neutral probabilities are likely as well. This provides a criterion to choose the leverage coefficient. Still, note that as long as the leverage effect is considered in the D-CEV lattice, the probability issue can always be eased, as part (or even all) of the problematic probabilities found in the equivalent D-CRR lattice can fall within their appropriate range.

Das and Sundaram (2007) show that a change in the CEV leverage coefficient, c, in the CEV diffusion process has small quantitative effect on the prices of credit default swap spreads derived from the D-CEV lattice. In the remainder of our thesis we set a moderate $c = 0.5$ in the given examples and computations to considerably account for some of the leverage effect. We will see that this is an appropriate choice for our examples to ease the probability issue.

IV. Numerical Evaluations

We now examine how the models perform for default modeling purposes. The numerical experiments allow us to investigate several characteristics and compare across the models. Recall that our experiments are carried out through the following three steps:

1. Build the correlated equity binomial lattices for all reference issuers in the credit portfolio for the D-CRR model and the D-CEV model.
2. Calibrate the lattices to the credit default swap market.
3. Simulate default with the correlated lattices and extract the numbers of default as well as their respective default times.

4.1 Base Case Data

Now, consider a credit portfolio of 10 identical reference entities with the following base level parameters:

$$S = 50\,,\ \sigma = 0.3\,,\ \alpha = -0.5\,,\ \beta = 1\,,\ \gamma = 0.1\,,\ r_f = 0.03\,,\ RR = 0.5\,, \text{ and } \rho = 0.5$$

The term horizon is 5 years with monthly time steps $h = 1/12$ (year). This is the same data in Bandreddi et al. (2007) used for the D-CRR model. It should be noted that the calibrated values of $\alpha,\ \beta,\ \gamma$ will actually be different between the D-CRR and D-CEV model even though a same term structure of credit default swap spreads is given. Hence we first assign these values to the D-CRR model and then compute the credit spreads implied from these parameters. The resulting spreads are treated as hypothesized market spreads, which are then used to calibrate our D-CEV model. This is to ensure we are applying the same set of data to the D-CEV model for our comparison. The fitted parameters are shown in Table 1. We can see that the D-CEV model fits well the hypothesized spreads.

Then we vary different parameters and simulate 10000 paths for each case and each model. In this chapter we will assume that only default intensities are correlated, whereas default events are not. Therefore, we draw independent random numbers from a uniform distribution between 0 and 1 to determine default along the simulated paths. The total numbers of defaults and their respective time of default are derived to form default frequency distributions. We further calculate the moments of the distributions for our investigation. The methodology we use is similar to that in Bandreddi et al. (2007), but instead of adding up all intensities in a simulated path and then drawing only one random number for each path to determine default events, such as in their work, we make default draws period by period as an extension to their approach. In this way we assure that the findings from our investigation can apply to pricing products that are dependent on the timing of defaults, such as CDOs.

Table 1
Parameters for Different Models and Their Fitted Spreads

Model	Parameters			Fitted spreads (bps)				
	α	β	γ	1yr	2yr	3yr	4yr	5yr
D-CRR	–0.50	1.00	0.10	60.39	68.09	74.84	81.68	88.87
D-CEV	–0.76	0.93	0.18	60.46	68.08	74.77	81.63	88.93

The base parameters $\alpha = -0.5$, $\beta = 1$, $\gamma = 0.1$ are first given to the D-CRR model, implying market spreads of 1yr=60.39 (bps), 2yr=68.09 (bps), 3yr=74.84 (bps), 4yr=81.68 (bps), and 5yr=88.87 (bps), i.e., assuming these were the spreads observed in the market, we get from the D-CRR model the parameters $\alpha = -0.5$, $\beta = 1$, $\gamma = 0.1$. We calibrate the D-CEV model to this data and obtain its intensity function parameters.

4.2 The Impact of Equity Correlation

First, we vary equity correlation. The moments of the default-frequency distributions for different models are shown in Table 2. We find that in both models, the mean and variance of the number of defaults does not vary much under different correlations. This is consistent with most intensity-based models in which loss distributions are less sensitive to intensity correlations when default events are assumed to be independent than when they are assumed to be correlated, conditional on the correlation between intensities. We will show in Chapter V how imposing conditional correlation of default events besides intensity correlation can make default frequency distributions much more sensitive to correlation assumptions.

A critical finding here is that, the D-CEV has slightly higher mean numbers of defaults than the D-CRR model. This is interesting since we claimed that there exists an unused area where the stock price cannot reach on the D-CRR lattice. The area is associated with lower stock prices and hence higher default probabilities. Therefore, avoiding this area will underestimate the number of defaults during the simulation. Nevertheless, the D-CEV model is mostly free of the issue, and its resulting higher mean numbers of defaults proves that the probability problem indeed causes significant influence on the simulation results.

Above, the original base case was considered. We will show later how the resulting difference of the two models widens with higher credit levels and different volatility assumptions.

Table 2

Moments of Loss Distributions when Correlation is Varied

This table presents the moments of the number of defaults under different models when equity correlation is varied. Assuming 10 identical firms with the base level parameters: $S = 50$, $\sigma = 0.3$, and $RR = 0.5$. The term structure of credit default swap spreads is 1yr=60.39 (bps), 2yr=68.09 (bps), 3yr=74.84 (bps), 4yr=81.68 (bps), and 5yr=88.87 (bps), as from Table 1's D-CRR row. The risk free rate is 0.03. The term horizon is 5 years with monthly time steps $h = 1/12$ (year). We simulated 10000 sample paths and computed the moments with various equity correlation assumptions for the D-CRR model and the D-CEV model.

D-CRR

Correlation	Mean	Variance	Skewness	Kurtosis
0	0.80	0.74	0.64	2.11
0.1	0.80	0.75	0.64	2.06
0.4	0.79	0.75	0.66	2.17
0.8	0.80	0.75	0.68	2.31
1	0.79	0.74	0.72	2.38

D-CEV

Correlation	Mean	Variance	Skewness	Kurtosis
0	0.82	0.75	0.61	2.05
0.1	0.81	0.75	0.64	2.12
0.4	0.82	0.76	0.68	2.28
0.8	0.80	0.75	0.68	2.32
1	0.81	0.74	0.61	1.98

4.3 The Impact of Equity Volatility

Second, we fix the equity correlation $\rho = 0.5$, same as in Section 4.1, and vary the equity volatility. This test was done also in Bandreddi, et al. (2007), where they claim that their outcomes are consistent with structural-form models in which the mean number of defaults in a portfolio increases with higher volatility. Nevertheless, they use a wrong methodology to yield this expected result. Indeed, in their work they neglect the true effect of volatility since they vary volatility without simultaneously re-adjusting the intensity-function parameters. But since volatility is determined at the beginning of the contract and held constant throughout the term in the model, the calibrated values of intensity-function parameters should be affected not only by the term structure of credit default swap spreads, but also by different volatility assumptions.

To rectify their sensitivity analysis, we simultaneously recalibrate the parameters every time the volatility is varied. In this recalibration methodology we generate results that concur with the empirical evidence on the joint movements between credit spreads and stock option implied volatilities. This has been discussed extensively in the literature (for example, Hull, Nelken and White (2004), Carr and Wu (2006)). It is pointed out that the credit default swap market and the stock options market contain overlapping information on the market and credit risk of the company, and moreover suggested that credit spreads are positively correlated with stock options' implied volatilities. Our goal is to see if the D-CRR model and the D-CEV model capture this empirical fact.

Table 3 shows our results. We stated that the analysis for equity volatility in Bandreddi, et al. (2007) is flawed. Indeed, following our method, D-CRR no longer yields the relationship whereby the mean number of defaults increases with volatility. Instead, the mean number of defaults drops as volatility increases, contrary to the claim of consistency with structural-form models in Bandreddi, et al. (2007). Even more, when volatility becomes higher, the difference in mean grows larger between that produced by the D-CRR model and by the D-CEV model. We believe this is caused by the rising numbers of problematic probabilities while volatility increases. We can see that the difference in moments and calibrated parameters between the two models is smaller with lower volatility. It is likely because there are fewer problematic probabilities found in the D-CRR model when volatility is low.

From the results for the D-CEV model, we observe that the fitted parameters and the resulting mean numbers of defaults are almost the same for different volatility assumptions. However, to match the claim of consistency to structural-form models the mean numbers of defaults should rise with volatility. So does the D-CEV fail to corroborate the evidence of higher default numbers given increased volatility as in structural-form models? The reason the simulation results are not sensitive to volatility is that the term structure of credit defaults swap spreads is held constant in our example. In the real world, however, higher equity volatility often comes with higher credit spreads for a firm, and hence higher probability of default. We can check in Table 4 the effect from different term structure of credit default swap spreads on simulation results as we vary the base case term structure of credit default swap spreads given in Section 4.1. It is clear from the table that increased spreads lead to increased mean numbers of defaults. Hence, combining both results from Table 3 and

Table 3

Moments of Loss Distributions when Volatility Is Varied

This table shows the moments of the number of defaults under different models when equity volatility is varied. Assuming 10 identical firms with the base level parameters: $S = 50$, $\rho = 0.5$, and $RR = 0.5$. The term structure of credit default swap spreads is 1yr=60.39 (bps), 2yr=68.09 (bps), 3yr=74.84 (bps), 4yr=81.68 (bps), and 5yr=88.87 (bps), as given in Table 1. Intensity function parameters α, β, and γ are recalibrated to both the credit default swap term structure and equity volatility. The risk free rate is 0.03. The term horizon is 5 years with monthly time steps $h = 1/12$ (year). We simulated 10000 sample paths and computed the moments with various volatility assumptions for the D-CRR model and the D-CEV model.

D-CRR

Volatility	Moments		Calibrated Parameters			Number of Problematic Probabilities
	Mean	Variance	α	β	γ	
0.1	0.82	0.76	–0.79	0.92	0.17	0
0.2	0.81	0.76	–0.67	0.96	0.15	1
0.3	0.78	0.77	–0.50	1.00	0.10	60
0.4	0.75	0.76	–0.45	1.01	0.03	107
0.5	0.72	0.78	–0.41	1.03	–0.06	110

D-CEV

Volatility	Moments		Calibrated Parameters			Number of Problematic Probabilities
	Mean	Variance	α	β	γ	
0.1	0.83	0.78	–0.76	0.93	0.18	0
0.2	0.82	0.78	–0.76	0.93	0.18	0
0.3	0.82	0.76	–0.76	0.93	0.18	0
0.4	0.82	0.76	–0.76	0.93	0.18	0
0.5	0.82	0.76	–0.76	0.93	0.18	0

Table 4

Moments of Loss Distributions When The Term Structure of Credit Default Swap Spreads Is Varied

This table shows the moments of the number of defaults under different models when the term structure of credit default swap spreads is varied. Assuming 10 identical firms with the base level parameters: $S = 50$, $\sigma = 0.3$, $\rho = 0.5$, and $RR = 0.5$. The base case term structure of credit default swap spreads is 1yr=61.09 (bps), 2yr=68.73 (bps), 3yr=75.42 (bps), 4yr=82.26 (bps), and 5yr=89.52 (bps), as given in Table 1. The base case is then varied by -20 (bps), -10 (bps), +10 (bps), and +20 (bps). Intensity function parameters α, β, and γ are always recalibrated to both the credit default swap term structure and equity volatility for each case. The risk free rate is 0.03. The term horizon is 5 years with monthly time steps $h = 1/12$ (year). We simulated 10000 sample paths and computed the moments with various term structure of credit default swap spread assumptions for the D-CEV model.

D-CEV

	Term Structure of Credit Default Swap Spreads (bps)					Moments	
	1yr	2yr	3yr	4yr	5yr	Mean	Variance
+20 (bps)	81.09	88.73	95.42	102.26	109.52	0.98	1.10
+10 (bps)	71.09	78.73	85.42	92.26	99.52	0.88	1.01
Base Case	61.09	68.73	75.42	82.26	89.52	0.80	0.94
-10 (bps)	51.09	58.73	65.42	72.26	79.52	0.73	0.85
-20 (bps)	41.09	48.73	55.42	62.26	69.52	0.65	0.79

Table 4, we know that in our D-CEV model (1) higher credit spreads will lead to higher mean numbers of defaults and (2) higher volatility will not have any significant influence on the mean number of defaults. It is thus evident that as both credit spreads and equity volatility jointly increase in line with empirical facts, the mean number of defaults will increase in the D-CEV model. For the D-CRR model this will not always hold. Indeed, in our example the positive effect from higher term structure of credit default swap spreads on the mean numbers of defaults can be offset by a negative effect from higher volatility. Hence, we conclude that, when applied in practice with empirical data, the D-CEV model resembles structural-form models in their relationship between the equity volatility and the mean number of defaults in a better way than the D-CRR model. That is, for the D-CEV model the mean number of defaults can increase with higher volatility as in structural-form models given the condition that credit spreads increase with volatility.

One last thing to mention in Table 3 is that, the number of problematic probabilities is reduced to zero in our example. This is because the leverage coefficient c we chose satisfies the inequality given in Eq. (6). It should be noted that the D-CEV model may not completely eliminate the problem for some cases where the inequality condition does not hold. Still, whether this condition is imposed or not, by accounting for stochastic volatility with the leverage effect and adjusting part of the probabilities, the D-CEV model has this problem to a less extent.

4.4 The Impact of Intensity Function Parameters

We now return to the intensity function parameters and reveal their effect on the term structure of credit default swap spreads. These parameters contain important information about the term structure of credit default swap spreads.

We plot the term structure of credit default swaps spreads for different parameter cases, in Figure 4, Figure 5, and Figure 6. Here we present the results for the D-CEV model. A base case is considered where $\alpha = -0.76$, $\beta = 0.93$, and $\gamma = 0.18$ within the D-CEV model. These are the parameters from the D-CEV row in Table 1. In Figure 4 we vary α, in Figure 5 we vary β, and in Figure 6 we vary γ. The parameters are individually varied by -0.5, -0.3, -0.1, $+0.1$, $+0.3$, $+0.5$.

We can clearly see in Figure 4 that α primarily captures parallel movements of the curve, where higher values of α lead to higher levels of credit spreads. It is thus regarded as a "level" modulator. Figure 5 shows that β is responsible for part of the level and slope, but holds an inverse relationship with the spreads. Since β lies on the denominator of the intensity function described in Eq. (3), a high value of β leads to lower intensity, as well as lower levels and slopes of the term structure of credit default swap spreads. As for γ, we can see in Figure 6 that the slope and convexity of the curve increases with γ, thus the shape is also captured. Consequently, with these parameters we have sufficient degrees of freedom to describe the level, slope, and shape of the term structures of credit default swap spreads. This serves a direct link to credit markets and a convenient way to measure how valuation of credit products is affected by changes in overall credit market conditions such as those reflected in the term structure of credit default swaps spreads.

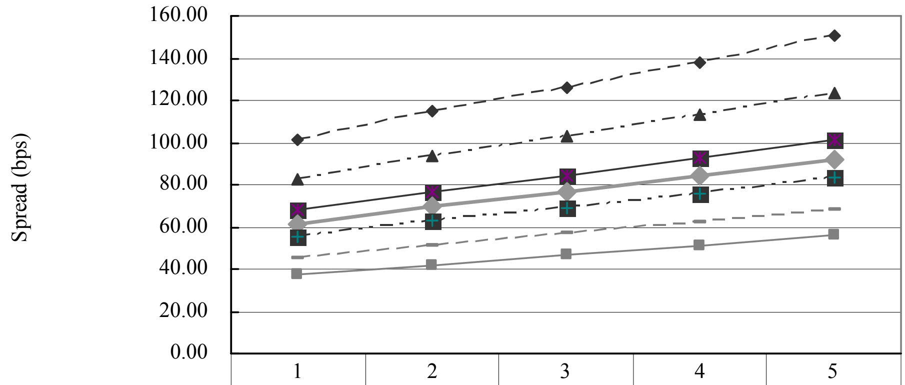
To have a better understanding of the effects of different intensity-function parameters on simulation, Table 5 shows what happens to the moments of the default frequency distributions when the intensity-function parameters are changed. The results are intuitive now that we have looked into the implications of the parameters. We see that with higher α and γ the level of credit spreads increases, indicating lower credit quality and higher mean numbers of defaults. On the other hand, lower mean numbers of defaults come with higher β's.

Figure 4

Term Structure of Credit Default Swap Spreads When Alpha Is Varied

This figure shows the impact of the intensity function parameter α. The D-CEV model is applied for this examination. We consider a base case where $\alpha = -0.76$, $\beta = 0.93$, and $\gamma = 0.18$. The parameter α is varied by -0.5, -0.3, -0.1, $+0.1$, $+0.3$, $+0.5$. We plot the respective spread quotes for the credit default swaps of maturities one year to five years, i.e., the term structure of credit default swap spreads.

Term Structure of Credit Default Swap Spreads



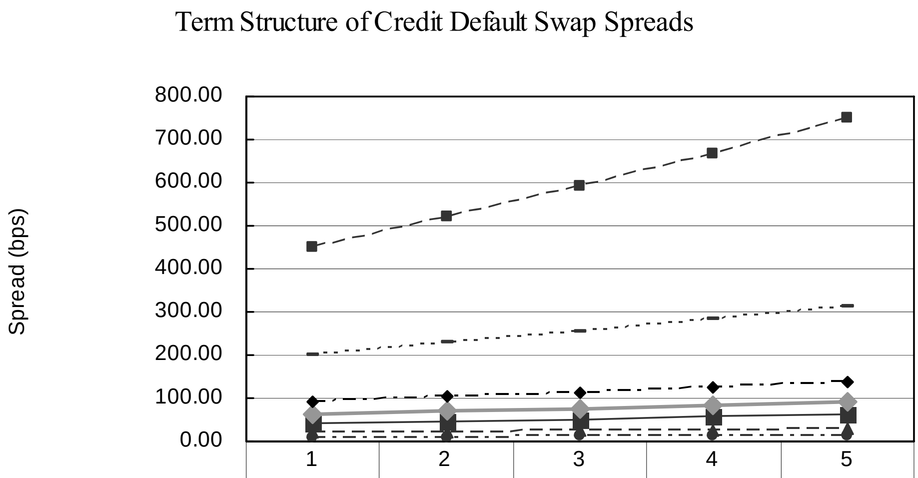
	1	2	3	4	5
Alpha+0.5	101.25	114.50	126.22	138.25	151.01
Alpha+0.3	82.87	93.71	103.33	113.22	123.76
Alpha+0.1	67.83	76.71	84.59	92.73	101.41
Base Case	61.37	69.40	76.54	83.91	91.79
Alpha-0.1	55.52	62.79	69.26	75.93	83.08
Alpha-0.3	45.45	51.40	56.70	62.18	68.06
Alpha-0.5	37.20	42.08	46.42	50.92	55.75

Time to Maturity

Figure 5

Term Structure of Credit Default Swap Spreads When Beta Is Varied

This figure shows the impact of the intensity function parameter β. The D-CEV model is applied for this examination. We consider a base case where $\alpha = -0.76$, $\beta = 0.93$, and $\gamma = 0.18$. The parameter β is varied by -0.5, -0.3, -0.1, $+0.1$, $+0.3$, $+0.5$. We plot the respective spread quotes for the credit default swaps of maturities one year to five years, i.e., the term structure of credit default swap spreads.



	1	2	3	4	5
Beta+0.5	8.65	9.75	10.73	11.74	12.83
Beta+0.3	18.92	21.36	23.51	25.72	28.10
Beta+0.1	41.44	46.82	51.58	56.50	61.75
Base Case	61.37	69.40	76.54	83.91	91.79
Beta-0.1	90.93	103.02	113.81	124.97	136.91
Beta-0.3	200.52	228.88	254.75	281.81	310.97
Beta-0.5	447.95	521.55	592.17	668.10	751.53

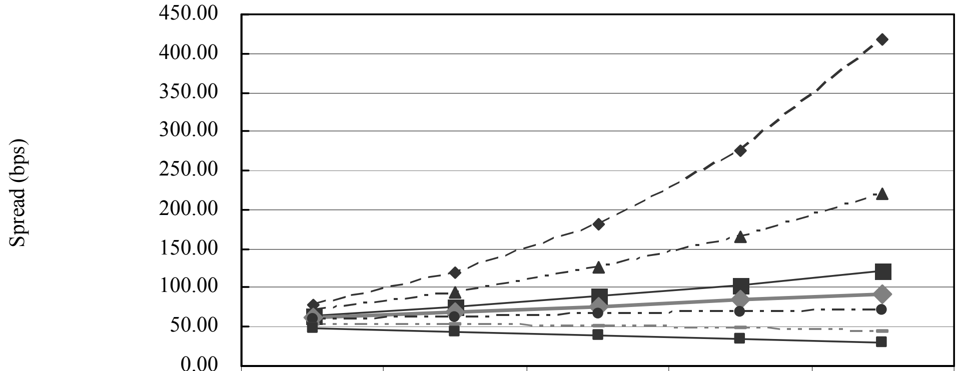
Time to Maturity

Figure 6

Term Structure of Credit Default Swap Spreads When Gamma Is Varied

This figure shows the impact of the intensity function parameter γ. The D-CEV model is applied for this examination. We consider a base case where $\alpha = -0.76$, $\beta = 0.93$, and $\gamma = 0.18$. The parameter γ is varied by -0.5, -0.3, -0.1, $+0.1$, $+0.3$, $+0.5$. We plot the respective spread quotes for the credit default swaps of maturities one year to five years, i.e., the term structure of credit default swap spreads.

Term Structure of Credit Default Swap Spreads



	1	2	3	4	5
Gamma+0.5	78.38	118.97	180.24	275.03	418.12
Gamma+0.3	70.90	95.01	125.57	166.36	220.90
Gamma+0.1	64.34	76.82	89.66	104.25	121.15
Base Case	61.37	69.40	76.54	83.91	91.79
Gamma-0.1	58.58	62.90	65.80	68.36	70.81
Gamma-0.3	53.51	52.19	49.71	47.14	44.68
Gamma-0.5	49.04	43.86	38.65	34.19	30.46

Time to Maturity

Table 5

Moments of Loss Distributions When Intensity Function Parameters Are Varied

This table presents the moments of the number of defaults under the D-CEV model varying intensity function parameters. Assuming 10 identical firms with the same parameter values: $S = 50$, $\rho = 0.4$, and $RR = 0.5$. The base case for intensity function parameters is set as: $\alpha = -0.5$, $\beta = 1$, and $\gamma = 0.1$. The risk free rate is 0.03. The term horizon is 5 years with monthly time steps $h = 1/12$ (year). We simulated 10000 sample paths and computed the moments for various parameter assumptions.

D-CEV

Alpha	Mean	Variance	Skewness	Kurtosis
–0.5	0.66	0.62	0.55	1.56
0	1.03	0.93	0.78	3.17
0.5	1.61	1.34	0.87	5.40

D-CEV

Beta	Mean	Variance	Skewness	Kurtosis
0	9.85	0.14	–0.13	0.18
0.5	3.72	2.35	0.64	15.51
1	0.65	0.61	0.55	1.58
2	0.01	0.01	0.01	0.02

D-CEV

Gamma	Mean	Variance	Skewness	Kurtosis
–0.1	0.41	0.39	0.35	0.70
0	0.51	0.48	0.43	1.09
0.1	0.66	0.60	0.48	1.35
1	7.09	2.11	–0.95	12.82

V. Further Research and Applications

We mentioned in the previous chapter that loss frequency distributions are less sensitive to intensity correlations when default events are assumed to be independent after conditioning on intensity correlation. Therefore, intensity correlations are insufficient to fully describe the joint movements in credit portfolios. In this chapter, we will first discuss about basket default swaps. Then we will look into different kinds of correlations that should be taken with care, specifically the correlation between intensities and the conditional correlation between default events. We further present some results for the valuation of *n*th-to-default contracts and CDOs.

5.1 Basket Default Swaps

A basket default swap is similar to a single entity credit default swap except that the underlying is a portfolio of entities rather than one single entity. One popular type of basket default swaps is an *n*th-to-default swap. For an *n*th-to-default swap whenever the *n*th default occurs in the reference portfolio, the buyer stops paying the periodic swap premium and receives the loss-given-default amount, $1-RR$. The premium does not stop until the *n*th default, even if there are already defaults in the portfolio.

The cost of protection for a basket default swap depends on its probability of being triggered within a specific time, i.e., the probability that the seller has to payout the loss-given-default amount. Correlation assumptions between the reference entities are key elements to the evaluation of such probabilities. They determine how credit risk in the reference portfolio is distributed among different types of default swaps.

In the following section, we will explore the correlation between intensities and the correlation between actual default events.

5.2 Intensity Correlation vs. Conditional Correlation

As we have introduced in Chapter I, Das, Duffie, Kapadia, and Saita (2005) show empirical evidence that corporate defaults cluster in time and that the doubly-stochastic assumption is rejected. The doubly-stochastic assumption says that, conditional on the path of the underlying state process determining default intensities, the respective default times are the first event times of independent Poisson arrivals. The assumption rules out the presence of contagion or frailty (incompletely observed default covariates not captured by the correlation in intensity processes across firms). We will test how our model accommodates the default contagion effect observed in empirical research and thus goes beyond the doubly-stochastic assumption.

We first describe what the conditional dependence and the conditional independence between default events are and how they are applied, as described below:

1. Conditional independence: After making intensities correlated, no further correlation is injected. Draw independent uniform random numbers for each issuer at each time-period in the simulation path and then compare these with the default probabilities to determine default occurrence.

2. Conditional dependence: Intensities are correlated, as well as default events, where the additional correlation between default events is called "conditional correlation." Draw *correlated* random numbers with normal distribution (or other distributions) and then compare the cumulative distribution functions of these numbers with the default probabilities to determine default occurrence. As we have mentioned in section 2.3, we will draw correlated random variables with the one-factor Gaussian copula model discussed in Appendix B.

To investigate which type of correlation matters more, we examine a simple portfolio with two reference entities where the parameters are set to be the same as in Section 4.1. Two experiments are conducted to test the effects on the default frequency distribution from conditional independence and conditional dependence for the D-CEV model. First, we impose correlation between intensities and assume independence between default events. Second, we impose both intensity correlation and conditional correlation. Note that in our D-CEV model, both intensity correlation and conditional correlation use the equity correlations between the reference entities as proxies. Then, by extracting the proportions of triggering both first-to-default and second-to-default contracts out of 10000 simulated paths, we are able to calculate the statistical probability of triggering each *n*th-to-default contract.[3]

Figure 7 exhibits our results. In the first case where intensity correlation is imposed without conditional correlation (which is equivalent to the doubly stochastic assumption), we see that both the probabilities of hitting the first-to-default and second-to-default contracts are not sensitive to correlation variations. It is obvious we cannot price correlation-dependent products under this assumption as the probabilities of default are similar across different correlation assumptions. We therefore move to the second case where intensity correlation and conditional correlation are both

[3] The statistical probability (or relative frequency probability) is the interpretation of probability that defines an event's probability as the limit of its relative frequency in a large number of trials.

imposed. We see that as the correlation value for both types of correlations increases, the probability of hitting the first-to-default contract decreases and the probability of default for the second contract increases. Indeed, when the default events of the two reference entities are assumed to be uncorrelated, all first defaults are absorbed by the first-to-default contract, and there is a lower chance to have two defaults at a time since defaults are independent. This leads to higher default probabilities for the first contract compared to that of the second contract. On the other hand, when the two entities are perfectly positively correlated, then this is the situation where either both default or none defaults. The probability of two firms defaulting is the same as the probability of just one firm defaulting. In such a case the probability of no firms defaulting should be the least among all correlation assumptions. We can see in Figure 7 that when the correlation coefficient is one, both contracts have the same probability of being triggered, which is approximately 0.08.

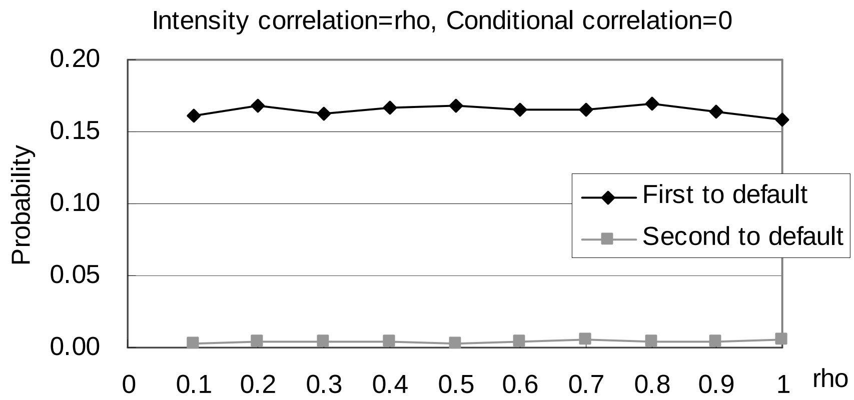
In the following sections, we will impose both the intensity correlation and the conditional correlation in the models to perform numerical experiments for the valuation of *n*th-to-default contracts and CDOs.

Figure 7

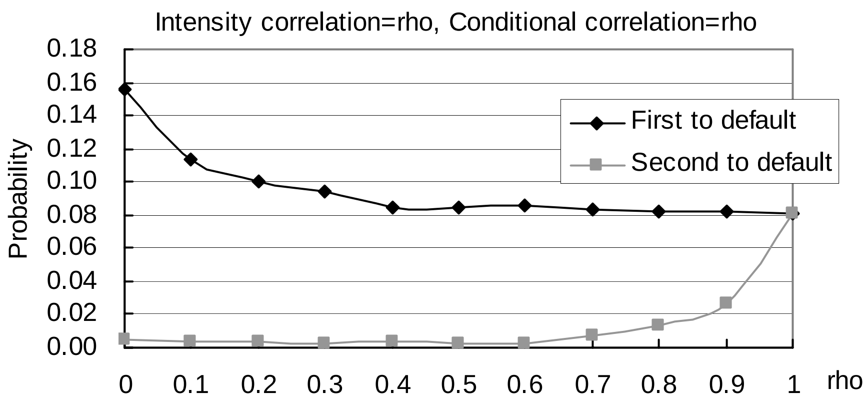
Intensity Correlation vs. Conditional Correlation

This numerical experiment was carried out with the D-CEV model. Assuming two identical firms with the base level parameters: $S = 50$, $\sigma = 0.3$, $\alpha = -0.5$, $\beta = 1$, $\gamma = 0.1$, and $RR = 0.5$ .The risk free rate is 0.03. The term horizon is 5 years with monthly time steps $h = 1/12$ (year). We simulated 10000 sample paths and extracted the probability of triggering first-to-default and second-to default contracts out of the total number of simulated paths.

Panel A



Panel B



5.3 Results for *n*th-to-Default Contracts

In this section, we present some numerical results to examine how correlation assumptions affect loss frequency distributions and the probability of triggering *n*th-to-default credit default swaps. We then make a comparison across the models discussed in this thesis to see how they perform. Consider a 5-year *n*th-to-default credit default swap on a basket of 10 reference entities. The initial data are assumed to be the same across all entities, where the initial stock price $S = 50$, volatility $\sigma = 0.3$, and recovery rate $RR = 0.5$. The risk-free interest rate is constant at 0.03, and the term structures of credit default swap spreads across all entities are also the same, given by: 1yr=258.02 (bps), 2yr=287.88 (bps), 3yr=311.53 (bps), 4yr=333.04 (bps), and 5yr=352.69 (bps). We obtained all results by simulating 10000 paths for each case.

We start by investigating the frequency distribution of the number of defaults and the probability of triggering *n*th-to-default swaps for the D-CEV model to understand the basic pattern for these figures. The results are exhibited in Figure 8.

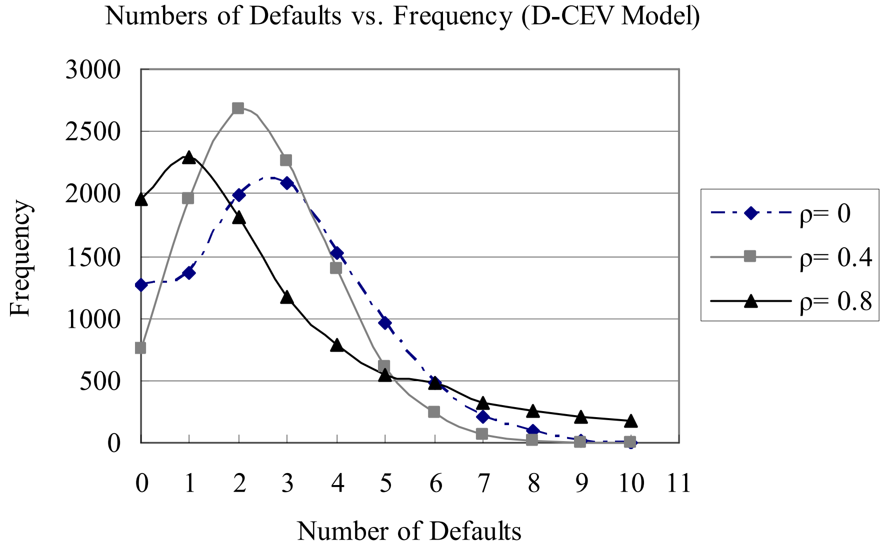
Panel A in Figure 8 shows the impact of changing correlation assumptions toward loss frequency distributions, where we plotted the distributions for $\rho = 0$, $\rho = 0.4$, and $\rho = 0.8$. To analyze this figure we look into the skewness of the distributions. We can observe from this panel that the higher the correlation, the more positively skewed the loss distribution. This implies that when the reference entities are highly correlated, lower numbers of defaults appear more frequently than when correlation is low. In other words, when correlation is low, there is higher probability that more defaults will happen. This is the basic structure for the frequency of the number of defaults in a portfolio, but what we are interested in is how the default risk in a portfolio is distributed among *n*th-to-default swaps for the pricing of such contracts.

For Panel B in Figure 8, the statistical probability of triggering *n*th-to-default contracts for different *n* is depicted for $\rho = 0$, $\rho = 0.4$, $\rho = 0.8$, and $\rho = 1$. We can see that increasing the correlations between all entities lowers the probability of triggering an *n*th-to-default contract if *n* is small and increases the probability if *n* is large. To understand this, consider how correlations move from the two limiting cases, zero to one. When correlation is zero, the probability of being triggered is a decreasing function of *n*. That is, the higher *n* is, the lower the probability of triggering an *n*th-to-default contract. On the other hand, in a perfectly correlated environment all entities default at the same time, which infers the same probabilities of being triggered for all *n*. Clearly seen from the figure, we are progressing from a negatively sloped function to a flat curve as correlation increases, and the probability decreases for low *n* and increases for high *n*.

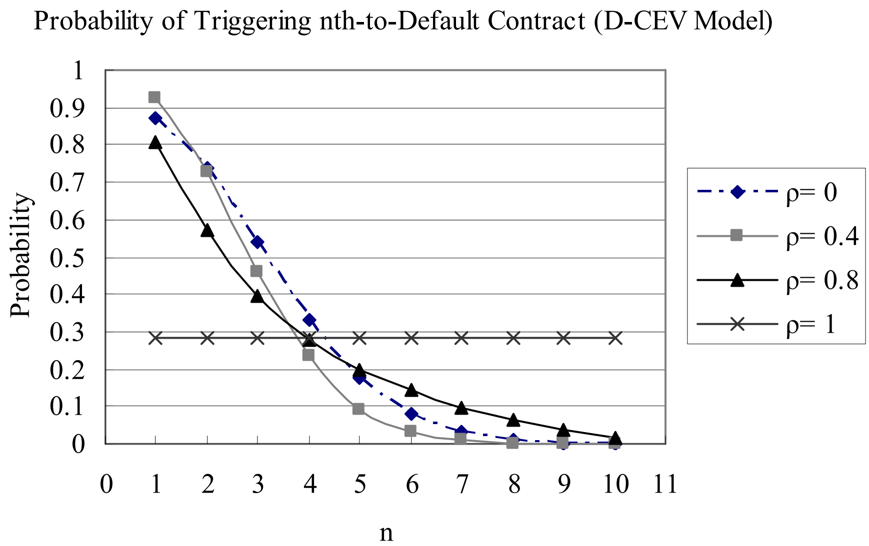
Figure 8

This figure shows the loss frequency distribution and probability of triggering *n*th-to-default contracts on a basket of 10 reference entities. The initial data are assumed to be the same across all entities, where the initial stock price $S = 50$, volatility $\sigma = 0.3$, and recovery rate $RR = 0.5$. The term is five years. The risk-free interest rate is constant at 0.03, and the term structures of credit default swap spreads across all entities are also the same, given by: 1yr=258.02 (bps), 2yr=287.88 (bps), 3yr=311.53 (bps), 4yr=333.04 (bps), and 5yr=352.69 (bps).

Panel A



Panel B



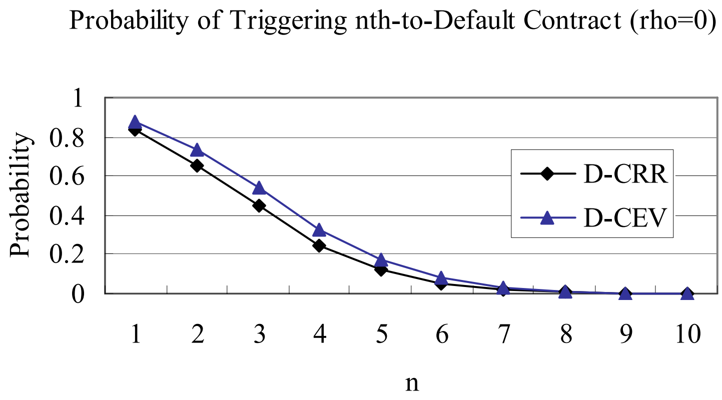
We now make a comparison across the D-CRR model and the D-CEV model. We present the statistical probability of triggering various *n*th-to-default contracts out of the 10000 trials in Figure 9 and Figure 10. The data used for Figure 9 are the same with those used in Figure 8. We varied correlation assumptions as different scenarios, showing their respective results in different panels of the figure. As for Figure 10, we changed to a downward-sloping term structure of credit default swap spreads, where the quoted spreads are 1yr=352.69 (bps), 2yr=333.04 (bps), 3yr=311.53 (bps), 4yr=287.88 (bps), and 5yr=258.02 (bps). We consider this downward-sloping term structure of credit default swap spreads just to ensure the shape of this curve is not a main determinant of our experiments.

From Figure 9 and Figure 10 we find that the statistical probabilities (i.e., frequencies) are always lower for all *n*th-to-default contracts within the D-CRR model than the D-CEV model, and the difference between the two models is quite marked. This holds for all correlation assumptions and different shapes of the term structure of credit default swap spreads. We claim that the D-CRR model undervalues these statistical probabilities which lead to undervalued contracts. This is due to the problematic probabilities for stock movements on the lattices mentioned in previous chapters. In the next section we will see how the actual prices for CDOs yield similar conclusions.

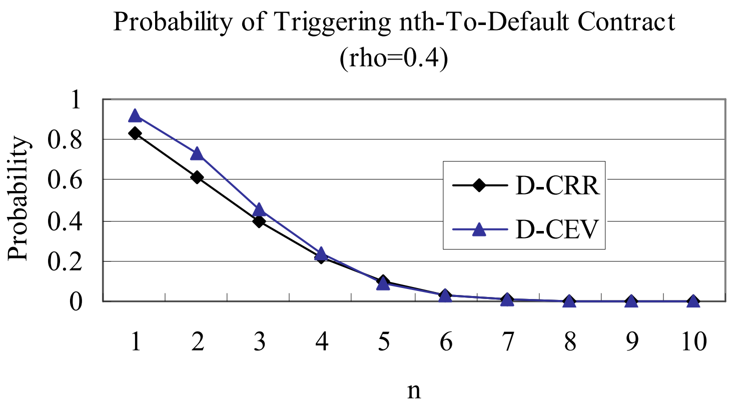
Figure 9

Statistical Probability of Triggering *n*th-to-Default Contracts With Different Correlation Values and An Upward-Sloping Term Structure of Credit Default Swap Spreads

Panel A



Panel B



Panel C

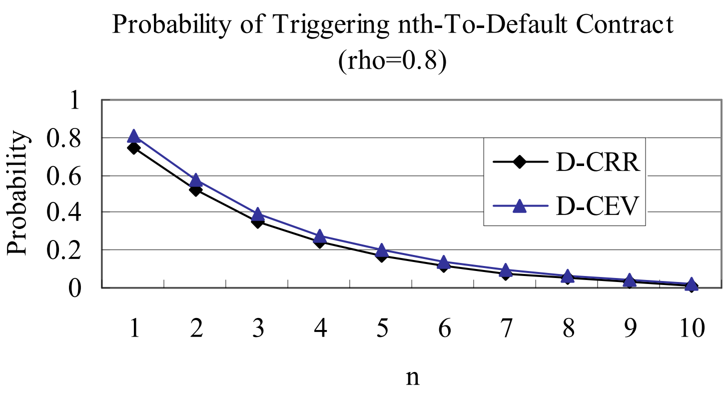
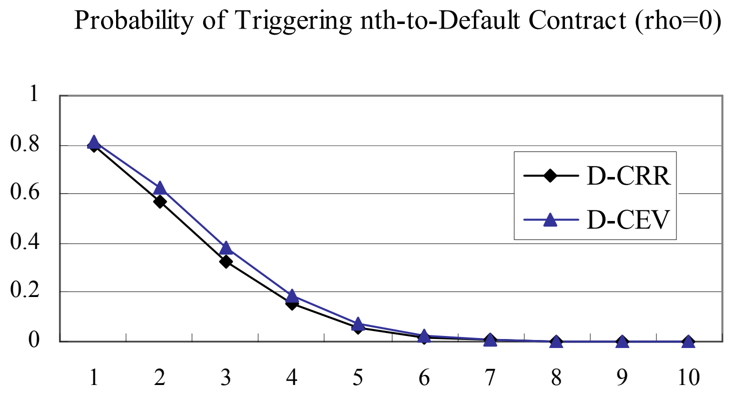


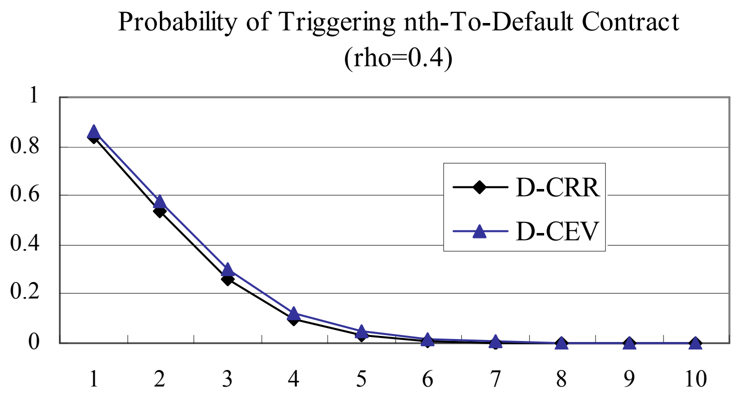
Figure 10

Statistical Probability of Triggering *n*th-to-Default Contracts With Different Correlation Values and A Downward-Sloping Term Structure of Credit Default Swap Spreads

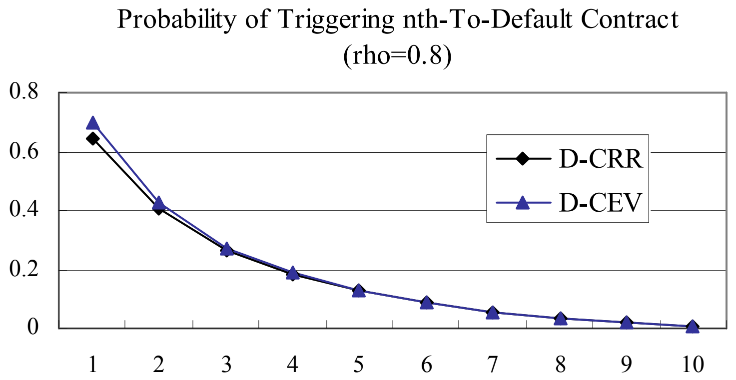
Panel A



Panel B



Panel C



5.4 The Valuation of CDO Tranches

A CDO structure distributes credit risk to investors by creating different tranches. A CDO tranche is a fixed-income security defined with reference to a portfolio of credit exposures and specified with a particular range in which it is responsible for within the total notional of the portfolio. This range is defined by an attachment point and a detachment point. For example, the successive tranches can be responsible for 0% to 3%, 3% to 7%, 7% to 10%, and so on of the total notional. When the percentage of total cumulative losses in the reference portfolio reaches the attachment point for a specific tranche, it starts to absorb the subsequent default losses until its detachment point is reached.

To illustrate how a CDO tranche is priced, suppose there are N entities in the portfolio backing a CDO each with the same notional principal L, and the same recovery rate RR. The attachment point and detachment point for each tranche can be mapped into the number of losses. A tranche with attachment point a_L and detachment point a_H is responsible for the n_L-th to n_H-th loss where $n_L = a_L N/(1-RR)$ and $n_H = a_H N/(1-RR)$. For example, if $N = 100$, $RR = 40\%$, $a_L = 5\%$ and $a_H = 15\%$, then $n_L = 8.33$ and $n_H = 25$. In this case the initial notional for the tranche is $(0.15-0.05)\times N \times L$. This is the remaining notional as long as there have been 8 or fewer defaults. This tranche bears 66.67% of the cost of the 9th default, and all costs from the 10th to 25th default.

To generalize this, let $P\left(j, a_L, a_H\right)$ be the remaining notional of a tranche with attachment point a_L and detachment point a_H after j defaults in a portfolio. A general expression for the outstanding tranche notional is

$$P\left(j, a_L, a_H\right) = \begin{cases} \left(a_H - a_L\right) NL & j < m\left(n_L\right) \\ a_H NL - j\left(1-RR\right)L & m\left(n_L\right) \le j < m\left(n_H\right), \\ 0 & j \ge m\left(n_H\right) \end{cases}$$

where $m(x)$ is the smallest integer greater than x.

The investor of a tranche is often referred to as the "protection seller." If defaults occur over the lifetime of the contract, the protection seller is required to make payments to its counterparty, the protection buyer, for cumulative losses on the reference portfolio that are in the range between the attachment and detachment points. In exchange for such coverage of cumulative default losses, the protection buyer of a tranche is required to make periodic spread payments on its remaining tranche notional to the protection seller.

A CDO is quoted as the spread paid by the protection buyer, which equates the expected present value of default costs to be borne by the protection seller ("protection leg") to the expected present value of investing in the tranche ("premium leg"). The value of the premium leg is the present value of the spread payments the protection seller receives from the protection buyer. Assuming M payment days, $t = t_1,\ t_2,\ \ldots,\ t_M$, on which the buyer of protection makes payments to the seller. Note that payments are only made as long as the (uncertain) effective notional of the tranche at time t_i, denoted by $P(t_i)$, is positive. Assume also that investors discount expected future income streams using the discount factors $D(0,t_i)$. Given the tranche premium S, the expected present value of the premium leg is:

$$V_{prem} = S \times E\left[\sum_{i=1}^{M} D(0,t_i) \times P(t_i)\right]$$

The expected tranche sizes depend on the number and timing of any future defaults and the expected costs of these future defaults. The present value of the premium leg is lower if (1) the premium is low, (2) the recovery rate is low, and (3) default losses are incurred early. The expected present value of the protection leg is:

$$V_{prot} = E\left[\sum_{i=1}^{M} D(0,t_i) \times \left(P(t_i) - P(t_{i-1})\right)\right]$$

The present value of the protection leg is lower if (1) the tranche size does not change, (2) the recovery rate is high, and (3)_defaults occur late during the contract period. The tranche premium is found by solving $V_{prem} = V_{prot}$ for S:

$$S = \frac{E\left[\sum_{i=1}^{M} D(0,t_i) \times \left(P(t_i) - P(t_{i-1})\right)\right]}{E\left[\sum_{i=1}^{M} D(0,t_i) \times P(t_i)\right]} \tag{7}$$

5.5 Results for CDO Tranches

We consider a CDO with a reference portfolio of 100 entities. We assume that the entities have the same set of parameter values, where initial stock price $S = 50$, volatility $\sigma = 0.5$, recovery rate $RR = 0.4$, and the principal of each entity $L = 100$. The contract term is set as five years and the time interval between nodes is $h = 1/12$ (year). We also assume that the risk-free interest rate is constant at 0.03, and the term structure of credit default swap spreads for all entities are the same, given by 1yr=60.39 (bps), 2yr=68.09 (bps), 3yr=74.84 (bps), 4yr=81.68 (bps), and 5yr=88.87 (bps). This is the set of parameters we investigated in Table 3 of Chapter IV. We investigate five tranches, each responsible for the range 0% to 3%, 3% to 7%, 7% to 10%, 10% to 15%, and 15% to 30% of the notional principle, respectively. In practice, tranches are often defined by different "classes." For example, the 0% to 3% tranche, which absorbs the first portion of losses, is usually referred to as the "equity tranche." The 3% to 7%, 7% to 10%, and 10% to 15% tranches are called "mezzanine tranches." As for tranches with the highest attachment point, in this case the 15% to 30% tranche, we usually call it a "senior tranche."

To attain the spread for each tranche, we apply the pricing methodology discussed previously with both the D-CRR model and the D-CEV model. The expectations in Eq. (7) are calculated by simulating 10000 trials, extracting the timing and numbers of defaults, and computing the statistical mean. We varied correlation assumptions from 0.1, 0.2, …, to 1 to investigate its impact toward different tranche spreads. The results are exhibited in Figure 11.

From Figure 11 we first observe that the D-CEV model produces higher spread quotes than the D-CRR model for all tranches, which again points out the significant difference in the simulation results for the two models. Moreover, we see that the difference in spreads is larger when correlation is low for tranches that absorb earlier losses, and when correlation is high for tranches that absorb later losses.

We then look into the implications of this sensitivity analysis for each individual tranche. We can see that for the equity tranche, the tranche spread decreases as correlation increases. This relationship is similar to that of *n*th-to-default contracts with low *n*. The reason is that higher default correlation increases the chance that no defaults will occur, and the influence to spreads is especially notable for tranches that absorb earlier losses. Therefore, with this relationship between equity tranche spreads and correlation, we often say that investors (or speculators) in equity tranches are taking a "long credit" and "long correlation" position. We can think of this with an example by assuming that an equity tranche investor is receiving a spread of 100 (bps). If correlation increases through time, the probability that the tranche absorbs defaults will decrease and the spread that a same tranche in the market is paying also decreases, for instance, to 50 (bps). The equity tranche investor now only needs to bear lower risk that is worth periodic payments of 50 (bps) while still receiving 100 (bps). . Therefore, if the investor chooses to unwind the credit position when correlation increases, he will realize a gain due to a lower new breakeven spread that he should pay by taking a short position of the same equity tranche to unwind his position.

By contrast, the results for the senior tranche reflect its greater exposure to the risk of losses when defaults are more clustered. We can see this apparently from the 15% to 30% tranche, where the spread increases monotonously with correlation. Thus, the investors in senior tranches are taking a "long credit" and "short correlation" position. The lower the correlation moves, the higher their realized gains.

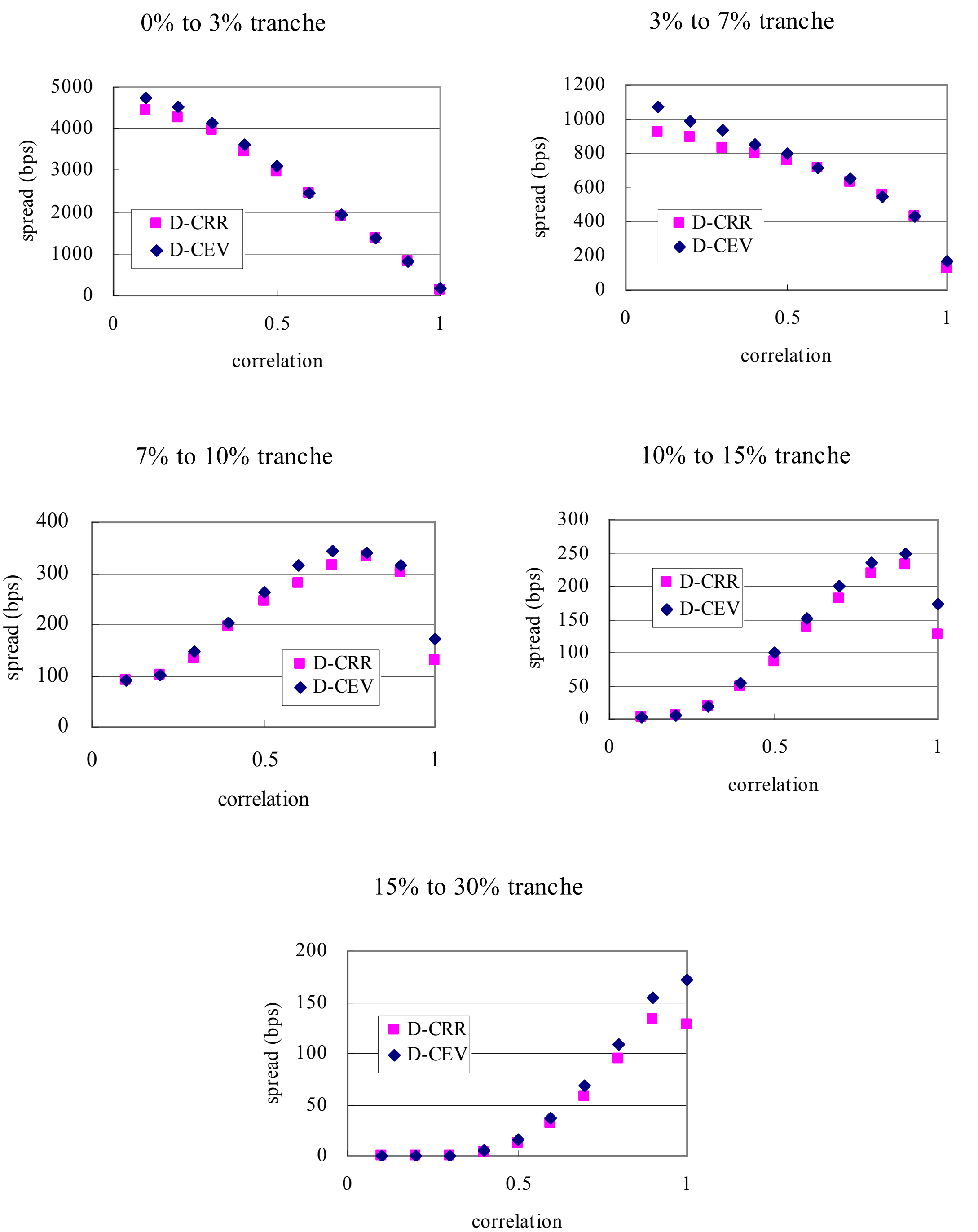
Finally, unlike the equity and senior tranches, the prices of mezzanine tranches are generally not a monotonic function of default correlation. In the 7% to 10% tranche in our example, we can obviously see that with both high and low correlations, there is higher probability that this tranche will survive intact, which is reflected by its spread value. However, for medium levels of correlation, there is higher risk that the tranche will suffer substantial losses.

We should understand that the spread vs. correlation pattern for all tranches will vary with different assumptions, such as different credit default swap information, different set attachment and detachment points, and different recovery rates, etc. To truly understand how the tranches of a particular CDO is affected from different parameter values, it is crucial to perform a price sensitivity analysis as we have done for correlation.

Figure 11

Price Sensitivity of CDO Tranches to Correlation Values

The CDO has a reference portfolio of 100 entities, each with the same parameter values where $S = 50$, $\sigma = 0.5$, $RR = 0.4$, and the principal of each entity $L = 100$. The term is five years, with time intervals $h = 1/12$ (year). The risk-free rate is 0.03, and the term structure of credit default swap spreads for all entities are given by 1yr=60.39 (bps), 2yr=68.09 (bps), 3yr=74.84 (bps), 4yr=81.68 (bps), and 5yr=88.87 (bps).



VI. Conclusions

In our thesis, we explored the D-CRR model and the simulation method developed in Das and Sundaram (2004) and Bandreddi, et al. (2007). Their model is a new method applied for the pricing of portfolio credit derivatives that combines stylized characteristics of both structural-form models and reduced-form models, where observable equity prices are used along with an intensity-base model to simulate default.

We introduced an alternative model, the D-CEV model, where the “leverage effect” is accounted for. We find that our suggested model addresses many undesirable properties found in the D-CRR model. First, taking account of the leverage effect, we are able to ease the issue of problematic probabilities in the D-CRR model. Second, we find unsustainable simulation results in the D-CRR model where higher volatility values lead to lower mean numbers of defaults simulated. Our proposed D-CEV model, in contrast, can avoid this shortcoming and produce more reasonable results, though not completely eliminating the problem for all cases. Thus compared with the D-CRR model, the D-CEV model attains better resemblance to structural-form models in the relationship between the equity volatility and the number of simulated defaults. Third, we find that simulating defaults with our model attains higher mean numbers of defaults, which lead to higher credit derivatives prices compared with those produced by the D-CRR model. Numerical results confirm our arguments.

We also investigated how our framework accommodates several empirical features, such as the joint movements between credit spreads and equity volatilities, and the default contagion effect.

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Appendix A

The Recovery of Market Value Condition (RMV)

The RMV condition is inspired by the recovery rules of OTC derivatives (see Schönbucher (2003)). The ISDA master agreement for swap contracts specifies that at default of one counterparty, the other counterparty's claim is the market value of a non-defaulted, but otherwise equivalent, security (if the value of this security is positive). He is paid a fraction $1-RR$ of this claim.

Now in order to price a credit default swap, we need to follow some procedure. First, we assume that defaultable zero-coupon bonds are the underlying for default swaps. Define the price of a defaultable zero-coupon bond at time t as $Z(t)$. Thus in mathematical terms, the pricing recursion under the RMV condition is:

$$\begin{aligned} Z(t) &= e^{-rh}\left\{q(t)Z^{+}(t+h)+(1-q(t))Z^{-}(t+h)\right\}\left(1-\pi(t)+\pi(t)RR\right) \\ Z(T) &= 1 \end{aligned}$$

Here, $q(t)$ is the probability of an up-move conditioned on no default occurring, $Z^{+}(t+h)$ and $Z^{-}(t+h)$ are the two states that the bond price evolves to in the next period, $\pi(t)$ is the probability of default, and *RR* is the recovery rate. Notice that the recovery value is a fraction of the value of the bond, if and when default occurs. Next, we compute the default leg of the default swap contract with the following recursive equation:

$$\begin{aligned} PV_L(t) &= e^{-rh}\left\{q(t)PV_L^{+}(t+h)+(1-q(t))PV_L^{-}(t+h)\right\}\left[1-\pi(t)\right]+Z(t)(1-RR)\pi(t) \\ PV_L(T) &= 0 \end{aligned}$$

The first part above is the present value of future possible losses on the default swap, given that default has not occurred at time *t*. The second part is the expected value of the loss-given-default. As for the fixed spread leg of the contract, the expected present value of a $1 payment at each node is calculated, defined below:

$$\begin{aligned} PV_1(t) &= e^{-rh}\left\{q(t)PV_1^{+}(t+h)+(1-q(t))PV_1^{-}(t+h)+1\right\}\left[1-\pi(t)\right] \\ PV_1(T) &= 0 \end{aligned}$$

Finally, the spread is calculated with Eq. (4) in Chapter II.

Appendix B

One-Factor Gaussian Copula

Copulas are functions that express dependence among random variables. The one-factor Gaussian copula is constructed by the standard multivariate normal distribution with correlation ρ. The method is summarized below:

1. Let $X_1, X_2, ..., X_N$ be N independent random variables, each distributed as $N(0,1)$.
2. Define random variables $Y_1, Y_2, ..., Y_N$ as

$$Y_i = a_i M + \sqrt{1-a_i^2} X_i, \quad i = 1, 2, ..., N$$

   where $M \sim N(0,1)$, independent of all X_i. In Gaussian copula models, it is often assumed that the variables have the same pair-wise correlation ρ, such that $a_i = a_j = \sqrt{\rho}$.
3. Given the default probability p at a specific time t, firm i defaults if $\Phi(Y_i) \le p$.

The systematic risk factor M can be viewed as an indicator of the state of the business cycle. The idiosyncratic factor X_i is a firm-specific factor, which is used to describe the quality of the management, or the financial situation of the firm, etc. The relative sizes of the systematic and idiosyncratic components are controlled by the correlation coefficient ρ. If $\rho = 0$, then the market condition has no direct influence on the firms. While if $\rho = 1$, then M is the only driver of defaults, and the individual firm has no control.

Note here that we are assuming ρ is positive so that the coefficient of M for the Y_i's is a real number. Effectively, we assume that all firms in the same economy are positively related to the macroeconomic factor M, since there are nearly few industries that do not comply with the business cycle. When the market is good, there should be lower chance of default, and vice versa. The correlation between two firms is actually attributed to the systematic factor, and has nothing to do with their idiosyncratic counterparts in this model. Still, there exist realistic cases where the default behavior of two firms may be negatively related. An example is the existence of the "competition effect" (see Jorion and Zhang (2007)), which refers to the situation where a firm benefits from the demise of its rival. However, one-factor Gaussian copulas cannot directly capture this effect. The idiosyncratic components for each firm are assumed to be independent in the one-factor Gaussian copula model.