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相關性微笑曲線模型之比較分析

Comparative Analyses of Correlation Skew Models

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Comparative Analyses of Correlation Skew Models

本論文係蘇雍智 君（R95723060）在國立臺灣大學財務金融學系、所完成之碩士學位論文，於民國 97 年 7 月 28 日承下列考試委員審查通過及口試及格，特此證明

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摘要

本文旨在提供擔保債權憑證評價模型的比較分析。所比較的評價模型都建構在單因子關聯結構的架構下，並利用 Hull and White (2004)所提出之機率杓斗法則(probability bucketing method)建構標的資產之違約損失分配，進而求算分券之信用價差。所考慮的模型有 NIG copula，隨機相關模型(stochastic correlation model)，局部相關模型(local correlation model)。此分析會對各個模型的市場配適度進行比較。有鑑於次級房貸風暴對於信用衍生性商品市場造成巨大的衝擊，該風暴對模型配適度的影響也會在本文中討論。最後，本文也會對各模型參數的穩定性進行比較。

關鍵字：合成型擔保債券憑證(synthetic CDO)、相關性微笑曲線(correlation smile)、機率杓斗法則(probability bucketing)、因子關聯結構(factor copula)

Abstract

In this work, we present a comparative analysis of correlation skew models for pricing of CDOs. All of these models are based on the factor copula pricing framework and can generate correlation skews. The models compared are normal inverse Gaussian copula, stochastic correlation model and local correlation model. By using Gaussian copula as benchmark, the fitness of these models to market data will be tested. Because the subprime mortgage crisis causes structural changes on the credit derivatives market, the fitness before the crisis and after the crisis is compared. Finally, the stability of parameter values over time will be given.

Keywords: synthetic CDO, correlation smile, probability bucketing, NIG copula, stochastic correlation, local correlation

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Chapter 1

Introduction

Since the first time credit default swaps (CDS) were introduced in the 1990s, there have been rapid developments in credit derivatives markets. As the market has grown, basket credit derivatives such as first-to-default CDS as well as synthetic CDOs emerged. In 2004, CDS indices iBoxx and Trac-X merged into iTraxx, and standardized tranches linked to these indices began to be actively quoted. Since then, the credit products depending on default correlations have become even more popular. However, in the late 2007, the subprime mortgage crisis in US hit the credit derivatives market and triggered a global financial crisis. The inevitable credit crunch made things even worse. Though the causes of the crisis are complicated, these events obviously raise doubts about current approaches to credit risk modeling and pricing, especially for CDOs.

The standard approach to pricing basket credit derivatives is one-factor Gaussian copula. The use of copula functions to describe the dependence structure among default times is pioneered by Li (2000). This approach allows independent specification of the dependence structure among defaults and the single-name credit curves. It is advantageous since the traditional reduced-form model can calibrate single-name credit curves accurately. When coupled with the factor approach, a semi-analytical formula for pricing CDOs can be obtained. If the large homogenous pool (LHP) assumption proposed by Vasicek (1987) is adopted, a closed-form solution can be achieved under Gaussian copula setting. When standardized index tranches market emerged, it becomes possible to calibrate the correlation parameter used in one-factor Gaussian copula model from market quotes by assuming identical correlation among all reference entities. Since the default processes of reference entities do not depend on any specific tranche characteristics, the correlation parameter should be the same across tranches. However, in reality, different correlation parameters are needed to match market quotes of different tranches exactly. Often, the correlation implied by the senior tranche and the equity tranche is higher than the correlation implied by the mezzanine tranche. This phenomenon is well known by market participants and is called correlation smile or correlation skew. It shows that the distribution of portfolio loss implied by Gaussian copula is inconsistent with the market implied loss distribution and further challenges the standard approach adopted by the industry. To address this issue, base correlation is proposed by McGinty, Beinstein, Ahluwalia, and Watts (2004) from JPMorgan. Nonetheless, this ad hoc method does not resolve the fundamental inconsistency exhibited by the Gaussian copula approach and cannot price all tranches using a single parameter set. Therefore, there continue to be works on correlation skew modeling, and this field is still being actively researched.

The aim of this thesis is to provide a comparison of some correlation skew models that have been proved accurate and to examine their effectiveness after the subprime mortgage crisis. Only the factor copula approach is considered since it provides a semi-analytical framework for pricing CDOs and facilitates the comparisons among models. The models under study are (1) normal inverse Gaussian copula (Kalemanova, Schmid and Werner, 2007), (2) stochastic correlation model (Burtschell et al., 2005), and (3) local correlation model (Andersen and Sidenius, 2004). By using one-factor Gaussian copula as benchmark, we will test the fitness of each model by comparing their absolute pricing errors and the sum of error squared across tranches. The change of market fitness due to the subprime mortgage crisis will be closely examined. Finally, the stability of the calibrated parameter values of each model will be investigated.

The thesis is organized as follows. Chapter 2 reviews basic knowledge about pricing CDOs and reviews how to value CDOs under the factor copula framework. Chapter 3 describes the standard market model and correlation skew. Chapter 4 details the models under comparisons. Chapter 5 shows the numerical results. Chapter 6 concludes.

Chapter 2

Valuation of CDOs

2.1 CDS, CDOs, and Index Tranches

Since credit default swap is the basic building block for synthetic CDOs as well as one of the most used instruments in the credit derivatives market, a description about credit default swap is given first. A credit default swap (CDS) is a credit derivative used to transfer the credit risk of a reference entity from one party to another. In a standard CDS contract, one party (the protection buyer) purchases credit protection from the other party (the protection seller) to cover the loss of the face value of an asset following a credit event. A credit event is usually either a default of the reference entity or other specified events defined in the ISDA agreements. This protection lasts until the maturity date specified in the contract. For this protection, the protection buyer periodically pays CDS spread based on the notional of the contract to the protection seller until a credit event or maturity, whichever occurs first. If a credit event occurs before the maturity date of the contract, the protection seller pays the difference between par and the post-default price of the assets of the reference entity based on the notional of the contract, and receives the accrued spread up to the event time. The above loss compensation and the accrued spread are assumed to be settled at the time the credit event occurs. The loss payment can be made by physical settlement or cash settlement.

CDS contracts are often traded in unfunded format. Namely, no exchange of notional is made at the initiation date and the maturity date. Only when a credit event occurs are loss payments required to be made by the protection seller to the protection buyer. Thus CDS contracts have counterparty risks. The payments involved during the life of a CDS contract is illustrated in Figure 2.1. Notice that the periodic spread payments to the protection seller are often called the “premium leg,” and the contingent loss payments to the protection buyer are often called the “protection leg.” The same terminologies are also used in the payment structure of synthetic CDOs.

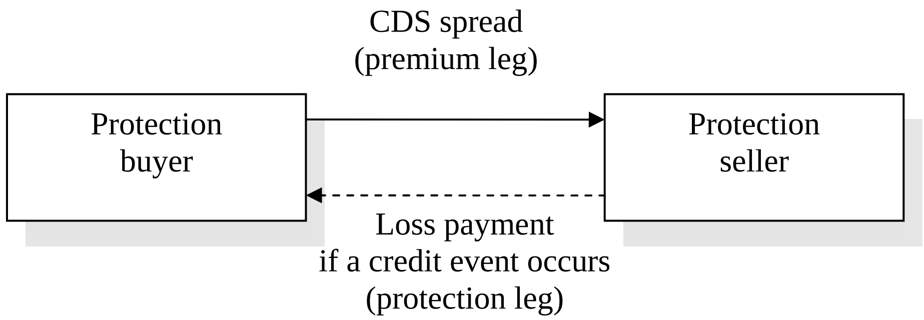


Figure 2.1: The structure of a CDS contract.

A collateralized debt obligation (CDO) is a securitization of a portfolio of defaultable instruments such as loans, bonds, etc. CDO investors will bear the losses resulting from the defaults of the instruments in the underlying portfolio in return for periodic payments. The underlying portfolio is transferred from the originator to a special purpose vehicle (SPV) that issues securities on the portfolio in several tranches with different seniorities. The cash flows generated from the underlying portfolio are arranged such that the most senior tranche is paid before mezzanine tranches are paid and with any residual cash flow to the equity tranche. When credit events occur, losses of the portfolio are absorbed first by the equity tranche and then by the next tranche, and so on before they reach the most senior tranche. The structure of CDOs is illustrated in Figure 2.2. Consider the example illustrated in Figure 2.2. Each tranche is defined by an attachment point and a detachment point. The investors of a specific tranche will bear all losses in the portfolio in excess of the attachment point and up to the detachment point in percentage of the total principal of the portfolio. For example, the equity tranche in Figure 2.2 has 10% of the total principal and covers all losses from the portfolio during the life of the CDO until they have reached 10% of the total principal. The mezzanine tranche has 20% of the total principal and absorbs all losses in excess of 10% of the principal up to a maximum of 30% of the principal. The senior tranche has 70% of the principal and bears all losses in excess of 30% of the principal. Notice that the interest rates paid to tranche investors are based on the balance of the principal remaining in the tranche after losses have been paid. Take the equity tranche for example. At the outset, the 30% interest rate is paid on the total amount invested by the equity tranche investor. If 5% losses of the total portfolio have been experienced, the equity tranche investors have lost 50% of their initial investment and the interest rate is paid on only 50% of the original amount invested.

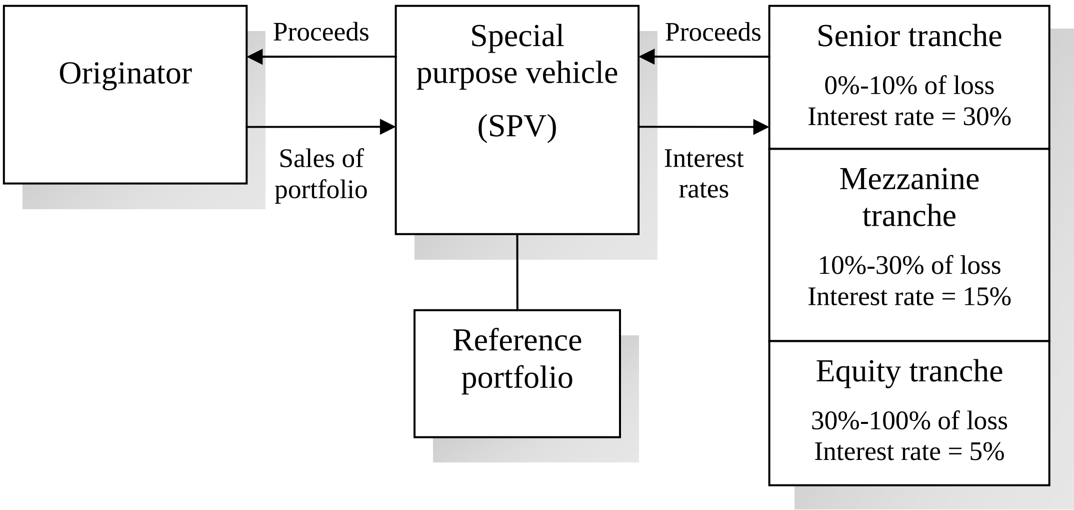


Figure 2.2: The structure of cash CDOs.

In recent years, synthetic CDOs emerge as a flexible and low-cost tool for transferring credit risk off balance sheets. The difference between cash CDOs mentioned above and synthetic CDOs relies on the ownership of underlying portfolios. While in the former a portfolio of bonds or loans are securitized and the ownership is transferred from the originator to an SPV, in the latter the exposure is obtained synthetically through credit default swaps or other credit derivatives and the underlying portfolio remains on the originator's balance sheet. This is illustrated in Figure 2.3.

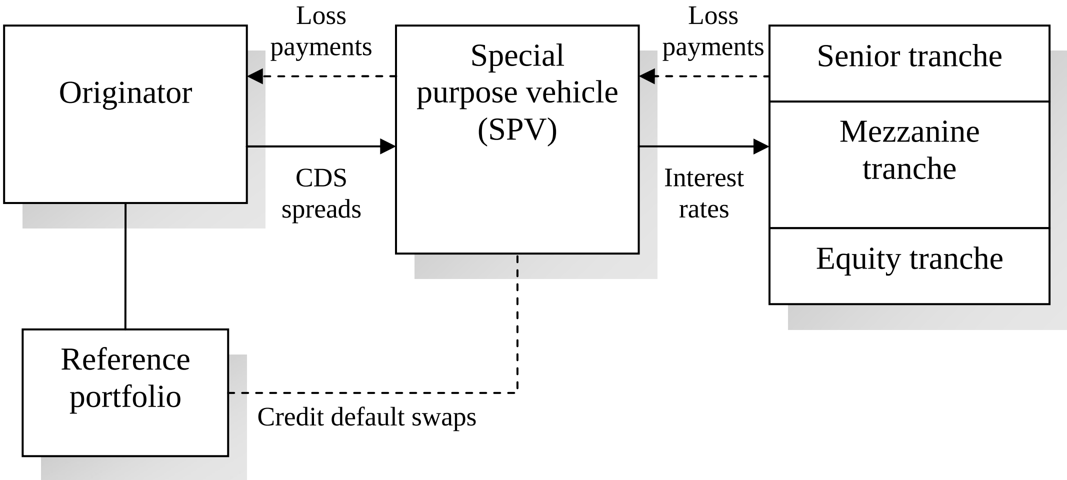


Figure 2.3: The structure of synthetic CDOs.

Synthetic CDOs may be either funded or unfunded. When issued in a funded format, the proceeds provided by investors at the time of investment are often invested in high-quality, liquid assets until a credit event occurs. The returns from these investments plus the premium from the CDS counterparty provide the cash flows to pay interests to the investors. When a credit event occurs and a payout to the CDS counterparty is required, the required payment is made from the reserve account that holds the liquid investments. In contrast, when issued in an unfunded format, the investors receive periodic payments but do not place any capital in the CDO when entering into the investment. Instead, the investors retain funding exposures and may have to make a payment to the CDO in the event the losses of the portfolio reach the attachment point of the tranche. In the rest of this thesis, synthetic CDOs are implicitly assumed to be unfunded.

The indices have been developed to track CDS spreads. The iTraxx is the family of CDS index products owned, managed, compiled and published by International Index Company (IIC). Nowadays, they form a large share of the overall credit derivative market. The indices are constructed on a set of rules according to the liquidity of the underlying CDS. The iTraxx are rebalanced every six months known as “rolling” the index. The index after rebalancing is called a new series. The composition of a new series of iTraxx is determined as follows. Index composition is initially set to be the same as the previous series. Ineligible entities (defaulted or merged) are excluded. Any entities with the highest CDS trading volume over the previous 6 months and not already in the index are added until the CDS’ remain in the final composition of the index have highest liquidity. The roll dates are March 20th and September 20th each year. These indices are tradable instruments in their own right with pre-determined fixed rates. The iTraxx Europe is one of the most popular CDS indices representing an equally-weighted portfolio of the most liquid 125 credit default swaps on investment-grade European companies. The iTraxx Europe is traded at 3, 5, 7 and 10-year maturities. The iTraxx Europe is also split into traded sector indices (autos, consumer, energy, industrial, non-financials, TMT, financial senior and financial sub) and a HiVol index composed of companies from iTraxx Europe with the top 30 highest CDS spreads. A Crossover index comprising the 50 most liquid sub-investment grade European companies is also traded. The iTraxx product family is illustrated in Figure 2.4.

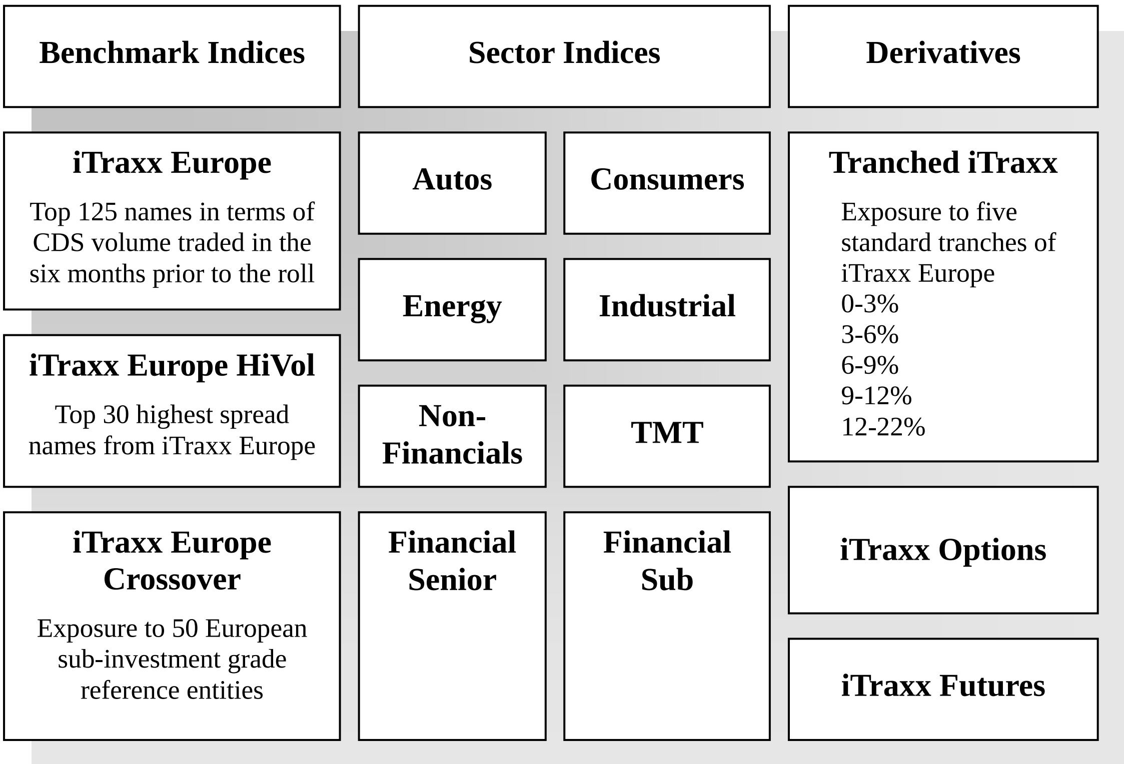


Figure 2.4: The iTraxx product family.

The iTraxx Europe index is also used to define standardized index tranches similar to the tranches of a CDO. In Figure 2.4, the tranched iTraxx investors who are essentially the protection seller are responsible for all losses on an underlying index portfolio of CDS in excess of a respective tranche attachment point up to the detachment point. Thus, an index tranche is economically equivalent to a synthetic CDO tranche. In return for covering the losses, the investors receive a running spread quarterly. Once default occurs, the notional amount upon which the running spread is charged is reduced with losses, dollar for dollar. All tranches except the equity tranche have a predetermined running spread; the equity tranche (0-3%) has an upfront fee. Unlike other tranches, the equity tranche has a contractually set running spread of 500 basis points per annum and the upfront fee is negotiated in the market. Market quotes of iTraxx Europe 5 year index and its tranches on April 7, 2008 is presented in Table 2.1. In Table 2.1, all tranche spreads are quoted in basis points per annum except the 0-3% tranche, which is quoted as an upfront payment in percentage of the tranche notional.

iTraxx Europe 5 Year, Series 9 on 7 April 2008						Source: Bloomberg
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	Index
Spread/Upfront fee	30%	335 bp	190 bp	135 bp	62 bp	85 bp

Table 2.1: Market quotes of iTraxx Europe 5 year on April 7, 2008.

2.2 General Pricing Formula for CDOs

In this section, general formulae for determining the market value and the fair spread of a CDO tranche will be derived. These formulae are model-independent and thus are general. The fair spread of a CDO tranche is the spread such that the marked-to-market value of the contract is zero. Namely, the present value of the premium payments is equal to the present value of the contingent loss payments. The premium payments are called the “premium leg” and refer to spreads received by the protection seller or the tranche investor. The contingent loss payments are called the “protection leg” and refer to cash flows that cover losses affecting the specific tranche and are paid by the protection seller. Here, only unfunded CDO is considered. However, the same concept can be extended to fully funded CDO.

In this thesis, we assume that there exists a risk-neutral probability measure Q such that all discounted price processes are martingales under this measure. All expectations in the following formulae are taken with respect to this measure. In addition, the total notional of CDO is assumed to be one unit of currency.

For convenience, the notations used in this section are listed below.

- ap: The attachment point of the CDO tranche as a percentage of total notional.
- dp: The detachment point of the CDO tranche as a percentage of total notional.
- $\Delta_{i-1,i}$: The year fraction between two payment dates t_{i-1} and t_i.
- $B(0,t_i)$: The discount factor at time 0 for cash flow occurring at t_i.
- T: The time to maturity of CDO as a fraction of year.

- $L(t)$: The aggregate portfolio loss as a percentage of total notional at time t.

- $L^{tr}(t)$: The tranche loss as a percentage of total notional at time t.

- MTM: The market value of a CDO tranche at time 0.

- *S*: The fair spread of the CDO tranche.

A tranche suffers a loss only if the total portfolio loss in percentage of total notional exceeds the attachment point of this tranche and the maximum loss of a tranche is the trance's size. The tranche loss in percentage of total notional at time t can be expressed as follows.

$$L^{tr}(t) = \max\{\min\{L(t) - ap, dp - ap\}, 0\}$$

Then the present value of a protection leg in percentage of total notional can be calculated by taking the expectation with respect to the risk-neutral probability measure *Q*. It is expressed as follows:

$$\text{PV(Protection Leg)} = E^Q\left[\int_0^T e^{-\int_0^t r_s ds} dL^{tr}(t)\right]$$

On the other hand, given the payment dates $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$, the present value of premium leg in percentage of total notional depends on the remaining tranche notional at time t and can be written as follows:

$$\text{PV(Premium Leg)} = E^Q\left[\sum_{i=1}^{n} S\Delta_{i-1,i} e^{-\int_0^{t_i} r_s ds}\left(dp - ap - L^{tr}(t_i)\right)\right]$$

Therefore, the marked-to-market value of a CDO tranche from protection sellers' view can be expressed below:

$$\text{MTM} = E^Q\left[\sum_{i=1}^{n} S\Delta_{i-1,i} e^{-\int_0^{t_i} r_s ds}\left(dp - ap - L^{tr}(t_i)\right)\right] - E^Q\left[\int_0^T e^{-\int_0^t r_s ds} dL^{tr}(t)\right]$$

The fair spread can be obtained by choosing a spread such that the above formula is equal to zero. It is expressed below:

$$S = \frac{E^Q\left[\int_0^T e^{-\int_0^t r_s ds} dL^{tr}(t)\right]}{E^Q\left[\sum_{i=1}^{n} \Delta_{i-1,i}\, e^{-\int_0^{t_i} r_s ds}\left(dp - ap - L^{tr}(t_i)\right)\right]}$$

For ease of implementation, it is furthermore assumed that the interest rate is stochastically independent of the occurrences of credit events in the reference portfolio. The integral appearing in the protection leg is discretized by assuming the credit events can only occur at the payment dates. Then the marked-to-market value of a CDO tranche and its fair spread can be rewritten as follows:

$$\begin{aligned}\mathrm{MTM} = &\sum_{i=1}^{n} S\Delta_{i-1,i} B(0,t_i)\left(dp - ap - E^Q\left[L^{tr}(t_i)\right]\right) \\ &-\sum_{i=1}^{n} B(0,t_i)\left(E^Q\left[L^{tr}(t_i)\right] - E^Q\left[L^{tr}(t_{i-1})\right]\right)\end{aligned} \tag{2.1}$$

$$S = \frac{\sum_{i=1}^{n} B(0,t_i)\left(E^Q\left[L^{tr}(t_i)\right] - E^Q\left[L^{tr}(t_{i-1})\right]\right)}{\sum_{i=1}^{n} \Delta_{i-1,i} B(0,t_i)\left(dp - ap - E^Q\left[L^{tr}(t_i)\right]\right)} \tag{2.2}$$

2.3 Review of Copula

In order to obtain the fair spread of a CDO tranche, it is essential to determine the aggregate portfolio’s loss distribution. The factor copula approach has proved to be powerful since it provides a semi-analytical framework for pricing CDOs. In this section, some basic concepts about copula will be reviewed and then the details about the factor copula approach will be given in the next section.

For ease of exposition, only bivariate copula is introduced. However, the same concepts can be extended to the multivariate case. To start with, the notions of groundedness and the 2-increasing property should be given first, which allow copulas to respect the properties of the distribution function.

Consider two non-empty subsets A_1 and A_2 of $\Re$ and a function $G: A_1 \times A_2 \to \Re$. Denote with a_i the least element of $A_i, i = 1,2$. The function G is said to be grounded if for every (v,z) of $A_1 \times A_2$, $G(a_1, z) = G(v, a_2) = 0$.

The function G is called 2-increasing if the following condition holds for every rectangle $[v_1, v_2] \times [z_1, z_2]$ whose vertices lie in $A_1 \times A_2$:

$$G(v_2, z_2) - G(v_2, z_1) - G(v_1, z_2) + G(v_1, z_1) \geq 0, \forall v_1 \leq v_2, z_1 \leq z_2.$$

Note that the left hand side measures the mass of the rectangle $[v_1, v_2] \times [z_1, z_2]$ according to the function G. In other words, the 2-increasing property requires that the functions assign non-negative mass to every rectangle in their domain.

A bivariate subcopula is a real function $\overline{C}: A \times B \to [0,1]$, where A and B are non-empty subsets of $I = [0,1]$ containing both 0 and 1 such that (1) it is grounded, (2) 2-increasing, and (3) for every (v,z) of $A \times B$, $\overline{C}(v,1) = v$, $\overline{C}(1,z) = z$. A bivariate copula C is a bivariate subcopula with $A = B = [0,1]$. Notice that, from the definition, copulas are joint distribution functions of standard uniform random variables. Suppose that the distribution functions associated with random variable X and Y are $F_1(x)$ and $F_2(y)$, respectively. Through the inverse probability integral transforms, a copula computed at $F_1(x)$ and $F_2(y)$ gives a joint distribution function at (x,y) thus,

$$\begin{aligned} C(F_1(x), F_2(y)) &= P\{U_1 \leq F_1(x), U_2 \leq F_2(y)\} \\ &= P\{F_1^{-1}(U_1) \leq x, F_2^{-1}(U_2) \leq y\} \\ &= P\{X \leq x, Y \leq y\} = F(x,y) \end{aligned}$$

The link between distribution functions and copulas allows us to consider a copula a dependence function. This relationship is essentially the spirit of Sklar’s theorem, which says that not only do copulas evaluated at $F_1(x)$ and $F_2(y)$ give joint distribution functions at (x,y) but the converse also holds true. To wit, joint distribution functions can be represented by the marginal distributions and a unique subcopula, which in turn can be extended (not unique, in general) to a copula. Sklar's theorem is stated formally below.

Let $F_1(x)$ and $F_2(y)$ be two marginal distribution functions. Then for every $(x,y)\in\Re^2$,

- **(i)** If $\overline{C}$ is any subcopula whose domain contains $\text{Range}(F_1)\times\text{Range}(F_2)$, then $\overline{C}(F_1(x),F_2(y))$ is a joint distribution function with marginal distributions $F_1(x)$ and $F_2(y)$.

- **(ii)** If $F(x,y)$ is a joint distribution with marginal distribution $F_1(x)$, $F_2(y)$, then there exists a unique subcopula $\overline{C}:\text{Range}(F_1)\times\text{Range}(F_2)\to[0,1]$ such that $F(x,y)=\overline{C}(F_1(x),F_2(y))$. If $F_1(x)$, $F_2(y)$ are continuous, the subcopula is a copula. If not, there exists a copula C such that $C(v,z)=\overline{C}(v,z)$, for every $(v,z)\in\text{Range}(F_1)\times\text{Range}(F_2)$.

By splitting the joint distribution into the marginal distributions and a copula, marginal behavior as represented by marginal distributions can be separated from the association as represented by a copula. That is why copulas can be thought of as dependence functions. The use of copulas gives great flexibility when modeling joint default processes in the reference portfolio of a CDO. Since there has been accurate ways to model single-name credit, it suggests that we can price CDOs by modeling single-name credits using existing techniques and then choosing an appropriate copula to model the dependence structure among credits.

Before closing this section, a useful corollary is stated below. This corollary allows us to construct a copula from the marginal distributions and the joint distribution function by inversion of Sklar's theorem. This will help explain the link between copulas and the factor copula framework described in the next section.

Given part (ii) of Sklar's theorem, the subcopula such that $F(x,y)=\overline{C}(F_1(x),F_2(y))$ is $\overline{C}(v,z)=F(F_1^{-1}(v),F_2^{-1}(z))$. If $\text{Range}(F_1)=\text{Range}(F_2)=[0,1]$, then the subcopula is a copula.

2.4 The Factor Copula Pricing Framework

Although the use of copula functions allows separate specifications and calibrations of single-name credit curves and the dependence structure among credits, it is rather slow when there are a large number of credits involved in a CDO. The main feature of the factor copula approaches is that default events are independent conditioned on some latent state variables. This eases the computation of aggregate loss distributions through dimensionality reduction and provides a semi-analytic solution to the pricing of CDOs. This factor approach is nicely suited for high-dimensional problems. For the sake of simplicity, only one-factor model is considered since it is parsimonious with respect to the number of parameters, which will ease model calibration. Nonetheless, this technique applies to multi-factor models as well.

Consider an underlying portfolio containing debt instruments of n companies and suppose that the marginal risk-neutral probabilities of default can be obtained for each company. One approach to back out the risk-neutral probabilities of default is shown in Appendix A. To model default times jointly, we define latent random variables X_i below:

$$X_i = a_i M + \sqrt{1-a_i^2}Z_i, -1 \le a_i \le 1, i = 1, 2, ..., n$$

where M and $\{Z_i\}_{i=1}^n$ are stochastically independent and all of them have zero mean and unit variance. Notice that X_i also has zero mean and unit variance and the correlation between X_i and X_j is $a_i a_j$.

To proceed, define the following notations:

- τ_i : The default time of the *i*-th company.

- $Q_i(t)$: The cumulative risk-neutral probability that company i will default before time t.

- $F_i(x)$: The marginal distribution function of X_i.

- $F_{\mathbf{X}}(x_1, x_2, ..., x_n)$: The joint distribution function of X_i's.

Further suppose that X_i's are continuous. So there exists a unique copula to represent the joint distribution of X_i's. Then the copula specifying the dependence structure among X_i's can be constructed by applying the corollary stated in the previous section. That is,

$$C_{\mathbf{X}}(u_1, u_2, ..., u_n) = F_{\mathbf{X}}\left(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_n^{-1}(u_n)\right).$$

Applying this copula to represent the joint probabilities of default times, the following formula is obtained:

$$\begin{aligned}
&Q\left(\tau_1 \le t_1, \tau_2 \le t_2, ..., \tau_n \le t_n\right) \\
&= C_{\mathbf{X}}\left(Q_1(t_1), Q_2(t_2), ..., Q_n(t_n)\right) \\
&= F_{\mathbf{X}}\left(F_1^{-1}\left[Q_1(t_1)\right], F_2^{-1}\left[Q_2(t_2)\right], ..., F_n^{-1}\left[Q_n(t_n)\right]\right) \\
&= F_{\mathbf{X}}(x_1, x_2, ..., x_n)
\end{aligned}$$

where $x_i = F_i^{-1}[Q_i(t_i)], i = 1, 2, ..., n$. Under this copula model, the event that the *i*-th company defaults before time t_i is the same as the event that X_i falls below a threshold x_i. Intuitively, we may think of latent variables X_i's as the firm values of companies and consider x_i's as default thresholds of companies. When the firm value of a company falls below the default threshold, its total assets cannot fulfill the obligations and thus the default occurs. Viewed in this way, the occurrences of defaults agree with the definition of default in structural-form models. This interpretation also provides some economic insights on the factor copula approach.

Since default events can be represented by X_i's and X_i's are stochastically independent given the common factor M, it is straightforward to build up the portfolio loss distribution by conditioning on the common factor. After the loss distribution by time t conditioned on the common factor M is calculated, it can be used to compute the expected tranche loss conditioned on M, i.e., $E^Q[L^{tr}(t)|M]$. Then the expected tranche loss by time t can be determined by integrating the conditional expected tranche loss numerically with respect to the common factor M. Plugging the expected tranche loss by each payment date into formula (2.2), the fair spread of the CDO tranche can be obtained.

Therefore, the problem reduces to how to construct the portfolio loss distribution conditioned on the common factor M by time t. Several approaches have been proposed to build up the portfolio loss distribution. The approach described here is the probability bucketing method of Hull and White (2004) since it is intuitive and easy to understand. This method works by constructing a bucketed distribution to approximate the true portfolio loss distribution. The bucketed distribution is constructed by dividing the true portfolio loss distribution into several buckets. The probability associated with each bucket is assumed to be concentrated at the mean loss conditional that the loss is in the bucket.

Suppose that all potential losses are divided into the following ranges: [0, b_0), [b_0, b_1), … , [b_{K-1}, ∞). We designate [0, b_0) as the 0th bucket, [b_{k-1}, b_k) as the *k*-th bucket ($1 \le k \le K-1$), and [b_{K-1}, ∞) as the *K*-th bucket. Denote p_k as the probability that the loss conditioned on the common factor M by time t will be in the *k*-th bucket and let A_k be the mean loss by time t conditional that the loss is in the *k*-th bucket ($0 \le k \le K$). Then the distribution function of the bucketed distribution used to approximate the true portfolio loss distribution can be expressed as follows:

$$\widehat{F}(x) = \sum_{k=0}^{K} p_k \mathbf{1}_{\{x \ge A_k\}}$$

The p_k's and A_k's in the above formula are calculated iteratively by introducing one debt instrument at a time. In the iterative procedure, it is assumed that the probability associated with bucket k is concentrated at the current value of A_k. After all of debt instruments are introduced, the bucketed distribution is completed. Though the probability bucketing method allows the recovery rates to be stochastic, it is assumed that the recovery rates are constant but need not be identical among all credits. We proceed to describe the details of the method below.

Initially, there is no debt instrument. Hence, $p_0 = 1$, $p_k = 0$ for $k > 0$ and $A_0 = 0$. The initial values of A_k's for $k > 0$ are set arbitrarily as $A_k = \frac{(b_{k-1} + b_k)}{2}$ for $1 \le k \le K-1$ and $A_K = b_{K-1}$. Suppose that p_k's and A_k's are determined when the first i–1 debt instruments are introduced, the loss given default from the i-th debt instrument is LGD_i, and the default probability conditioned on M by time t is $p_t^{i|M}$. Let $H(z)$ be the distribution function of Z_i's. Then the conditional default probability for the i-th debt instrument can be obtained under the one-factor copula model as follows:

$$
\begin{aligned}
p_t^{i|M} &= Q(\tau_i \le t \mid M) = Q(X_i \le x_i \mid M) = Q(a_i M + \sqrt{1-a_i^2} Z_i \le x_i \mid M) \\
&= H\left[\frac{x_i - a_i M}{\sqrt{1-a_i^2}}\right] = H\left[\frac{F_i^{-1}(Q_i(t)) - a_i M}{\sqrt{1-a_i^2}}\right]
\end{aligned}
$$

Define $u(k)$ as the bucket containing $(A_k + LGD_i)$ for $0 \le k \le K$. Since each bucket may be updated several times when one debt instrument is introduced, for the sake of clarity, we denote $p_k^{(j)}$ and $A_k^{(j)}$ as the values of p_k and A_k after j updates. In particular, $p_k^{(0)}$ and $A_k^{(0)}$ are the initial values before any updates. In addition, to update the conditional mean losses correctly, a variable $B_k^{(j)}$ is introduced for each bucket k to denote the mean loss by time t after j updates that the default of the i-th debt instrument will move the aggregate loss from other buckets to bucket k. The $B_k^{(0)}$'s are set to zero for each time one debt instrument is introduced. Then the updating scheme can be determined as follows. If $p_k^{(j)} = 0$, then no update is made for bucket k, i.e., p_k and A_k remain unchanged. Since the probability that the aggregate losses fall on bucket k is zero, the calculation of conditional mean loss A_k is unnecessary. If $u(k) = k$, then

$$p_k^{(j+1)} = p_k^{(j)}$$

$$A_k^{(j+1)} = \frac{p_k^{(0)} A_k^{(0)} + B_k^{(j)}}{p_k^{(j+1)}}$$

If $u(k) > k$, then

$$p_k^{(j+1)} = p_k^{(j)} - p_k^{(0)} p_t^{i|M}$$

$$A_k^{(j+1)} = \frac{p_k^{(0)} (1 - p_t^{i|M}) A_k^{(0)} + B_k^{(j)}}{p_k^{(j+1)}}$$

$$p_{u(k)}^{(j+1)} = p_{u(k)}^{(j)} + p_k^{(0)} p_t^{i|M}$$

$$B_{u(k)}^{(j+1)} = B_{u(k)}^{(j)} + p_k^{(0)} p_t^{i|M} (A_k^{(0)} + LGD_i)$$

Notice that when $u(k)$ is not equal to *k*, the addition of the *i*-th debt instrument will move some amount of probability from bucket *k* to bucket $u(k)$ because only when no default occurs on the *i*-th debt instrument do the aggregate losses fall on the *k*-th bucket. After all debt instruments are added, the bucketed loss distribution is obtained.

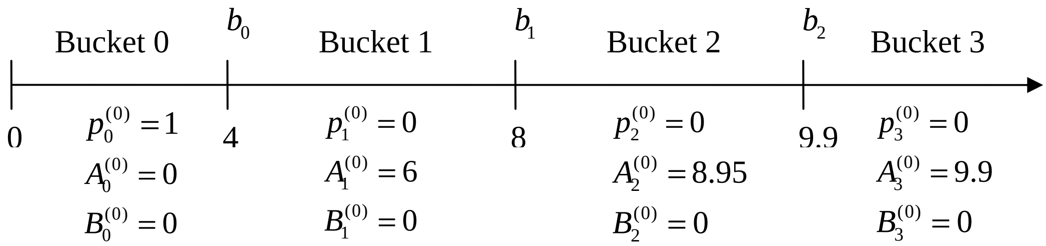
In the following, we will give a numerical example to illustrate the updating scheme. Suppose that there are three debt instruments in the portfolio. The notional of each debt instrument is 15 units of currency. Their default probabilities conditioned on *M* by time *t* and losses given default are listed below:

- Debt instrument 1: $p_t^{1|M} = 0.2$, $LGD_1 = 4$.

- Debt instrument 2: $p_t^{2|M} = 0.4$, $LGD_2 = 4$.

- Debt instrument 3: $p_t^{3|M} = 0.6$, $LGD_3 = 4$.

Further suppose that the tranche's attachment point is 12% and the detachment point is 22% and all potential losses are divided into the following ranges: [0, 4), [4, 8), [8, 9.9), [9.9, ∞). Initially, p_k's, A_k's and B_k's are set as follows:



Consider adding the debt instrument 1 into the portfolio. When $k = 0$, $A_0^{(0)} + LGD_1 = 4$, $u(0) = 1$. Bucket 0 and bucket 1 are updated as follows:

$$p_0^{(1)} = p_0^{(0)} - p_0^{(0)} p_t^{1|M} = 1 - 1 \times 0.2 = 0.8$$

$$A_0^{(1)} = \frac{p_0^{(0)}(1 - p_t^{1|M}) A_0^{(0)} + B_0^{(0)}}{p_0^{(1)}} = \frac{1 \times (1 - 0.2) \times 0 + 0}{0.8} = 0$$

$$p_1^{(1)} = p_1^{(0)} + p_0^{(0)} p_t^{1|M} = 0 + 1 \times 0.2 = 0.2$$

$$B_1^{(1)} = B_1^{(0)} + p_0^{(0)} p_t^{1|M} (A_0^{(0)} + LGD_1) = 0 + 1 \times 0.2 \times (0 + 4) = 0.8$$

When $k = 1$, $A_1^{(0)} + LGD_1 = 6 + 4 = 10$, $u(1) = 3$. Bucket 1 and bucket 3 are updated as follows:

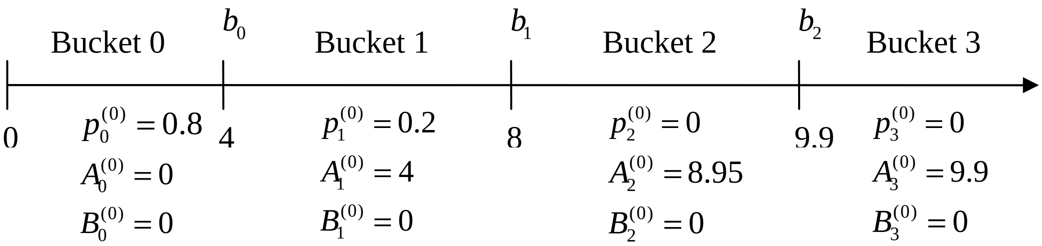
$$p_1^{(2)} = p_1^{(1)} - p_1^{(0)} p_t^{1|M} = 0.2 - 0 \times 0.2 = 0.2$$

$$A_1^{(1)} = \frac{p_1^{(0)}(1 - p_t^{1|M})A_1^{(0)} + B_1^{(1)}}{p_1^{(2)}} = \frac{0 \times (1 - 0.2) \times 6 + 0.8}{0.2} = 4$$

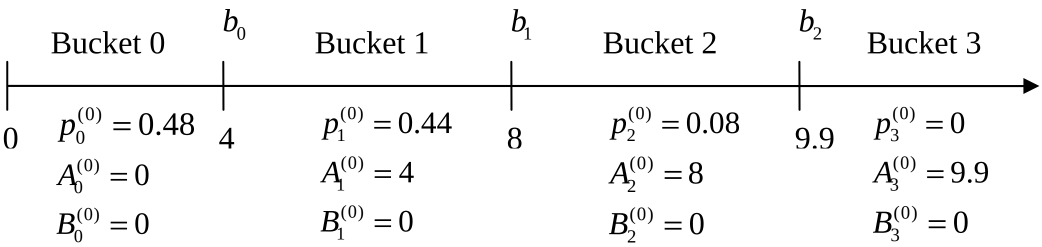
$$p_3^{(1)} = p_3^{(0)} + p_1^{(0)} p_t^{1|M} = 0 + 0 \times 0.2 = 0$$

$$B_3^{(1)} = B_3^{(0)} + p_1^{(0)} p_t^{1|M} (A_1^{(0)} + LGD_1) = 0 + 0 \times 0.2 \times (6 + 4) = 0$$

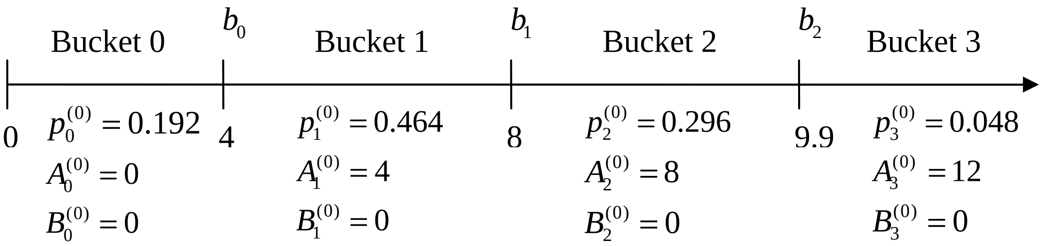
When $k = 2$, $p_2^{(0)} = 0$, no update is made for bucket 2. When $k = 3$, $p_3^{(1)} = 0$, no update is made for bucket 3. The bucketed loss distribution after debt instrument 1 is added is listed below:



Follow the same procedure to add debt instrument 2. The bucketed loss distribution after debt instrument 2 is added is listed below:



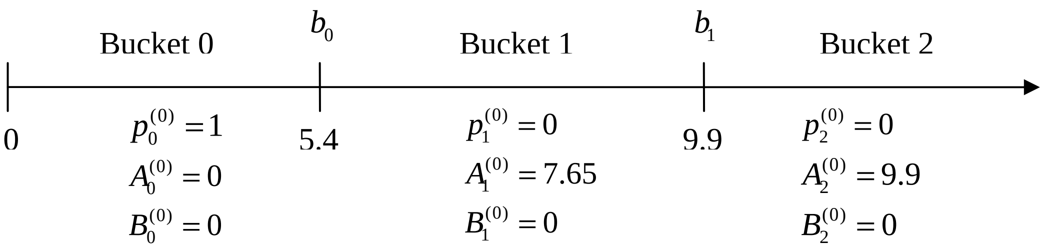
The bucketed loss distribution after debt instrument 3 added is listed below:



After all of debt instruments are added into the portfolio, the expected tranche loss conditioned on M by time *t* can be calculated as follows:

$$E^Q\left[L^{tr}(t)\middle|M\right] = 0\times 0.192 + 0\times 0.464 + 2.6\times 0.296 + 4.5\times 0.048 = 0.9856$$

Before we close this chapter, an implementation issue is to be noted. According to Hull and White (2004), the probability bucketing method is not sensitive to the bucket widths because it keeps track of the mean loss for each bucket and thus allows wide buckets to be used for losses not corresponding to the tranche. However, the bucket widths cannot be arbitrarily wide for buckets whose range is below the attachment point. Wide buckets may cause all of potential aggregate losses to fall wrongfully within the buckets whose range is below the attachment point. Consider the previous numerical example. If all potential losses are divided into the following ranges: [0, 5.4), [5.4, 9.9), [9.9, ∞), then the initial bucketed loss distribution can be expressed as follows:

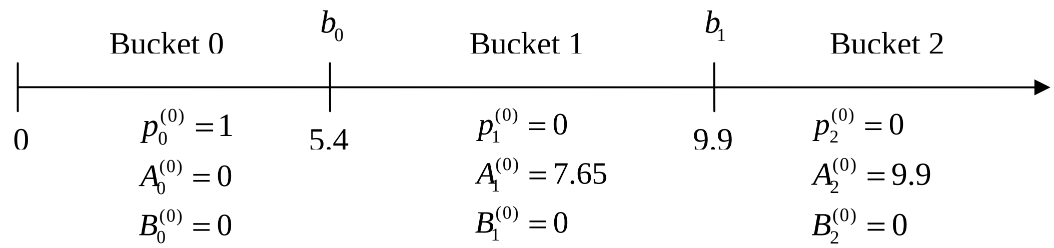


Consider adding the debt instrument 1 into the portfolio. When $k = 0$, $A_0^{(0)} + LGD_1 = 4$, $u(0) = 0$. Bucket 0 is updated as follows:

$$p_0^{(1)} = p_0^{(0)} = 1$$

$$A_0^{(1)} = \frac{p_0^{(0)} A_0^{(0)} + B_0^{(0)}}{p_0^{(1)}} = \frac{1 \times 0 + 0}{1} = 0$$

When $k = 1$, $p_1^{(0)} = 0$, no update is made for bucket 1. When $k = 2$, $p_2^{(0)} = 0$, no update is made for bucket 2. The bucketed loss distribution after debt instrument 1 is added is listed below:



Clearly, bucket 0 is so wide that $A_0^{(0)} + LGD_i$ is always within bucket 0. To address this problem, the greatest common divisor of losses given default is chosen as the bucket width. In addition, since the tranche losses for buckets whose range is above the detachment point are always the notional of the tranche, the exact values of A_k's for these buckets are not important with respect to the calculation of the tranche losses. Consider using only one bucket to accommodate for the potential aggregate losses above the detachment point and dividing the total losses below the detachment point into buckets whose width is the greatest common divisor of losses given default. The p_k's for buckets whose range is below the detachment point are correct since this choice of bucket width can avoid the pitfall mentioned above. Furthermore, the p_k's associated with each bucket are always summed to one, i.e., $\sum_{k=0}^{K} p_k^{(0)} = 1$ since the aggregate losses must fall within one of these buckets. The correctness of p_k's for buckets whose range is below the detachment point ensures that the p_k for the bucket whose range is above the detachment point is correct. Therefore, only one bucket is needed to accommodate for the potential aggregate losses above the detachment point.

Chapter 3 Correlation Skew

3.1 Standard Market Model

Standard approach to pricing CDOs is the one-factor Gaussian copula model with constant pairwise correlations, constant CDS spreads, and constant default intensities for all companies in the reference portfolio. Under this model, the recovery rates can be estimated from data published by rating agencies and is often assumed to be 40% (the recovery rate of unsecured senior debts). Since most of time only the CDS index spread can be observed in the market, this model also assumes that the CDS spreads for all companies are the same and can be represented by their average CDS spread. Also, the default process for each company is assumed to be driven by a Poisson process with constant intensity. The risk-neutral default probability for each company can be estimated from CDS spreads as described in Appendix A. The standard market model can be expressed as follows:

$$X_i = \sqrt{\rho} M + \sqrt{1-\rho} Z_i, 0 \le \rho \le 1, M \sim N(0,1), \{Z_i\}_{i=1}^n \sim N(0,1), i = 1,2,...,n \quad (3.1)$$

In the above equation, M and $\{Z_i\}_{i=1}^n$ are stochastically independent, standard normal distributions. The correlation between X_i and X_j is ρ. Since the normal distribution is closed under convolution, X_i is also a standard normal distribution. Therefore, the default threshold x_i can be determined as follows:

$$x_i = F_i^{-1}[Q_i(t)] = \Phi^{-1}[Q_i(t)] = \Phi^{-1}[Q(t)], i = 1,2,...,n$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. Since this model assumes that the CDS spread for each company equals the average CDS spread, the cumulative probabilities of default for each time are identical among credits. Therefore, the subscript i for Q is eliminated in the above formula. The default probability conditioned on M by time t for each company can be written below:

$$p_t^{i|M} = H\left[\frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}\right] = \Phi\left[\frac{\Phi^{-1}[Q(t)] - \sqrt{\rho}M}{\sqrt{1-\rho}}\right], i = 1, 2, ..., n$$

Once the correlation ρ is available, the fair spreads of CDO tranches can be calculated as described in section 2.4.

3.2 Correlation Skew

As noted in section 2.4, the latent variables X_i's can be thought of as the asset values of companies. Therefore, the correlation parameter ρ in (3.1) can be viewed as the correlation between asset values of two companies. Since the asset value of a company cannot be directly observed in the market, it is often assumed that the correlation between their equities equals the correlation between their asset values. If we allow the correlation parameter ρ to be different for each company, then the standard market model can be rewritten as follows:

$$X_i = \sqrt{\rho_i}M + \sqrt{1-\rho_i}Z_i, 0 \le \rho_i \le 1, M \sim N(0,1), \{Z_i\}_{i=1}^n \sim N(0,1), i = 1, 2, ..., n \quad (3.2)$$

Since the correlation between X_i and M is $\sqrt{\rho_i}$, the parameter $\sqrt{\rho_i}$ for each company can be estimated by the correlation between the equity return of the company and the return of a market index, i.e., the beta coefficient of the company's equity. However, as Walker (2005) points out, the default correlation under risk-neutral measure can be very different from the default correlation under real world measure. Hence, the use of the equity correlations in real world measure is inadequate for pricing basket credit derivatives..

As the credit derivatives market grows, it becomes possible to calibrate risk-neutral correlation ρ in (3.1) from observed market prices of credit products directly, which avoids the hypothesis that the risk-neutral default correlation is the same as the real world default correlation. Since index tranches are standardized credit products and have been actively traded, dependence calibration from credit derivatives has been massively used with index tranches. Because index tranches involve a large number of reference entities while dependence measures are bivariate, default correlations have been assumed to be the same among all reference entities in the portfolio, i.e., $\rho_i = \rho, i = 1, 2, \cdots, n$. This leads to formula (3.1).

However, when the correlation parameter is implied through inverting the standard market model, different correlation parameter is needed to match each tranche spreads even if those tranches have the same underlying index portfolio. The correlation found by matching the model generated spreads to market quoted spreads is called implied correlation or compound correlation.

iTraxx Europe 5 Year, Series 6 on 4 January 2007						Source: Bloomberg
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%	Index
Spread/Upfront fee	10%	44 bp	12 bp	4 bp	1 bp	22 bp

Table 3.1: Market quotes of iTraxx Europe 5 year on January 4, 2007.

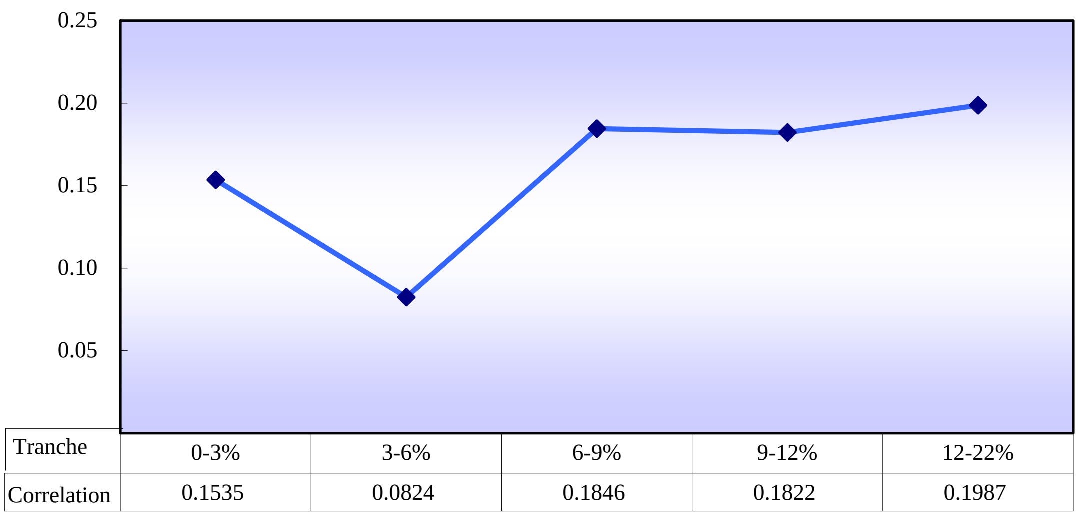


Figure 3.1: Compound correlations of iTraxx Europe 5 year on January 4, 2007.[1]

Consider the market quotes of iTraxx Europe 5 year on January 4, 2007 shown in Table 3.1. The compound correlations of index tranches are presented in Figure 3.1. Because the mezzanine tranche typically has a lower compound correlation than the

[1] When searching the compound correlations of 3-6% tranche and 6-9% tranche, there are two values such that the marked-to-market values of these two tranches have zero values. The values are 0.0824 and 0.9646 for 3-6% tranche and 0.1846 and 0.9393 for 6-9% tranche. Because 0.9646 and 0.9393 are unreasonably high compared to the compound correlations of other tranches, they are ruled out and only 0.0824 and 0.1846 are reported.

equity or senior tranche, this phenomenon is called correlation smile or correlation skew. Although the standard market model is simplistic in that the default correlations are assumed to be the same among all reference entities, the correlation parameter ρ should not depend on the attachment point and the detachment point of the tranche priced in the model. Theoretically, one would expect an almost flat implied correlation curve among tranches. However, the existence of correlation smile shows that there must be some problems in the standard market model.

To investigate the meanings of correlation smile further, it is necessary to

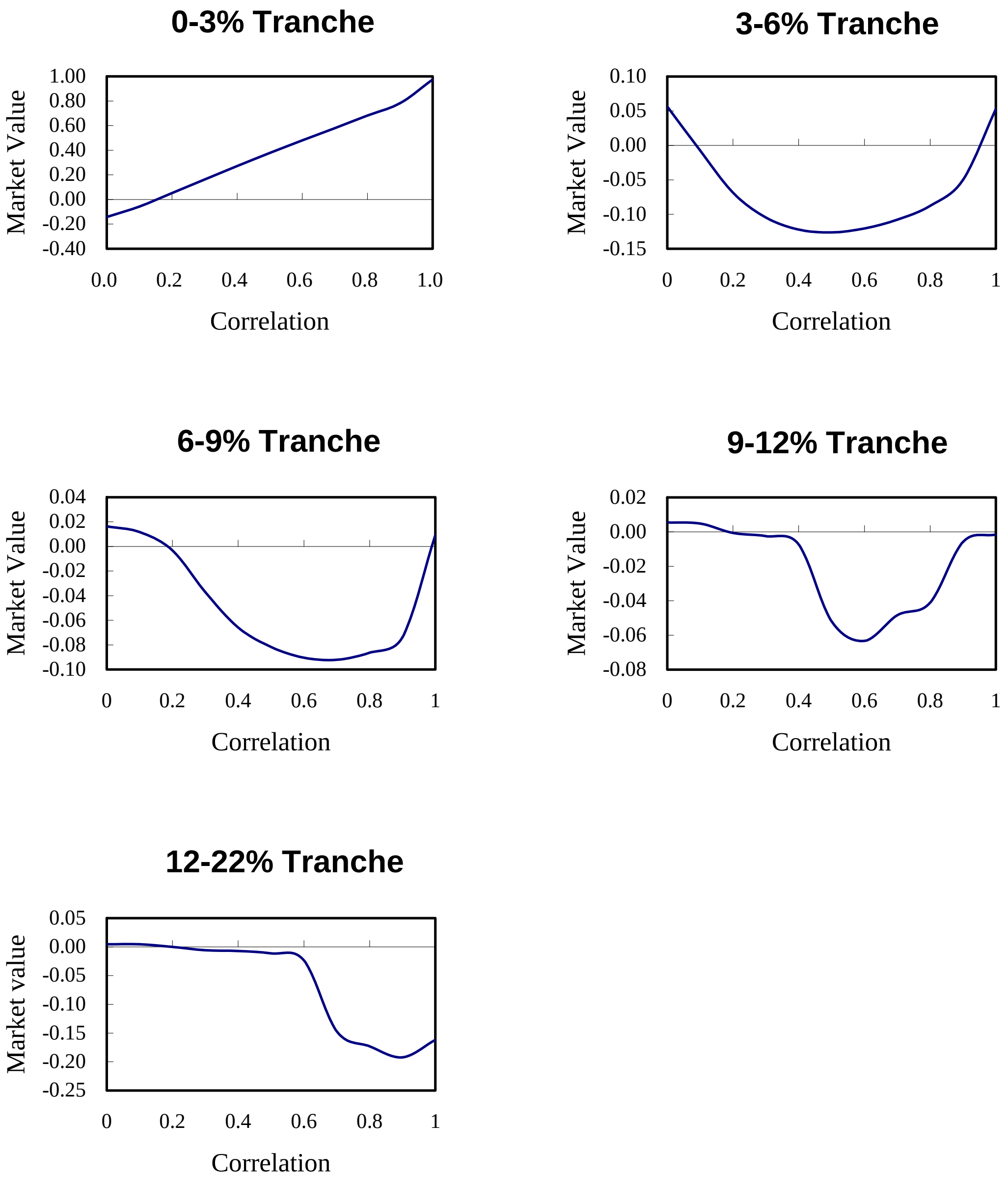
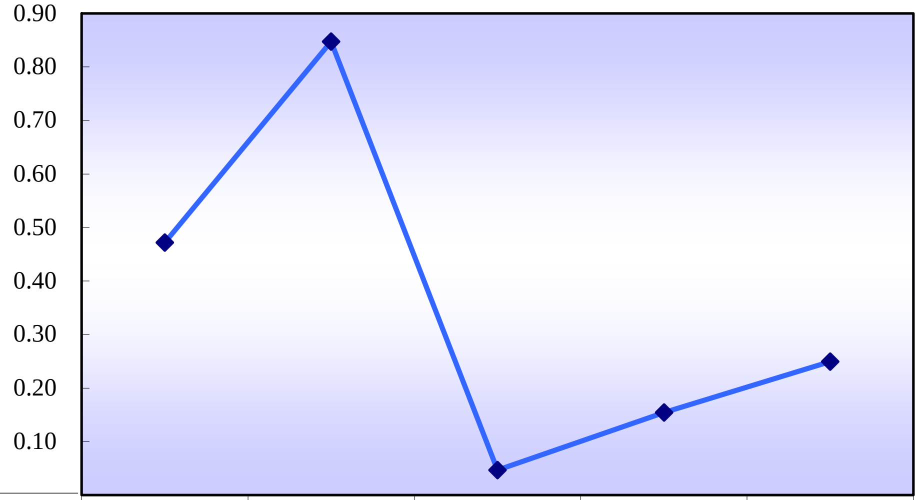


Figure 3.2: The relationships between index tranches and correlation.

understand the relationships between the market values of tranches and the correlation parameter in formula (3.1). The relationships between the market value of each tranche and the correlation parameter are plotted in Figure 3.2. Formula (2.1) is used to determine the market values of tranches under the standard market model with the market spreads or the upfront fee presented in Table 3.1 applied. The total notional is assumed to be one unit of currency, and the recovery rate is taken as 40% for all of reference entities in Figure 3.2. From Figure 3.2, it is observed that when the correlation increases, the market value of the equity tranche (0-3%) will rise. It is caused by the fact that higher correlation leads to higher probability of joint default occurrence. In other words, high correlation implies that there will be either few defaults or many and thus have positive effect on the market value of equity tranche. This observation holds true generally for the equity tranche. That's why market participants often call investing equity tranche as long correlation.

If the one-factor Gaussian copula model uses the compound correlation of the mezzanine tranche to price the equity tranche, the market value of the equity tranche will be underestimated. It is because the compound correlation of the equity tranche is higher than the compound correlation of the mezzanine tranche and the value of the equity tranche is an increasing function of correlation. Therefore, the shape of correlation smile means essentially that Gaussian copula model underestimates the chance of observing a very high or very low number of defaults. Since a fat-tailed distribution has a higher probability to observe extreme events than the normal distribution, it means that the market implied loss distribution is fat-tailed.

Nonetheless, the subprime mortgage crisis not only has negative impact on CDOs market but also changes the shape of correlation smile as well. Taking the market quotes in Table 2.1 for example, we plot the corresponding compound correlations in Figure 3.3. From Figure 3.3, it is observed that the shape and the level of correlation smile have changed a lot compared to Figure 3.1 after the subprime mortgage crisis. Actually, the 6-9% tranche has two values of compound correlations and they are 0.0466 and 0.9753; the 9-12% tranche have two values of compound correlations and they are 0.1540 and 1.0000. However, even if we take the multiple values into account, it cannot change the fact that the compound correlation of equity tranche and senior tranche increase a lot, especially for the 0-3% tranche. Thus, the subprime mortgage crisis must cause some structural changes in the CDO markets and raise doubts about the applicability of the current CDO pricing models.



Tranche	0-3%	3-6%	6-9%	9-12%	12-22%
Correlation	0.4716	0.8476	0.0466	0.1540	0.2495

Figure 3.3: Compound correlations of iTraxx Europe 5 year on April 7, 2008.

3.3 Problems of Correlation Skew

Since the standard market model has been well-known and widely used, the active trading of standardized index tranches leads to the market convention of pricing tranches in terms of compound correlation rather than in terms of spread. Market participants rely on compound correlation as it can facilitate a comparison of prices across tranches. This practice was inspired by the use of Black-Scholes implied volatilities of equity markets.

However, the compound correlation is different from the Black-Scholes implied volatilities because the compound correlation may not exist and may even have multiple values. Empirically, inverting pricing formula under the standard market model for spreads of index tranches on different dates, series, and maturities shows that the compound correlation for a mezzanine tranche does not always exist. Furthermore, the compound correlation is often not unique for the mezzanine tranche. This can be explained by looking at Figure 3.2. In Figure 3.2, the correlation that makes the marked-to-market values of 3-6% tranche and 6-9% tranche zero is not unique. The non-uniqueness problem is caused by the fact that the market values of mezzanine tranches are not monotonic functions of correlation. All of problems mentioned above make the compound correlation concept less applicable.

Chapter 4

Correlation Skew Modeling

4.1 Normal Inverse Gaussian Copula

The insufficiency of one-factor Gaussian copula model to match market quotes has already been shown in previous section. The shape of correlation skew (before the subprime mortgage crisis) indicates that the market allocates consistently higher probability to high default scenarios than the one-factor Gaussian copula model and assigns a lower probability of zero or few defaults than the one-factor Gaussian copula model. Since Sklar's theorem insures the existence of copula function that can represent the joint probabilities of default times, one natural way to model correlation skew is to specify fat-tailed distributions to the common factor M and idiosyncratic factors Z_i.

The model we consider here is normal inverse Gaussian copula model proposed by Kalemanova, Schmid and Werner (2007). In this model, the distribution of the common factor and idiosyncratic factors are normal inverse Gaussian distribution, a subclass of the generalized hyperbolic distributions. The choice of NIG distribution is due to their particular versatility and ability to cope with heavy-tailed processes. It can provide an extensive range of shapes of the distribution if their parameters are appropriately chosen. In the following, the definition and basic properties of NIG distribution will be reviewed. Then the specification of NIG copula model will be given.

The normal inverse Gaussian distribution is a mixture of normal and inverse Gaussian distributions. A non-negative random variable Y has inverse Gaussian (IG) distribution with parameters $\alpha > 0$ and $\beta > 0$ if its probability density function is of the following form:

$$f_{IG}(y;\alpha,\beta) = \begin{cases} \dfrac{\alpha}{\sqrt{2\pi\beta}} y^{-\frac{3}{2}} \exp\left(\dfrac{-(\alpha-\beta y)^2}{2\beta y} \right) & , y > 0 \\ 0 & , y \le 0 \end{cases}$$

A random variable X follows a normal inverse Gaussian (NIG) distribution with parameters α, β, μ and δ if

$$X \left| Y = y \sim N(\mu + \beta y), Y \sim IG(\delta\gamma, \gamma^2)\right.$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$, $0 \le |\beta| < \alpha$ and $\delta > 0$. We write $X \sim NIG(\alpha, \beta, \mu, \delta)$ and designate the probability density function and the distribution functions as $f_{NIG}(x; \alpha, \beta, \mu, \delta)$ and $F_{NIG}(x; \alpha, \beta, \mu, \delta)$, respectively.

The probability density function of NIG distribution can be written as follows:

$$f_{NIG}(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha \exp\{\delta\gamma + \beta(x-\mu)\}}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)$$

where $K_1(\cdot)$ is the modified Bessel function of the third kind with order one and the meaning of γ is the same as in the previous formula.

The useful properties of NIG distribution are the scaling property and the closure under convolution for two independently NIG-distributed random variables. These properties are listed below:

Scaling property: If $X \sim NIG(\alpha, \beta, \mu, \delta)$, then $cX \sim NIG\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta\right), c > 0$.

Closure property: If $X \sim NIG(\alpha, \beta, \mu_1, \delta_1)$, $Y \sim NIG(\alpha, \beta, \mu_2, \delta_2)$ and X and Y are mutually independent, then $X + Y \sim NIG(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2)$.

The mean and variance of NIG distribution are

$$\mathrm{E(X)} = \mu + \delta\frac{\beta}{\gamma}$$

$$\mathrm{Var(X)} = \delta\frac{\alpha^2}{\gamma^3}$$

After the basic concepts of NIG distribution are reviewed, the NIG copula model can be constructed by replacing Gaussian random variables in the one-factor Gaussian copula model with NIG random variables. The closure property of NIG distribution ensures that this simple replacement can be achieved. The model specification is listed as follows:

$$X_i = \sqrt{\rho} M + \sqrt{1-\rho} Z_i, 0 \le \rho \le 1$$

where $M \sim NIG(\alpha, \beta, \frac{-\beta\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2})$, $\{Z_i\}_{i=1}^n \sim NIG(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2})$, and $\gamma = \sqrt{\alpha^2 - \beta^2}$. In the above specification, M and $\{Z_i\}_{i=1}^n$ are stochastically independent. The last two parameters in NIG distribution are such that the means of M and Z_i's are zero and the variances of M and Z_i's are one. The correlation between X_i and X_j still remains ρ. It is not difficult to show that $X_i \sim NIG(\frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2})$ by applying the scaling property and the closure property of NIG distribution. After the distribution of X_i's are obtained, the default threshold x_i can be determined below:

$$x_i = F_{NIG}^{-1}\left[Q_i(t); \frac{\alpha}{\sqrt{\rho}}, \frac{\beta}{\sqrt{\rho}}, -\frac{1}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{1}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2} \right], i = 1, 2, \ldots, n$$

The default probability conditioned on M by time t for each company can be written below:

$$p_t^{i|M} = F_{NIG}\left[\frac{x_i - \sqrt{\rho}M}{\sqrt{1-\rho}}; \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\beta\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\gamma^3}{\alpha^2} \right], i = 1, 2, \ldots, n$$

After these two important quantities are determined, the fair spreads of CDO tranches can be obtained under the factor copula pricing framework described in section 2.4.

4.2 Stochastic Correlation Model

Another way to model correlation skew is to make the correlation parameter stochastic just like stochastic volatility models in option pricing field. One nice feature of stochastic correlation model is that the marginal distributions of the latent variables X_i's remain normally distributed. This eases calibration and implementation of the models. The general structure of a stochastic correlation model can be expressed as follows:

$$X_i = \sqrt{\widetilde{\rho_i}}M + \sqrt{1-\widetilde{\rho_i}}Z_i, 0 \le \widetilde{\rho_i} \le 1, M \sim N(0,1), \{Z_i\}_{i=1}^n \sim N(0,1), i = 1,2,...,n$$

where M and $\{Z_i\}_{i=1}^n$ are standard normally distributed and $\left\{\widetilde{\rho_i}\right\}_{i=1}^n$ are some random variables taking values in [0,1]. M, $\{Z_i\}_{i=1}^n$ and $\left\{\widetilde{\rho_i}\right\}_{i=1}^n$ are jointly independent. The independence between $\left\{\widetilde{\rho_i}\right\}_{i=1}^n$, M and $\{Z_i\}_{i=1}^n$ is particularly important as conditioning upon $\widetilde{\rho_i}$ the latent variables X_i's remain standard normally distributed. Therefore, the marginal distribution of X_i is also standard normal. This can be easily proved as follows:

$$\begin{aligned}
\mathrm{P}\{X_i \le x\} &= \mathrm{P}\{\sqrt{\widetilde{\rho_i}}M + \sqrt{1-\widetilde{\rho_i}}Z_i \le x\} \\
&= \mathrm{E}\left[\ \mathrm{P}\{\sqrt{\widetilde{\rho_i}}M + \sqrt{1-\widetilde{\rho_i}}Z_i \le x \mid \widetilde{\rho_i}\}\right] \\
&= \mathrm{E}[\Phi(x)] = \Phi(x)
\end{aligned}$$

The uniqueness of the distribution function ensures that X_i's are standard normally distributed. Hence, the default threshold x_i can be easily determined below:

$$x_i = \Phi^{-1}\left[Q_i(t)\right], i = 1, 2, \ldots, n$$

Also, the general formula for the default probability conditioned on M by time t for each company can be written as follows:

$$
\begin{aligned}
p_t^{i|M} &= Q\left(X_i \le x_i \,\middle|\, M\right) = Q\left(\sqrt{\widetilde{\rho}_i}M + \sqrt{1-\widetilde{\rho}_i}Z_i \le x_i \,\middle|\, M\right) \\
&= \mathrm{E}\left[\mathrm{Q}\left(\sqrt{\widetilde{\rho}_i}M + \sqrt{1-\widetilde{\rho}_i}Z_i \le x_i \,\middle|\, M, \widetilde{\rho}_i\right)\right] \\
&= \mathrm{E}\left[\Phi\left(\frac{\mathrm{x_i} - \sqrt{\widetilde{\rho}_i}M}{\sqrt{1-\widetilde{\rho}_i}}\right)\right], i = 1, 2, \ldots, n
\end{aligned}
$$

where the expectation appeared in the above formula is taken with respect to $\widetilde{\rho}_i$. In the following, one specification of $\widetilde{\rho}_i$ proposed by Burtschell et al. (2005) is considered since it is parsimonious and incorporates idiosyncratic risks as well as systemic risks into the model. The consideration of systemic risks in this model involves the incorporation of comonotonic state which corresponds to a state of 100% correlation. The effect of comonotonicity is to increase both equity and senior tranche premiums which accounts for the shape of correlation skew. The model can be written as follows:

$$
X_i = [(1-B_s)(1-B_i)\rho + B_s]M + (1-B_s)[\sqrt{1-\rho^2}(1-B_i) + B_i]Z_i, 0 \le \rho \le 1
$$

where $\widetilde{\rho}_i = [(1-B_s)(1-B_i)\rho + B_s]^2$ and B_s and $\{B_i\}_{i=1}^n$ are independent Bernoulli random variables with $Q(B_s = 1) = q_s$ and $Q(B_i = 1) = q$. Under this specification of $\widetilde{\rho}_i$, the distribution of $\widetilde{\rho}_i$ can be expressed as follows:

$$
\widetilde{\rho}_i = \begin{cases} 0 & , Q(\widetilde{\rho}_i = 0) = q(1-q_s) \\ \rho^2 & , Q(\widetilde{\rho}_i = \rho^2) = (1-q)(1-q_s) \\ 1 & , Q(\widetilde{\rho}_i = 1) = q_s \end{cases}
$$

It can be seen from above that the perfect correlation occurs whenever $B_s = 1$. Thus, the realized value of B_s determines if comonotonic state will be achieved. So, B_s represents the systemic risk exhibited in the reference portfolio. In addition, when $B_s = 0$, the realized values of B_i 's determine whether zero correlation will be achieved, i.e., independent defaults will happen. Hence, B_i 's represent the idiosyncratic risks among reference entities.

Under this specification of $\widetilde{\rho}_i$, the default probability conditioned on M by time t can be derived as follows:

$$
\begin{aligned}
p_t^{i|M} &= \mathrm{E}\left[\Phi\left(\frac{x_i - \sqrt{\widetilde{\rho}_i}M}{\sqrt{1-\widetilde{\rho}_i}}\right)\right] \\
&= q(1-q_s)\Phi(x_i) + (1-q)(1-q_s)\Phi\left(\frac{x_i - \rho M}{\sqrt{1-\rho^2}}\right) + q_s \mathbf{1}_{\{M \le x_i\}} \\
&= q(1-q_s)Q_i(t) + (1-q)(1-q_s)\Phi\left(\frac{\Phi^{-1}[Q_i(t)] - \rho M}{\sqrt{1-\rho^2}}\right) + q_s \mathbf{1}_{\{M \le \Phi^{-1}[Q_i(t)]\}}
\end{aligned}
$$

where $\mathbf{1}_{\{\cdot\}}$ the indicator function which is equal to one when the event {.} occurs and zero otherwise. Then the fair spreads of CDO tranches can be determined under the factor copula pricing framework described in section 2.4.

4.3 Local Correlation Model

Still another way to account for the correlation skew is to introduce the concept of local correlation. The term “local correlation” refers to the idea underlying a model that the correlation is a function of the common factor M. Under stochastic correlation model, the correlation is stochastic but independent of the common factor, whereas the local correlation is a deterministic function of the common factor and thus is also stochastic. These terminologies parallel stochastic volatility models and local volatility models used in option pricing.

The local correlation approach is introduced by Andersen and Sidenius (2004) with a random factor loading model and by Turc et al. (2005). These models attempt to explain correlation through the intuitive relationship between default correlation and the economic cycle. That is, default correlation tends to be higher during recession than during a growing economy period.

In the following, the random factor loading model (RFL) is chosen as the representative of local correlation models due to its simplicity and ease of calibration. This model can be expressed as follows:

$$X_i = m + (\alpha \mathbf{1}_{\{M \le \theta\}} + \beta \mathbf{1}_{\{M > \theta\}})M + vZ_i, \alpha > 0, \beta > 0$$

where $m = (\alpha - \beta)\varphi(\theta)$, $v = [1 + m^2 - \alpha^2(\Phi(\theta) - \theta\varphi(\theta)) - \beta^2(\theta\varphi(\theta) + 1 - \Phi(\theta))]^{\frac{1}{2}}$, $\varphi(\cdot)$ is the probability density function of standard normal distribution. Also, M and $\{Z_i\}_{i=1}^n$ are stochastically independent, standard normal distributions. The number m and v are fixed and structured so that $\mathrm{E}[X_i] = 0$ and $\mathrm{Var}[X_i] = 1$.

The ability of the random factor loading model to produce a correlation skew depends on the parameters α and β. If $\alpha > \beta$, the factor loading falls as M increases, i.e., good economic state lowers the default correlation among reference entities while bad economic state increases the default correlation among reference entities. In the special case $\alpha = \beta$, the model coincides with the one-factor Gaussian copula, but in general the latent variable X_i's do not follow a Gaussian distribution. The distribution function of X_i is listed below and the derivation is shown in Appendix B:

$$F_{X_i}(x) = \Phi_2\left(\frac{x-m}{\sqrt{v^2+\alpha^2}}, \theta; \frac{\alpha}{\sqrt{v^2+\alpha^2}}\right) + \Phi\left(\frac{x-m}{\sqrt{v^2+\beta^2}}\right) - \Phi_2\left(\frac{x-m}{\sqrt{v^2+\beta^2}}, \theta; \frac{\beta}{\sqrt{v^2+\beta^2}}\right)$$

where $\Phi_2(x, y; \rho)$ is the cumulative probability distribution function of bivariate standard normal distribution with correlation ρ. Then the default threshold can be obtained through inverting the distribution function of X_i and is expressed below:

$$x_i = F_{X_i}^{-1}[Q_i(t)], i = 1, 2, \ldots, n$$

The default probability conditioned on M by time t can be obtained as usual under random factor loading model and is listed below:

$$p_t^{i|M} = \Phi\left[\frac{F_{X_i}^{-1}[Q_i(t)] - m - (\alpha \mathbf{1}_{\{M \le \theta\}} + \beta \mathbf{1}_{\{M > \theta\}})M}{v}\right], i = 1, 2, ..., n$$

Then the fair spreads of CDO tranches can be determined under the factor copula pricing framework as before.

Chapter 5

Numerical Results

5.1 Data and Model Calibration

In this chapter, the capabilities of CDO pricing models mentioned in the previous chapter will be assessed. Since the one-factor Gaussian copula model is widely used in pricing CDOs, its results will be given as a benchmark. The data used are iTraxx Europe 5 year daily quotes from January 4, 2007 to April 7, 2008. The marginal default probabilities of each reference entity are assumed to be the same and are obtained by the methodology described in Appendix A. The recovery rate is assumed to be 40% and is applied to all of reference entities. The model parameters are calibrated for each day by the least squares method. To be more specific, the parameters are chosen such that the following error function has the minimum value.

$$\text{Sum of error squared} = \sum_{\text{tranches}} \left(\frac{\text{market tranche quote} - \text{model tranche fee/spread}}{\text{market tranche quote}} \right)^2$$

The Euro default-free rates are obtained from the data available on European Central Bank's website (http://www.ecb.eu/stats/money/yc/html/index.en.html). The ECB estimates zero rate curves for the European region from AAA-rated, existing zero coupon bonds and fixed coupon bonds issued by European central governments on a daily basis. The estimation of the curve is done by minimizing the sum of the squared difference between the yields that can be computed from the model and the yields actually measured. The model they used is the Svensson model, which is a parametric model specifying a functional form for the spot rate. Other details about the estimation of zero rate curves are available on ECB's website.

After the default-free rates, the recovery rates, the marginal default probabilities are determined, the model spreads/upfront fees can be computed under the factor copula framework described in section 2.4.

5.2 Market Fitness

As mentioned in section 3.3, the credit derivatives market has structural changes after the subprime mortgage crisis. Hence, the fitness of models may also change after the crisis. To take the impact of the crisis into account, the data are divided into three samples: one sample consists of data before the crisis and the other two samples consist of data after the crisis. The first sample includes the data from January 4, 2007 to July 10, 2007 and the second sample includes the data from July 11, 2007 to February 13, 2008. The third sample includes the rest of the data. July 10, 2007 is chosen as the division point since the index spreads and tranche spreads have big jumps after this date. In addition, the index spread is quite unstable after July 10, 2007 and has another surge at February 14, 2008. Therefore, two samples are split up further.

The results are summarized in Table 5.1, Table 5.2 and Table 5.3. All of numbers shown in the three tables are average values over the sub-sample. The first rows of the three tables show average market spreads/upfront fees. The remaining rows of the tables show the average spreads/upfront fees for the four models. The numbers in parentheses are average absolute pricing errors for each model and tranche. It is defined as follows:

$$\text{Absolute pricing error} = \frac{\sum_{\text{sample}} \left|\text{market tranche quote} - \text{model tranche fee/spread}\right|}{\text{Sample Size}}$$

In Table 5.1, it can be seen that before the crisis all of three models are better than Gaussian copula in the least squares sense. The average absolute pricing errors for the three models are small enough compared with the bid-ask spread of market quotes (not shown here) except the local correlation model for the 3-6% tranche. All models including Gaussian copula have large pricing errors for the mezzanine tranches, especially the 3-6% tranche. The difficulty of pricing mezzanine tranche accurately in a single parameter set comes from the fact that the mezzanine tranche is not a monotonic function of correlation as illustrated in Figure 3.2. This observation can also be made for Table 5.2 and Table 5.3.

	0-3%	3-6%	6-9%	9-12%	12-22%	Sum of Error Squared
Market	9.28%	47.70 bp	11.79 bp	4.60 bp	1.40 bp	
Gaussian Copula	10.70% (1.43%)	74.01 bp (26.31 bp)	14.96 bp (3.18 bp)	3.69 bp (0.90 bp)	0.41 bp (1.00 bp)	0.9522
NIG Copula	11.69% (2.41%)	46.95 bp (4.06 bp)	12.26 bp (0.61 bp)	4.87 bp (0.60 bp)	1.27 bp (0.18 bp)	0.1389
Stochastic Correlation Model	11.56% (2.29%)	52.86 bp (5.16 bp)	15.09 bp (3.30 bp)	5.02 bp (0.49 bp)	0.69 bp (0.71 bp)	0.4291
Local Correlation Model	10.62% (1.51%)	65.38 bp (17.67 bp)	13.81 bp (2.34 bp)	4.19 bp (0.78 bp)	0.83 bp (0.61 bp)	0.5570

Table 5.1: Average market spreads and average model spreads for iTraxx Europe 5 year from January 4, 2007 to July 10, 2007.

	0-3%	3-6%	6-9%	9-12%	12-22%	Sum of Error Squared
Market	21.80%	140.43 bp	68.80 bp	42.17 bp	23.70 bp	
Gaussian Copula	34.20% (12.40%)	292.40 bp (151.97 bp)	94.81 bp (27.82 bp)	38.90 bp (16.71 bp)	10.27 bp (14.47 bp)	2.5680
NIG Copula	29.06% (7.70%)	132.20 bp (17.90 bp)	65.42 bp (5.95 bp)	44.10 bp (4.03 bp)	26.85 bp (3.39 bp)	0.2941
Stochastic Correlation Model	36.53% (14.73%)	174.52 bp (34.09 bp)	89.41 bp (20.61 bp)	54.88 bp (12.71 bp)	21.41 bp (9.00 bp)	1.0067
Local Correlation Model	26.89% (6.75%)	202.26 bp (61.82 bp)	72.00 bp (18.58 bp)	41.12 bp (8.60 bp)	23.38 bp (5.49 bp)	0.6769

Table 5.2: Average market spreads and average model spreads for iTraxx Europe 5 year from July 11, 2007 to February 13, 2008.

	0-3%	3-6%	6-9%	9-12%	12-22%	Sum of Error Squared
Market	37.55%	464.27 bp	302.73 bp	210.18 bp	106.82 bp	
Gaussian Copula	18.65% (29.01%)	563.41 bp (311.27 bp)	385.38 bp (137.17 bp)	294.07 bp (83.89 bp)	200.63 bp (93.81 bp)	2.6516
NIG Copula	45.13% (8.26%)	461.15 bp (39.01 bp)	265.19 bp (37.54 bp)	199.87 bp (15.41 bp)	143.50 bp (36.68 bp)	0.2985
Stochastic Correlation Model	40.05% (2.56%)	481.49 bp (17.21 bp)	300.81 bp (16.05 bp)	220.20 bp (19.23 bp)	198.31 bp (91.49 bp)	0.9128
Local Correlation Model	29.27% (12.84%)	542.00 bp (151.22 bp)	328.09 bp (37.85 bp)	223.88 bp (44.81 bp)	152.99 bp (46.18 bp)	0.7702

Table 5.3: Average market spreads and average model spreads for iTraxx Europe 5 year from February 14, 2008 to April 7, 2008.

In Table 5.2 and Table 5.3, the three models still perform better than Gaussian copula. However, the fitness of all models worsens sharply, especially after February 14, 2008. The absolute pricing errors for each model jump to several dozen basis points after the subprime mortgage crisis. The only exception is the stochastic correlation model. The fitness of this model improves for every tranche except the 9-12% and 12-22% tranches. It shows that the systemic risk modeling in the stochastic correlation model is problematic. In Table 5.1, Table 5.2 and Table 5.3, it can be also seen that the normal inverse Gaussian copula outperforms other models in the least squares sense whether the subprime mortgage crisis occurs or not. Since after the crisis the idiosyncratic default risks as well as the systemic default risk is largely increased, it is advantageous to apply fat-tailed distribution for the common risk factor and idiosyncratic risk factors in the factor copula models. On the other hand, the local correlation model has consistently large pricing errors for the 3-6% tranche. It may be caused by the fact that only two-point distribution is used to account for the distribution of correlation in the local correlation model. A possible improvement is to use multinomial distribution instead of the binomial distribution to model the distribution of local correlation. However, this improvement complicates the distribution of latent variable X_i's and slows down the computation of default thresholds. So although the local correlation model may be meaningful in economic sense, it is not easy to implement in general.

5.3 Stability of Parameters

Another important aspect of pricing models is the stability of a model's parameter values. Since the models we considered in the thesis are all static, the stability of calibrated parameter values may be questionable. Therefore, the calibrated parameter values of Gaussian copula, NIG copula, stochastic correlation model, and local correlation model are plotted against time in Figure 5.1, Figure 5.2, Figure 5.3, and Figure 5.4, respectively. The first vertical line in Figure 5.1, Figure 5.2, Figure 5.3 , and Figure 5.4 indicates when the subprime mortgage crisis occurs, i.e., July 11, 2007. The second vertical line in these figures indicates the time at which the second jump occurs in iTraxx index spread, namely February 14, 2008. From those figures, it can be observed that the parameters of all models are quite stable over time before the subprime mortgage crisis. After the crisis, all parameter values become very unstable, especially after the second jump of iTraxx index spread. The instability of parameter values is phenomenal for NIG copula in return for its good fit to market quotes. In addition, it can be also found that the systematic default risk rises dramatically as evidenced by the increase of ρ in Gaussian copula, NIG copula and stochastic correlation models. It shows that the market-perceived systemic default risk increased after the subprime mortgage crisis. However, the worsening fitness of the models we consider in the thesis shows the inadequacy of systematic default risk treatments in these models. Hence, the appropriate approach to incorporate the systematic default risk into basket credit derivatives pricing models is still unsolved.

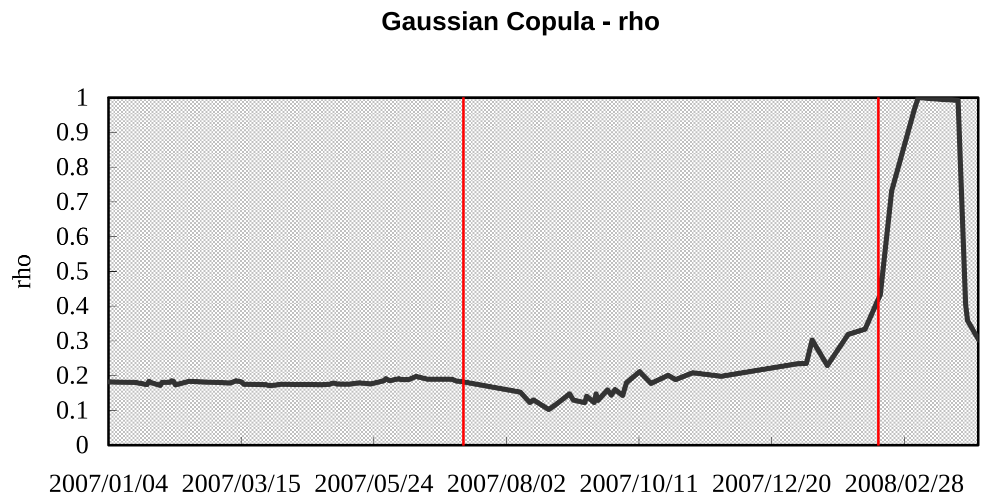


Figure 5.1: Parameter values of Gaussian copula from January 4, 2007 to April 7, 2008.

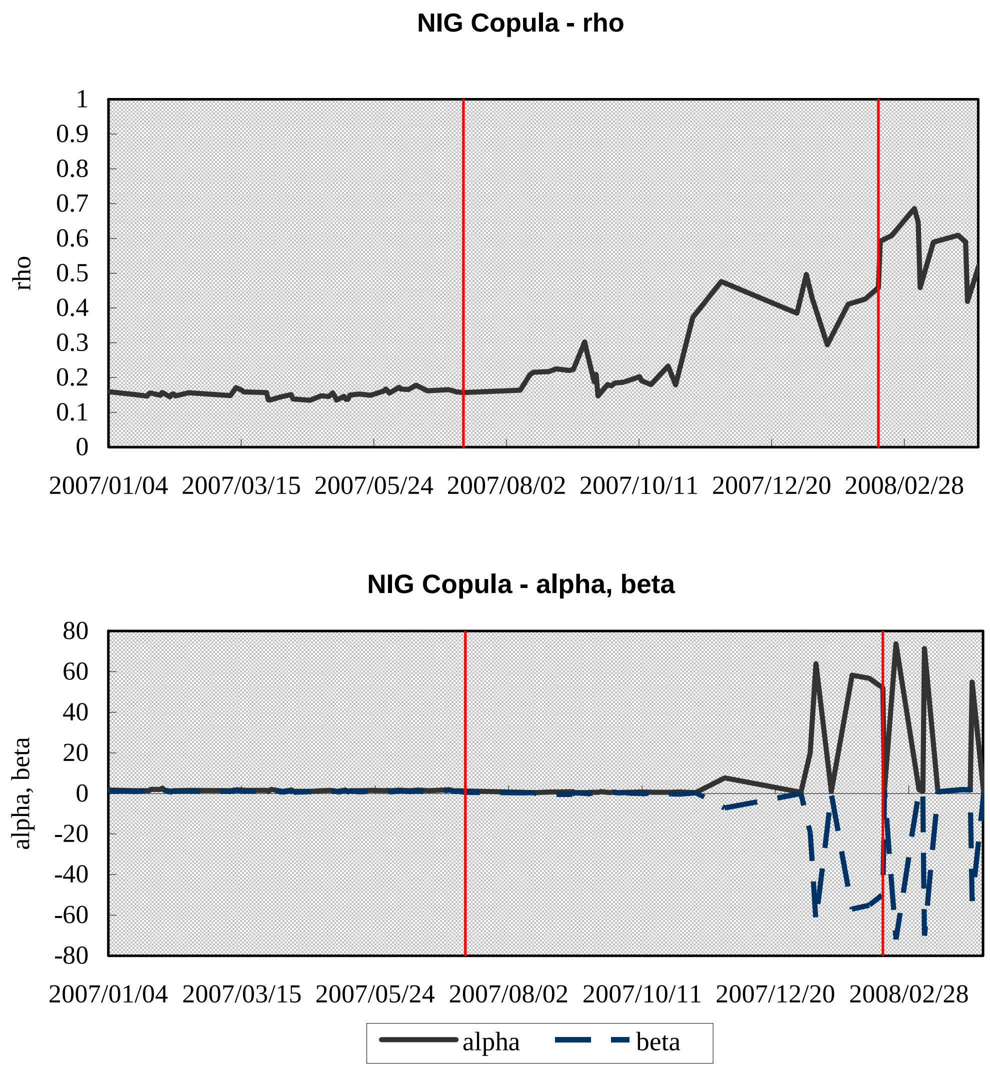


Figure 5.2: Parameter values of NIG copula from January 4, 2007 to April 7, 2008.

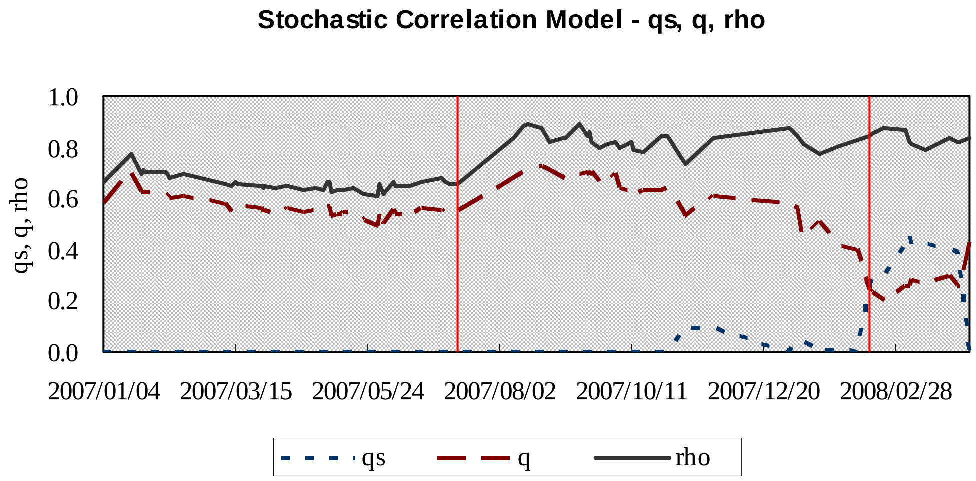
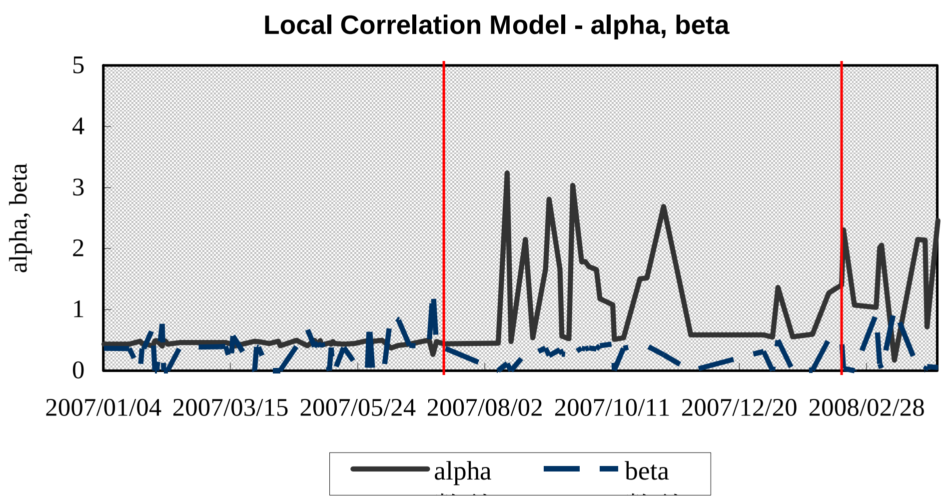


Figure 5.3: Parameter values of stochastic correlation model from January 4, 2007 to April 7, 2008.



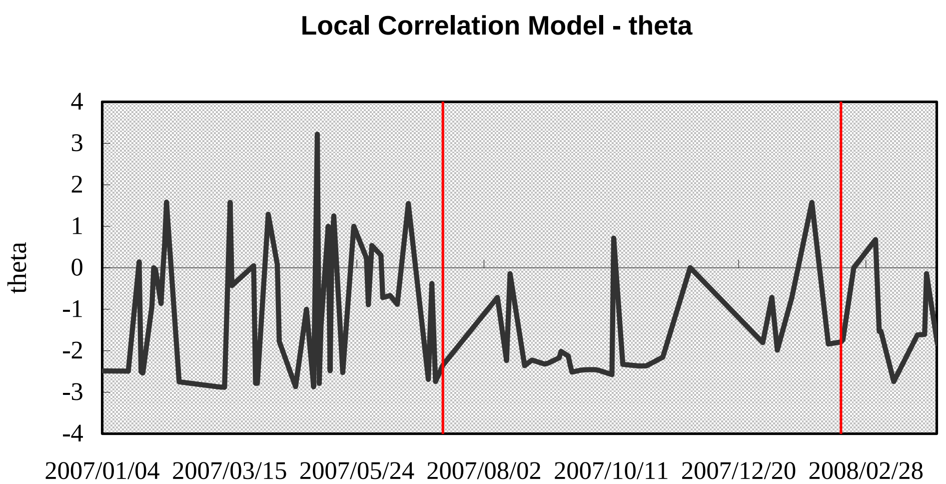


Figure 5.4: Parameter values of local correlation model from January 4, 2007 to April 7, 2008.

Chapter 6

Conclusions

In this thesis, we analyze three CDO pricing models and the standard Gaussian copula by examining their fitness to market quotes and the stability of calibrated parameter values over time. It is found that the occurrence of the subprime mortgage crisis not only changes the shape of correlation skew from “smile” type to “wave” type but also affects the accuracy of these models. These models perform quite well before the crisis and their fitness worsens a lot after it. Nonetheless, the normal inverse Gaussian copula still outperforms other models whether the crisis occurs or not. It suggests that the use of fat-tailed distribution in the factor copula models is effective. On the other hand, the local correlation model incorporates the relationship between default correlation and business cycle into model but fails to fit the market quotes of mezzanine tranches. It shows that although the local correlation model is meaningful in economic sense, this technique may not be appropriate for correlation modeling. Also, the stochastic correlation model performs well for the equity tranche and the mezzanine tranche but its fitness to the senior tranche is unacceptable after the subprime mortgage crisis. It indicates that the systematic default risk modeling in this model is not suitable for the senior tranche. Finally, the parameter values of these models are rather stable before the subprime mortgage crisis and become volatile after the crisis. It shows that the crisis disturbs the credit derivatives market and its impact still takes effect.

Bibliography

[1] Andersen, L., J. Sidenius and S. Basu (2003) All Your Hedges in One Basket. RISK, November, pp. 67–72.

[2] Andersen, L and J. Sidenius (2004) Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings. Journal of Credit Risk, 1 No.1, pp. 29–70.

[3] Brigo, D. and F. Mercurio (2006) Interest Rate Models — Theory and Practice: With Smile, Inflation and Credit, 2nd ed., Springer, Berlin, pp. 724–737.

[4] Burtschell, X., J. Gregory and J.-P. Laurent (2007) Beyond the Gaussian Copula: Stochastic and Local Correlation. Journal of Credit Risk, Vol. 3, No. 1, pp. 31–62.

[5] Burtschell, X., J. Gregory and J.-P. Laurent (2005) A Comparative Analysis of CDO Pricing Models. Working paper, ISFA Actuarial School and BNP Parisbas, Paris.

[6] Hull, John C. and White, A. (2004) Valuation of n-th to Default CDS Without Monte Carlo Simulation. Journal of Derivatives, pp. 8–23.

[7] Kalemanova, A., B. Schmid, and R. Werner (2007) The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing. Journal of Derivatives, Vol. 14, Iss. 3, pp. 80–93.

[8] Li, D. (2000) On Default Correlation: A Copula Approach. Journal of Fixed Income, Vol. 9, pp. 43–54.

[9] McGinty, L., E. Beinstein, R. Ahluwalia and M. Watts (2004) Introducing Base Correlations. Credit Derivatives Strategy, JPMorgan, London.

[10] Turc, J., P. Very and D. Benhamou (2005) Pricing CDOs with a Smile. SG Credit Research, Paris.

[11] Vasicek, O. (1987) Probability of Loss on Loan Portfolio. Technical report, KMV Corporation, San Francisco.

[12] Walker, M. (2005) Risk-neutral correlations in the pricing and hedging of basket credit derivatives. Journal of Credit Risk, Vol. 1, No. 1, pp. 131–139.

Appendix

A. Extraction of Default Probabilities from CDS Spreads

There are several approaches to extract default probabilities of a single credit. CDS spreads unlike corporate bonds only reflect the credit risks of the reference entities. Thus, CDS spreads are more suitable to be used to extract default probabilities. Since the default probabilities inferred from CDS spreads will be used to determine the fair spreads of CDO tranches, all of default probabilities obtained are in a risk-neutral sense. In the following, we will derive one version of CDS pricing formula adapted from Brigo, D. and F. Mercurio (2006).

To proceed, we need to determine the fair spread of a CDS contract first. The fair spread of a CDS contract is the spread such that the value of the contract is zero, i.e., the present value of the premium leg is equal to the present value of the protection leg.

For convenience, the notations used are listed below.

- τ : The time at which a credit event occurs.
- R : The recovery rate as a fraction of the notional.
- $\Delta_{i-1,i}$: The year fraction between two payment dates t_{i-1} and t_i.
- $B(0,t_i)$: The discount factor at time 0 for cash flow occurring at t_i.
- T : The time to maturity of the CDS contract as a fraction of a year.
- $Q(t)$: The risk-neutral probability that the company will default before time t.
- λ : The default intensity associated with the default process.
- S : The fair spread per annum of the CDS contract.

Suppose that the notional of the CDS contract is one unit of currency, the recovery rate is deterministic and the interest rates are stochastically independent of the default events. Given the payment dates $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, the payoff of a CDS contract from protection seller's view can be expressed as follows:

$$\underbrace{\sum_{i=1}^{n} S\Delta_{i-1,i} e^{-\int_0^{t_i} r_s ds} \mathbf{1}_{\{\tau \geq t_i\}}}_{\text{Spread}} + \underbrace{\sum_{i=1}^{n} S(\tau - t_{i-1}) e^{-\int_0^{\tau} r_s ds} \mathbf{1}_{\{t_{i-1} < \tau < t_i\}}}_{\text{Accrued Spread}} - \underbrace{(1-R) e^{-\int_0^{\tau} r_s ds} \mathbf{1}_{\{\tau \leq T\}}}_{LossPayments}$$

For simplicity, it is assumed that the defaults can only occur at the payment dates.[2] Hence, we do not need to consider the accrued spread. The present values of protection leg and premium leg can be derived as follows:

$$\begin{aligned}
&\text{PV(Protection Leg)} \\
&= \mathrm{E}^Q\left[(1-R) e^{-\int_0^{\tau} r_s ds} \mathbf{1}_{\{\tau \leq T\}}\right] = \mathrm{E}^Q\left[(1-R) E\left(e^{-\int_0^{\tau} r_s ds} \middle| \tau\right) \mathbf{1}_{\{\tau \leq T\}}\right] \\
&= \mathrm{E}^Q\left[(1-R) B(0,\tau) \mathbf{1}_{\{\tau \leq T\}}\right] = \int_0^T (1-R) B(0,t) dQ(t) \\
&\cong \sum_{i=1}^{n} (1-R) B(0,t_i)[Q(t_i) - Q(t_{i-1})]
\end{aligned}$$

$$\begin{aligned}
&\text{PV(Premium Leg)} \\
&= \mathrm{E}^Q\left[\sum_{i=1}^{n} S\Delta_{i-1,i} e^{-\int_0^{t_i} r_s ds} \mathbf{1}_{\{\tau \geq t_i\}}\right] \\
&= \sum_{i=1}^{n} S\Delta_{i-1,i} \mathrm{E}\left(e^{-\int_0^{t_i} r_s ds}\right) \mathrm{E}\left(\mathbf{1}_{\{\tau \geq t_i\}}\right) \\
&= \sum_{i=1}^{n} S\Delta_{i-1,i} B(0,t_i)[1 - Q(t_i)]
\end{aligned}$$

Therefore, the fair spread of the CDS contact can be obtained by equalizing protection leg and premium leg and is expressed as follows:

[2] Another usual approach to pricing CDS is to assume the default time always falls halfway between payment dates. However, the default probabilities obtained by these two assumptions are very similar in real applications.

$$S = \frac{\sum_{i=1}^{n} (1-R)[Q(t_i) - Q(t_{i-1})]B(0,t_i)}{\sum_{i=1}^{n} \Delta_{i-1,i}[1-Q(t_i)]B(0,t_i)} \tag{A.1}$$

Furthermore, if the reduced-form model is applied, then the default process can be represented as a Poisson process with constant intensity λ. Hence, the default time follows an exponential distribution and the risk-neutral default probability $Q(t)$ can be expressed as follows:

$$Q(t) = 1 - e^{-\lambda t} \tag{A.2}$$

Substituting (A.2) into (A.1), the fair spread can be rewritten as follows:

$$S = \frac{\sum_{i=1}^{n} (1-R)[e^{-\lambda t_{i-1}} - e^{-\lambda t_i}]B(0,t_i)}{\sum_{i=1}^{n} \Delta_{i-1,i} e^{-\lambda t_i} B(0,t_i)} \tag{A.3}$$

To back out risk-neutral default probabilities of the reference entity, it is needed to estimate the recovery rate and construct the discount curve from other data sources. Then the default intensity can be obtained by using the CDS spread in the market and inversion of formula (A.3) numerically. After the default intensity is determined, risk-neutral probabilities of default by each time can be calculated via formula (A.2).

B. Derivation of Distribution Function of the Latent Variable in Local Correlation Model

In this section, $\varphi(\cdot)$ denotes the probability density function of standard normal distribution, $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution and $\Phi_2(x,y;\rho)$ is the cumulative distribution function of bivariate standard normal distribution with correlation ρ. Also, $\mathbf{1}_{\{\cdot\}}$ is the indicator function which is equal to one when the event {.} occurs and zero otherwise. For clearness, $\mathrm{E}_X(\cdot)$ means the expectation taken with respect to random variable X. Some useful results are proved first to facilitate the derivation of the distribution function of X_i in random factor loading model. These results come from the appendix of Andersen, L and J. Sidenius (2004).

LEMMA For arbitrary real constants a, b and c,

$$\text{(i)}\ \int_{-\infty}^{\infty}\Phi(ax+b)\varphi(x)dx=\Phi\left(\frac{b}{\sqrt{1+a^2}}\right)$$

$$\text{(ii)}\ \int_{-\infty}^{c}\Phi(ax+b)\varphi(x)dx=\Phi_2\left(\frac{b}{\sqrt{1+a^2}},c;-\frac{a}{\sqrt{1+a^2}}\right)$$

PROOF

(i)

$$\int_{-\infty}^{\infty}\Phi(ax+b)\varphi(x)dx=\mathrm{E}_X\left[\Phi(aX+b)\right]=\mathrm{E}_X\left(\mathrm{E}_Z\left[\mathbf{1}_{\{Z\le aX+b\}}\right]\right)$$

$$=\mathrm{E}_X\left(\mathrm{E}_Z\left[\mathbf{1}_{\{Z\le aX+b\}}\mid X\right]\right)(\because\ Z \text{ and } X \text{ are independent.})$$

$$=\mathrm{E}_X\left(\mathrm{E}_S\left[\mathbf{1}_{\{S\le b\}}\mid X\right]\right)(\text{Let S = -aX+Z ~ N(0,1+a}^2))$$

$$=\mathrm{E}_S\left[\mathbf{1}_{\{S\le b\}}\right]=\mathrm{P}\{S\le b\}=\mathrm{P}\{\frac{S}{\sqrt{1+a^2}}\le b\}=\Phi\left(\frac{b}{\sqrt{1+a^2}}\right)$$

(ii)

$$\int_{-\infty}^{c}\Phi(ax+b)\varphi(x)dx=\int_{-\infty}^{c}\mathrm{P}\{Z\le ax+b\}\varphi(x)dx$$

$$=\int_{-\infty}^{c}\mathrm{P}\{Z\le ax+b\mid X=x\}\varphi(x)dx \text{ } (\because \text{ Z and X are independent.})$$

$$=\int_{-\infty}^{c}\mathrm{P}\{S\le b\mid X=x\}\varphi(x)dx \text{ (Let S = -aX+Z \sim N(0,1+a}^2\text{))}$$

$$=\mathrm{P}\{S\le b, X\le c\}=\mathrm{P}\{\frac{S}{\sqrt{1+a^2}}\le\frac{c}{\sqrt{1+a^2}}, X\le c\}$$

$$=\Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; -\frac{a}{\sqrt{1+a^2}}\right)$$

Now, the distribution function of X_i in random factor loading model can be derived as follows:

$$F_{X_i}(x)=Q\{X_i\le x\}$$

$$=Q\left\{m+(\alpha\mathbf{1}_{\{M\le\theta\}}+\beta\mathbf{1}_{\{M>\theta\}})M+vZ_i\le x\right\}$$

$$=\int_{-\infty}^{\infty}Q\left\{m+(\alpha\mathbf{1}_{\{M\le\theta\}}+\beta\mathbf{1}_{\{M>\theta\}})M+vZ_i\le x\middle|M\right\}\varphi(M)dM$$

$$=\underbrace{\int_{-\infty}^{\theta}Q\left\{m+(\alpha\mathbf{1}_{\{M\le\theta\}}+\beta\mathbf{1}_{\{M>\theta\}})M+vZ_i\le x\middle|M\right\}\varphi(M)dM}_{(1)}$$

$$+\underbrace{\int_{\theta}^{\infty}Q\left\{m+(\alpha\mathbf{1}_{\{M\le\theta\}}+\beta\mathbf{1}_{\{M>\theta\}})M+vZ_i\le x\middle|M\right\}\varphi(M)dM}_{(2)}$$

$$(1)=\int_{-\infty}^{\theta} Q\{m+\alpha M+vZ_i \le x \mid M\}\varphi(M)dM=\int_{-\infty}^{\theta}\Phi\left(\frac{x-m-\alpha M}{v}\right)\varphi(M)dM$$

$$=\int_{-\infty}^{\theta}\Phi\left(-\frac{\alpha}{v}M+\frac{x-m}{v}\right)\varphi(M)dM$$

$$=\Phi_2\left(\frac{\frac{x-m}{v}}{\sqrt{1+(-\frac{\alpha}{v})^2}},\theta;-\frac{(-\frac{\alpha}{v})}{\sqrt{1+(-\frac{\alpha}{v})^2}}\right) \text{ (Apply Lemma (i))}$$

$$=\Phi_2\left(\frac{x-m}{\sqrt{v^2+\alpha^2}},\theta;\frac{\alpha}{\sqrt{v^2+\alpha^2}}\right)$$

$$(2)=\int_{\theta}^{\infty} Q\{m+\beta M+vZ_i \le x \mid M\}\varphi(M)dM=\int_{\theta}^{\infty}\Phi\left(\frac{x-m-\beta M}{v}\right)\varphi(M)dM$$

$$=\int_{-\infty}^{\infty}\Phi\left(\frac{x-m-\beta M}{v}\right)\varphi(M)dM-\int_{-\infty}^{\theta}\Phi\left(\frac{x-m-\beta M}{v}\right)\varphi(M)dM$$

$$=\int_{-\infty}^{\infty}\Phi\left(-\frac{\beta}{v}M+\frac{x-m}{v}\right)\varphi(M)dM-\int_{-\infty}^{\theta}\Phi\left(-\frac{\beta}{v}M+\frac{x-m}{v}\right)\varphi(M)dM$$

$$=\Phi\left(\frac{\frac{x-m}{v}}{\sqrt{1+(-\frac{\beta}{v})^2}}\right)-\Phi_2\left(\frac{\frac{x-m}{v}}{\sqrt{1+(-\frac{\beta}{v})^2}},\theta;-\frac{(-\frac{\beta}{v})}{\sqrt{1+(-\frac{\beta}{v})^2}}\right) \text{ (Apply Lemma (i) (ii))}$$

$$=\Phi\left(\frac{x-m}{\sqrt{v^2+\beta^2}}\right)-\Phi_2\left(\frac{x-m}{\sqrt{v^2+\beta^2}},\theta;\frac{\beta}{\sqrt{v^2+\beta^2}}\right)$$

$$F_{X_i}(x) = (1) + (2)$$

$$= \Phi_2\left(\frac{x-m}{\sqrt{v^2+\alpha^2}}, \theta; \frac{\alpha}{\sqrt{v^2+\alpha^2}}\right) + \Phi\left(\frac{x-m}{\sqrt{v^2+\beta^2}}\right) - \Phi_2\left(\frac{x-m}{\sqrt{v^2+\beta^2}}, \theta; \frac{\beta}{\sqrt{v^2+\beta^2}}\right)$$