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利用隱含樹狀模型評價波動率及變異數交換

Pricing Volatility and Variance Swaps by Implied Trees

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Abstract

Equity-index volatility and variance swaps offer an efficient way for traders to take synthetic positions in pure volatility. General pricing method for volatility and variance swaps uses the replication method in Demeterfi, Derman, Kamal, and Zou (1999). In this thesis, we try to use the more direct and intuitive way to price volatility and variance swaps. Specifically, we will use implied trees introduced in Derman, Kani, Chriss (1994) and Derman, Kani (1996) which can match the implied local volatilities and variances. Then we employ these local volatilities and variances to price volatility and variance swaps. After using the implied tree to price, we also compare the result of this method to the general pricing method. We find out that using this method can also get the value of volatility and variance swaps just similar to the general method.

Contents

Chapter 1
Introduction

1. 1 Motivations

Equity-index volatility and variance swaps offer an efficient way for traders to take synthetic positions in pure volatility. There are several ways to use volatility and variance swaps. People who use volatility and variance swaps are trading volatility levels directionally, trading the spread between the realized and implied volatility levels, or hedging implicit volatility exposure.

General pricing method for volatility and variance swaps uses the replication method in Demeterfi, Derman, Kamal, and Zou (1999). Pricing variances is more direct than pricing volatilities. In general, we will price variances first, and then take the square roots of variances to price volatilities. But general pricing method is not a straightforward method. It is to try to use lots of options to replicate the value of variance swaps. In this thesis, we try to use the more direct and intuitive way to price volatility and variance swaps. Specifically, we will use implied trees which can match the implied local volatilities and variances and are introduced in Derman, Kani, Chriss

(1994) and Derman, Kani (1996). Then we employ these local volatilities and variances to price volatility and variance swaps.

1. 2 Organization of This Thesis

There are seven chapters in this thesis. In Chapter 1, a brief introduction and motivation of this thesis are presented. In Chapter 2, we will introduce some background information and also the concepts of volatilities. In Chapter 3, we will go through the basic definitions and pricing methods for volatility and variance swaps. In Chapter 4, we will explain the methodology used to price volatility and variance swaps. For that purpose, we will explain the structure of implied trees and how we use them to price volatility and variance swaps. Also we will give an example to compare the results between the general pricing method and our method. In Chapter 5, the strategies and applications of volatility and variance swaps will be briefly discussed. Finally, Chapter 6 concludes and points out future research.

Chapter 2
Background

2. 1 Literature Review

First of all, we introduce some concepts about volatilities. Derman, Kani, and Zou (1995) explain the local volatility surface, give examples of its applications, and propose several properties of volatilities for understanding the relation between local and implied volatilities. Kani, Derman, and Kamal (1996) outline a methodology for hedging and trading index volatilities. In the world of index options, local volatilities are the arbitrage-free volatilities at future times and market levels that can be locked in by trading options today. Derman, Kamal, Kani, McClure, Pirasteh, and Zou (1998) explain the definition of volatilities and how to invest in index volatilities. They also talk about the advantage of volatility contracts.

We now introduce volatility and variance swaps. Demeterfi, Derman, Kamal, and Zou (1999) explain the properties and the theories of both volatility and variance swaps, from an intuitive point of view and then more rigorously. Hull (2003) explains some basic definition of volatility and variance swaps. Neftci and Fame (2004) explain the concepts and pricing methods of volatility and variance swaps more precisely.

Finally, we introduce our methodology to price volatility and variance swaps based on the implied tree. Derman and Kani (1994) introduce the volatility smile and show how to extend the Black-Scholes model to a model which assumes that the index level executes a random walk with a constant volatility so as to make it consistent with the volatility smile. By this extension of the Black –Scholes model, they obtain a new model which is consistent with the volatility smile. They call it the implied binomial tree. Derman, Kani, and Chriss (1996) show how to build implied trinomial tree models that incorporate the volatility smile.

2. 2 Volatilities

Volatility plays an important role in option pricing and risk management. It is the simplest measure of its risk or uncertainty.

Stock investors think they know something about the direction of the stock market. So, they may have insight into the level of future volatility. If they think current volatility is too low, for the right price they may want to take a position that will profit when the volatility increases.

Volatility has a lot of definitions. People usually use the word “volatility” to denote several related but different concepts. We shall now clarify the different volatilities; specifically, we shall clarify exactly what “realized”, “implied”, and “local” volatilities mean.

The realized volatility of an index over some period is the annualized standard deviation of its daily returns over that period. The implied volatility of an index, as implied by the current price of a particular European-style option with strike K and expiration T, is the volatility parameter that, when entered into the Black-Scholes formula, equates the model value and the market option price. The local volatilities of an index at some future market levels and time levels are the future volatilities that the index must have at that market level and time in order to make current option prices fair.

The role of volatility in the option world is as important as the role of interest rate in the bond world. That is, the concepts between volatilities and options are similar to the concepts between interest rates and bond prices. So understanding different kinds of interest rates first is helpful for us to understand the characteristics of different kinds of volatilities.

The realized interest rate is the actual interest rate that comes to pass during some period. The realized volatility is similar to it. The yield to maturity of the bond is its implied yield. As implied volatility translates into an option price through the Black-Scholes option pricing formula, so the yield-to-maturity translates into a bond price through the present value formula. The forward rate is the future rate that must come to pass to justify current yield-to-maturity; it is the future rate that can be locked in by trading a bond portfolio. The local volatility is similar.

Chapter 3
Volatility and Variance Swaps

3. 1 Basic Definitions

The volatility or variance swap allows an investor to directly implement a view on the direction of future realized volatility or variance. Like an equity swap, in which two parties exchange cash flows based on the return of specified equity, the volatility or variance swap is characterized by the exchange of cash flows tied to the performance of realized volatility or variance.

In a volatility or variance swap, an investor agrees to receive or pay the realized volatility or variance of an equity index or single stock relative to an agreed strike level. Realized volatility is typically measured as the annualized standard deviation of the daily natural log returns of the stock or index. The formula for the realized volatility is as follows. Define

S_i = the stock price at each time level

u_i = the log return of the stock index at each time level

$\bar{u}$ = the average log return of the stock index

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \tag{1}$$

$$\sigma_R = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(u_i-\bar{u}\right)^2} \times \sqrt{252}$$

$$\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}u_i^2-\frac{1}{n\times(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2} \times \sqrt{252} \tag{2}$$

where σ_R is the annualized realized volatility.

Whereas, an equity swap is based on a specified number of shares, a volatility or variance swap is expressed in terms of the dollar value of each volatility point.

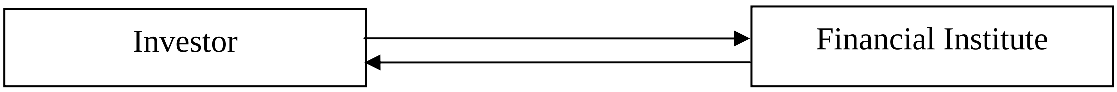
A volatility swap is sometimes called a realized volatility forward contract because it provides pure exposure to volatility. A stock volatility swap is a forward contract on realized volatility. And there is no initial exchange of cash flow between two parties, only an agreement upon the strike price. Its payoff is equal to

$$N\times(\sigma_R-K_{vol})\times 100 \quad , \tag{3}$$

where σ_R is the realized stock volatility over the life of the contracts, K_{vol} is the annualized volatility strike price, and N is the notional amount of the swap in dollars per annualized volatility point. The holder of a volatility swap at expiration receives N dollars for every point by which the stock's realized volatility has exceeded the volatility strike price. That is, the holder is swapping a fixed volatility for the actual future volatility which is floating. For example, the cash flow of buying a volatility swap is shown in Figure 1.

Figure 1. The Cash Flow of Buying a Volatility Swap.

If $\sigma_R < \sigma_K$, $N \times (\sigma_K - \sigma_R) \times 100$.



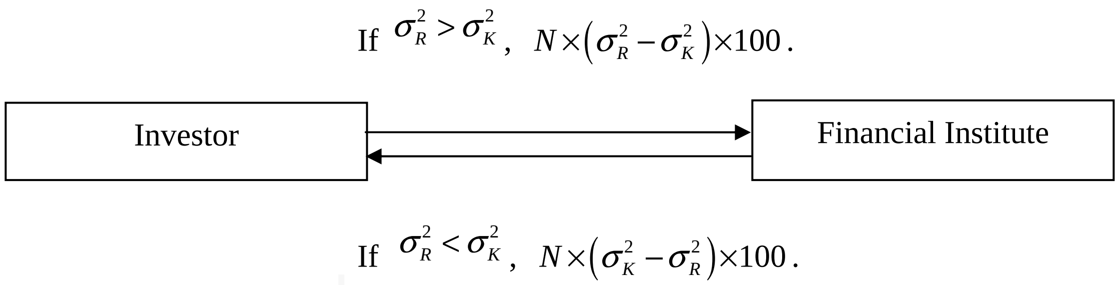
If $\sigma_R > \sigma_K$, $N \times (\sigma_R - \sigma_K) \times 100$.

A variance swap is similar to a volatility swap. A variance swap is a forward contract on realized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N \times \left(\sigma_R^2 - K_{\mathrm{var}}\right) \times 100 \quad , \tag{4}$$

where σ_R^2 is the realized stock variance over the life of the contracts, K_{var} is the annualized variance strike price, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap at expiration receives N dollars for every point by which the stock's realized variance has exceeded the variance strike price. That is, the holder is swapping a fixed variance for the actual future variance which is floating. For example, the cash flow of selling a variance swap is shown in Figure 2.

Figure 2. The Cash Flow of Selling a Variance Swap.



Who will use volatility and variance swaps? Volatility has several characteristics that make trading it attractive. It is likely to grow when uncertainty and risk increase. Similar to interest rates, volatilities appear to revert to the mean. And volatility is often negatively correlated with stock or index level. Derman, Kani, and Zou (1995) have explained this relationship. Given these characteristics, there are several kinds of users of volatility and variance swaps. The main users of volatility and variance swaps are those who are trading volatility levels directionally, who are trading the spread between the realized and implied volatility levels, and who are hedging implicit volatility exposure.

3. 2 General Pricing Method

General pricing method for volatility and variance swaps is to replicate the value of variance swaps first, and then take square roots of variance swaps to get the value of volatility swaps. In fact, volatility swaps are more straightforward for investors who want to hedge their volatility. According to the actual condition, why we price variance swaps first? There are two main reasons. First, variance swaps provide similar volatility exposure to straight volatility swaps. Second, variance is easier to calculate than volatility. It can serve as the basic building block for constructing volatility-dependent instruments. The fair value of a variance swap is the delivery price that makes the swap have zero value. It is determined by the cost of the replicating portfolio.

To price variance swaps, we assume that there is no jumps allowed for the stock or index process. Therefore, we assume that the stock price process is given by

$$\frac{dS_t}{S_t} = \mu(t,\cdots)\,dt + \sigma(t,\cdots)\,dZ_t \quad , \tag{5}$$

where μ is the drift, σ is the continuously-sampled volatility, and we assume that both of them are arbitrary functions of time and other parameters. For simplicity, we assume that the stock pays no dividends.

The theoretical definition of realized variance for a given price history is the continuous integral

$$V = \frac{1}{T}\int_0^T \sigma^2(t,\ldots)dt \quad . \tag{6}$$

To value a variance swap or forward contract is similar to valuing other derivative securities. The value of a forward contract F on future realized variance with strike price K is the expected present value of the future payoff in the risk-neutral world, or,

$$F = E[e^{-rT}(V-K)] \quad , \tag{7}$$

where r is the risk-free discount rate corresponding to the expiration T, and $E[\ ]$ denotes the expectation. The fair delivery value of future realized variance is the strike price K_{var} for which the contract has zero present value, or,

$$K_{\text{var}} = E[V] \quad . \tag{8}$$

If the future volatility is specified, then one way to calculate the fair value of variance is to calculate the risk-neutral expectation directly

$$K_{\text{var}} = E\left[\frac{1}{T}\int_0^T \sigma^2(t,\cdots)dt\right] \quad . \tag{9}$$

By applying Ito’s lemma to $\log S_t$, we find

$$d(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dZ_t \quad . \tag{10}$$

Subtracting the process of $\log S_t$ from the process of the stock price, we obtain

$$\frac{dS_t}{S_t} - d\left(\log S_t\right) = \frac{1}{2}\sigma^2 dt \quad , \tag{11}$$

in which all dependence on the drift μ has been cancelled. Integrating this result over all times from time 0 to time T, we obtain the continuously sampled variance

$$V = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \quad . \tag{12}$$

This equation identity dictates the replication strategy for variance. The first term in the bracket can be thought as the net outcome of continuous rebalancing a stock position so that it is always instantaneously long $1/S_t$ shares of stock worth \$1. The second term represents a static short position in a contract which pays the logarithm of the total return at expiration. We call this contract a log contract. It has a detailed explanation in Neuberger (1994). Following this rebalancing strategy captures the realized variance of the stock from initiation to expiration. In this equation, we note that no expectations or averages have been taken. So, it guarantees that variance can be captured no matter how the stock price moves, as long as it moves continuously.

According to the previous equation, Eq. (12), we can take the expected risk-neutral value of the right-hand side of it to obtain the cost of replication directly

$$K_{\text{var}} = \frac{2}{T}E\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \quad . \tag{13}$$

The expected value of the first term in this equation accounts for the cost of rebalancing. In a risk-neutral world with a constant risk-free rate r, the stock price process changes to

$$\frac{dS_t}{S_t} = rdt + \sigma(t,\cdots)dZ_t \quad , \tag{14}$$

so the risk-neutral price of rebalancing component of the hedging strategy is given by

$$E\left[\int_0^T \frac{dS_t}{S_t}\right] = rT \quad . \tag{15}$$

This equation represents the fact that a shares position, continuously rebalanced to be worth \$1, has a forward price that grows at the risk-less rate.

There is no actively traded log contract which pays the logarithm of the total return at expiration for the second term in the previous equation of the strike price K_{var}, Eq. (15). We must replicate the log payoff, at all stock price levels and expiration, by decomposing its shape into linear and curvature components, and then replicate each of these separately. The linear component can be replicate with a forward contract on the stock with expiration at time T. The curvature component, including the quadratic and higher order contributions, can be replicated by using lots of standard options with all possible strike levels and the same expiration time T.

For practical reasons, we will replicate the log payoff with liquid options, that is, with a combination of out-of-the-money calls for high stock values and out-of-the-money puts for low stock values. There is a new parameter S^* to represent the boundary between calls and puts. The log payoff can be written as

$$\log\frac{S_T}{S_0}=\log\frac{S_T}{S^*}+\log\frac{S^*}{S_0} \quad . \tag{16}$$

As the second term in this equation is a constant and independent of the final stock price S_T, only the first term in this equation has to be replicated.

The following equation which holds for future values of S_T suggests this decomposition of the log payoff

$$-\log\frac{S_T}{S^*}=-\frac{S_T-S^*}{S^*}+\int_0^{S^*}\frac{1}{K^2}\max\left(K-S_T,0\right)dK+\int_{S^*}^{\infty}\frac{1}{K^2}\max\left(S_T-K,0\right)dK \,. \tag{17}$$

This decomposition of a log payoff into a portfolio consisting of a short position in $1/S^*$ forward contracts struck at S^*, a long position in $1/K^2$ put options struck at K, for all strike price from 0 to S^*, and a similar long position in $1/K^2$ call options struck at K, for all strike price from 0 to S^*. All contracts expire at time T.

In summary, we can obtain the fair delivery value of variance swap as follows

$$K_{\text{var}}=\frac{2}{T}\left(rT-\left(\frac{S_0}{S^*}e^{rT}-1\right)-\log\frac{S^*}{S_0}+e^{rT}\int_0^{S^*}\frac{1}{K^2}P\left(K\right)dK+e^{rT}\int_{S^*}^{\infty}\frac{1}{K^2}C\left(K\right)dK\right), \tag{18}$$

where $P(K)$ and $C(K)$ denote the current fair value of a put and call option with strike price K, respectively. If we use the fair market prices of there options, we can obtain an estimate of the current market price of future variance.

At the beginning of this section, we focused on pricing and replicating variance swaps. But most market participants prefer to quote levels of volatility rather than variance because volatility swaps are more straightforward for investors who want to hedge their volatility. So we have to consider volatility swaps.

Unfortunately, there is no simple replication strategy for a volatility swap. The replication strategy for a volatility swap is difficult, and it is affected by changes in volatility and its value depends on the volatility of future realized volatility. Therefore, variance is the primary underlying and all other volatility payoffs, such as volatility swaps, are considered to be derivative securities with variance as underlying. From this point of view, volatility is a nonlinear function of variance and is therefore more difficult to value and hedge.

Basically, we will obtain the fair delivery price of a volatility swap by taking the square root of a variance swap rate. It is the simplest way to approximate the fair delivery price of a volatility swap. That is, the approximated value of a volatility swap is the square root of the value of a variance swap

$$K_{vol} = \sqrt{K_{\mathrm{var}}} \quad , \tag{19}$$

where K_{vol} is the fair delivery price of the volatility swap. This method will have some bias for the following reason. Our method first calculates the expected value of variances, and then takes square root of it to obtain the volatility. But by definition, we should take the square root of variances first to obtain the volatilities, and then calculate the expected volatility. Our method will overestimate the fair delivery price of the volatility swap as

$$K_{vol} = E\left[\sqrt{V}\right] < \sqrt{E[V]} = \sqrt{K_{\mathrm{var}}} \quad . \tag{20}$$

This bias is mentioned in Demeterfi, Derman, Kamal, and Zou (1999), who also introduce ways to reduce the bias. This thesis only uses Eq. (19) to approximate the fair delivery price of the volatility swap.

Chapter 4
Methodology

4. 1 Implied Binomial Trees

The market implied volatilities of stock index options often have a skewed structure, which is commonly called the volatility smile. It implies a negative relationship between implied volatilities and strike prices. So out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls. The implied binomial tree is an arbitrage-free model that fits the smile, is preference-free, avoids additional factors and can be used to value options from observable data.

We use forward induction and Arrow-Debreu prices to build an implied tree with identical time periods. The volatility function in the implied tree is deduced numerically from the volatility smile given by the prices of liquid options; the implied tree model is calibrated to be arbitrage-free relative to observed option prices. Use forward stock prices and option prices to solve unknown values (the implied stock prices and transition probabilities) and then find out implied local volatilities. Following are the notations we will use when we construct the implied tree.

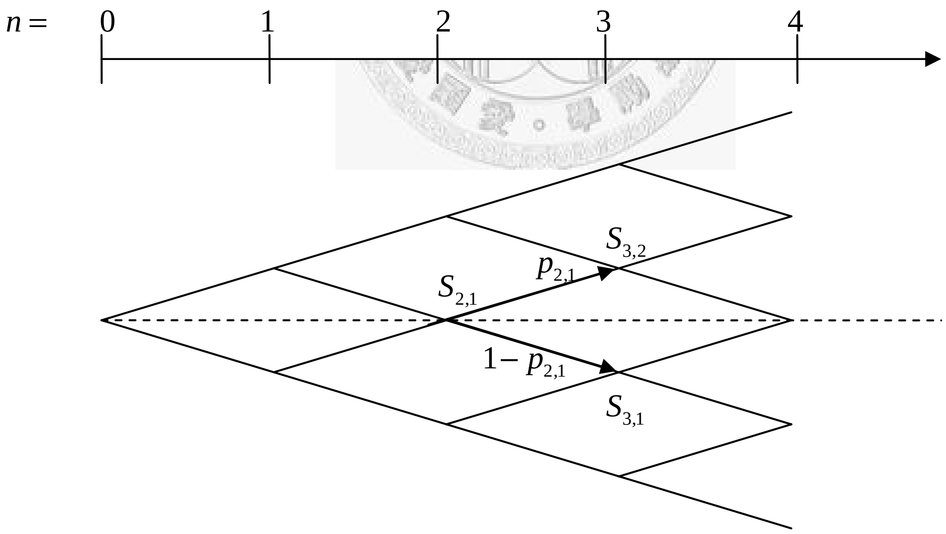
r	=	forward risk-less interest rate at each time step
$S_{n,i}$	=	stock price at time n and state of stock price i; the strike price for options expiring at time $n+1$
$S_{n+1,i+1}$	=	stock price at time $n+1$ and state of stock price $i+1$; the up node of $S_{n,i}$ at time $n+1$
$S_{n+1,i}$	=	stock price at time $n+1$ and state of stock price i; the down node of $S_{n,i}$ at time $n+1$
$F_{n,i}$	=	forward price of the known stock price $S_{n,i}$ at time n, where $F_{n,i} = e^{r\Delta t} S_{n,i}$
$\lambda_{n,i}$	=	Arrow-Debreu price at time n and state of stock price i
$p_{n,i}$	=	risk-neutral transition probability from node (n,i) to node $(n+1, i+1)$

First of all, we have to construct a binomial tree for the stock price which is proposed by Cox, Ross, and Rubinstein (1979). See Appendix A. Then use forward induction and Arrow-Debreu prices to build an implied tree and reconstruct the binomial tree of the stock price.

When we stand at time n which starts from 0 and wish to find the unknown parameters at time $n+1$, there are $2n+3$ unknown parameters to solve for, namely the $n+2$ unknown stock prices and the $n+1$ unknown transition probabilities from node (n,i) to node $(n+1, i+1)$. But we only have $2n+2$ known quantities, that is, $n+1$ known forward sock prices at time n, $F_{n,i} = e^{r\Delta t} S_{n,i}$ and $n+1$ known option prices all expiring at time $n+1$. Therefore, There are $2n+2$ equations to solve for $2n+3$ parameters.

For example, when we stand at time $n=2$, the stock prices $S_{2,i}$ for each i from 0 to 2 are known. We wish to find the unknown parameters at time $n=3$. There are 7 unknown parameters to solve for, namely the 4 unknown stock prices, $S_{3,i}$ for each i from 0 to 3, and the 3 unknown transition probabilities, $p_{2,i}$ from node $(2,i)$ to node $(3,i+1)$. But we only have 6 known quantities, that is, 3 known forward sock prices at time $n=2$, $F_{2,\,i}=e^{r\Delta t}S_{2,\,i}$ and 3 known option prices all expiring at time $n=3$. Therefore, There are 6 equations to solve for 7 parameters. See Figure 3.

Figure 3. An Example For Constructing The Implied Tree.



We use the one remaining degree of freedom to make the center of the tree coincide with the center of the standard CRR tree which has constant local volatilities.

This is explained as follows. If the number of nodes at a given time is odd, choose the center node's stock price to be equal to the spot today. If the number is even, make the average of the natural logarithms of the two center nodes' stock prices equal the logarithm of today's spot price. That is $S_{n+1,\, i} = S^2 / S_{n+1,\, i+1}$. Then, we will have $2n+3$ equations to solve for $2n+3$ unknown parameters.

Before starting to construct the implied tree, we have to understand the concept of Arrow-Debreu prices. An Arrow-Debreu price is a price today of a security that has a cash flow at the given time and state of stock price. The Arrow-Debreu price for the next time $n+1$ is given by

$$e^{r\Delta t}\lambda_{n+1,i} = \begin{cases} p_{n,n}\lambda_{n,n} & , \quad \text{when} \quad i = n+1 \\ p_{n,i-1}\lambda_{n,i-1} + \left(1 - p_{n,i}\right)\lambda_{n,i} & , \quad \text{when} \quad 1 \le i \le n+1 \\ \left(1 - p_{n,0}\right)\lambda_{n,0} & , \quad \text{when} \quad i = 0 \end{cases} \quad . \tag{21}$$

Now, we have to derive the theoretical values of forward stock prices. The implied tree is risk-neutral. Therefore, the expected value of stock at any node (n,i) one time step later must be its known forward price

$$F_{n,\, i} = p_{n,\, i} S_{n+1,\, i+1} + \left(1 - p_{n,\, i}\right) S_{n+1,\, i} \quad , \tag{22}$$

where $F_{n,i}$ is known, $F_{n,\, i} = e^{r\Delta t} S_{n,\, i}$. There are n of these forward equations.

We also have to derive the theoretical values of options. The strike price for the option at time n which is expiring at time $n+1$ is the known stock price at time n. The strike $S_{n,i}$ can go to the up or down node, $S_{n+1,i+1}$ or $S_{n+1,i}$, at the next time. This ensures that only the up (down) node and all nodes above (below, respectively) it contribute to a call (put, respectively) struck at $S_{n,i}$. Let $C\left(S_{n,i}, t_{n+1}\right)$ and $P\left(S_{n,i}, t_{n+1}\right)$ be the known market values for a call and put struck at $S_{n,i}$ and expiring at time $n+1$, respectively.

The theoretical binomial value of a call struck at K and expiring at time $n+1$ is given by the sum over all discounted probability nodes at time $n+1$ multiplied by the call payoff there. Alternatively, we represent it by Arrow-Debreu prices

$$C\left(K;T_{n+1}\right)=e^{-r\Delta t}\sum_{j=1}^{n}\left[p_{n,j}\lambda_{n,j}+\left(1-p_{n,j+1}\right)\lambda_{n,j+1}\right]\max\left(S_{n+1,j+1}-K,0\right) \quad . \tag{23}$$

When the strike price K equals $S_{n,i}$, Eq. (23) can be separated into three parts. First, contributions come from the up and down nodes of $S_{n,i}$. Because the down node of $S_{n,i}$, $S_{n+1,i}$, is out-of-the money to the call option, we ignore its contribution and only keep the contribution of the up node of $S_{n,i}$, $S_{n+1,i+1}$. Second, contributions come from up and down nodes of the nodes below $S_{n,i}$. That is, contributions come from up and down nodes of $S_{n,i-1}, S_{n,i-2} \cdots S_{n,0}$. Because all of them are out-of-the-money to the call option, we ignore them all. Third, the contribution from up and down nodes of the nodes above $S_{n,i}$. That is, contributions come from up and down nodes of $S_{n,i+1}, S_{n,i+2} \cdots S_{n,n}$. Because these nodes are all in-the-money to the call option, they can be written into the summation of the known Arrow-Debreu prices $\lambda_{n,j}$, the known stock prices $S_{n,j}$, and the known forward stock prices $F_{n,\mathrm{j}} = e^{r\Delta t} S_{n,\mathrm{j}}$. Therefore, Eq. (23) becomes to

$$e^{r\Delta t} C\left(S_{n,i};T_{n+1}\right) = p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - S_{n,i}\right) + \sum_{j=i+1}^{n} \lambda_{n,j}\left(F_{n,j} - S_{n,i}\right) \quad . \tag{24}$$

The first term depends on the unknown transition probability $p_{n,i}$ and the up node with unknown stock price $S_{n+1,i+1}$. The second term is the sum of known quantities.

Since we know both the forward stock prices and call option prices for the smile, we can solve the equations of these values to find out the stock prices which are above the center node at the next time and the transition probabilities. The equations are as follows

$$\begin{cases} e^{r\Delta t} C\left(S_{n,i};T_{n+1}\right) = p_{n,i}\lambda_{n+1,i}\left(S_{n+1,i+1} - S_{n,i}\right) + \sum_{j=i+1}^{n} \lambda_{n,j}\left(F_{n,j} - S_{n,i}\right) \\ F_{n,i} = p_{n,i} S_{n+1,i+1} + \left(1 - p_{n,i}\right) S_{n+1,i} \end{cases} \quad . \tag{25}$$

If the number of time steps n already solved is odd, $n+1$ is even, the initial central node is set equal to the central node of CRR tree. And then we get the value of stock prices above the center node

$$S_{n+1,i+1}=\frac{S_{n+1,i}\left[e^{r\Delta t}C\left(S_{n,i};T_{n+1}\right)-\Sigma\right]-\lambda_{n,i}S_{n,i}\left(F_{n,i}-S_{n+1,i}\right)}{\left[e^{r\Delta t}C\left(S_{n,i};T_{n+1}\right)-\Sigma\right]-\lambda_{n,i}\left(F_{n,i}-S_{n+1,i}\right)}\quad , \tag{26}$$

where

$$\Sigma=\sum_{j=i+1}^{n}\lambda_{n,j}\left(F_{n,j}-S_{n,i}\right)\quad , \quad F_{n,j}=S_{n,j}e^{(r-q)\Delta t}\quad .$$

Similar to using forward stock prices and call option prices to solve the stock prices above the center node, we use forward stock prices and put options to solve for the stock prices below the center node

$$S_{n+1,i}=\frac{S_{n+1,i+1}\left[e^{r\Delta t}P\left(S_{n,i};T_{n+1}\right)-\Sigma\right]+\lambda_{n,i}S_{n,i}\left(F_{n,i}-S_{n+1,i+1}\right)}{\left[e^{r\Delta t}P\left(S_{n,i};T_{n+1}\right)-\Sigma\right]+\lambda_{n,i}\left(F_{n,i}-S_{n+1,i+1}\right)}\quad , \tag{27}$$

where

$$\Sigma=\sum_{j=i+1}^{n}\lambda_{n,j}\left(F_{n,j}-S_{n,i}\right)\quad , \quad F_{n,j}=S_{n,j}e^{(r-q)\Delta t}\quad .$$

The transition probability at any time step and any state is given by

$$p_{n,i}=\frac{F_{n,i}-S_{n+1,i}}{S_{n+1,i+1}-S_{n+1,i}}\quad . \tag{28}$$

If the number of time steps n already solved is even and $n+1$ is odd, use the logarithmic CRR centering condition $S_{n+1,i}=S^2/S_{n+1,i+1}$ where S is today's stock price corresponding to the CRR-style center node at the previous time. Then we get the formula for the node above the two center nodes as

$$S_{n+1,i+1}=\frac{S\left[e^{r\Delta t}C\left(S;T_{n+1}\right)+\lambda_{n,i}S-\Sigma\right]}{\lambda_{n,i}F_{n,i}-e^{r\Delta t}C\left(S_{n,i};T_{n+1}\right)+\Sigma}\quad , \tag{29}$$

where

$$\Sigma = \sum_{j=i+1}^{n} \lambda_{n,j}\left(F_{n,j} - S_{n,i}\right) \quad , \quad F_{n,j} = S_{n,j} e^{(r-q)\Delta t} \quad .$$

Similarly, we can fix all the nodes below the center node at this time by using put options prices. We get the formula for these nodes

$$S_{n+1,i} = \frac{S_{n+1,i+1}\left[e^{r\Delta t} P\left(S_{n,i};T_{n+1}\right) - \Sigma\right] + \lambda_{n,i} S_{n,i}\left(F_{n,i} - S_{n+1,i+1}\right)}{\left[e^{r\Delta t} P\left(S_{n,i};T_{n+1}\right) - \Sigma\right] + \lambda_{n,i}\left(F_{n,i} - S_{n+1,i+1}\right)} \quad , \tag{30}$$

where

$$\Sigma = \sum_{j=i+1}^{n} \lambda_{n,j}\left(F_{n,j} - S_{n,i}\right) \quad , \quad F_{n,j} = S_{n,j} e^{(r-q)\Delta t} \quad .$$

According to the implied stock prices and transition probabilities, we can find out the implied local volatilities at each node:

$$\sigma_{n,i} = \frac{1}{\sqrt{\Delta t}} \sqrt{p_{n,i}\left(1 - p_{n,i}\right)} \ln\left(\frac{S_{n+1,i+1}}{S_{n+1,i}}\right) \quad . \tag{31}$$

The transition probabilities at any node in the implied tree must lie between 0 and 1. If $p_{n,i} > 1$, the stock price at the upper node will fall below the forward price. Similarly, if $p_{n,i} < 0$, the stock price at the lower node will fall above the forward price. Either these conditions allows risk-less arbitrage. Therefore, as we go through the tree node by node, we demand that each newly determined node's stock price lie between the neighboring forwards from the previous time, that is $F_{n,i} < S_{n+1,i+1} < F_{n,i+1}$. If the stock price at any node violates the above inequality, we override the option price that produced it. In its place, we choose a stock price that keeps the logarithm spacing between this node and its neighboring node the same as that between corresponding nodes at the previous time. This procedure removes arbitrage violations from input option prices, while keeping the implied local volatility function smooth.

4. 2 Implied Trinomial Trees

Implied trinomial trees have more parameters than implied binomial trees. Implied trinomial trees offer more flexibility and can match a larger class of volatility structure than implied binomial trees. We use these additional parameters to conveniently choose the state space of all node prices, and let the transition probabilities be constrained only by option prices. That is, we choose the state space independent of the transition probabilities and use option prices to solve for the transition probabilities only.

Similar to implied binomial trees, we have to construct a trinomial tree for the stock price. See Appendix B. We also use forward stock prices and option prices to find out the transition probabilities. And the option prices are presented by Arrow-Debreu prices. But we do not have to find out the newly implied stock price at each node because we assume that the state space at each node is independent of the transition probabilities.

We use forward stock prices and call option prices to solve for the transition probabilities of the node above the center node. Therefore the equations we have to solve are given by

$$\begin{cases} C\left(K;T_{n+1}\right)=e^{-r\Delta t}\sum_{j=1}^{n}\left[p_{n,j-2}\lambda_{n+1,j-2}+\left(1-p_{n,j-1}-q_{n,j-1}\right)+q_{n,j}\lambda_{n+1,j}\right]\max\left(S_{n+1,j}-K,0\right) \\ F_{n,i}=p_{n,i}S_{n+1,i+2}+\left(1-p_{n,i}-q_{n,i}\right)S_{n+1,i+1}+q_{n,i}S_{n+1,i} \end{cases}. \tag{32}$$

Similar to constructing the implied binomial trees, we can rewrite the equations to

$$\begin{cases} e^{r\Delta t}C\left(S_{n,i+1};T_{n+1}\right)=p_{n,i}\lambda_{n+1,i}\left(S_{n+1,i+2}-S_{n+1,i+1}\right)+\sum_{j=i+1}^{2n}\lambda_{n+1,j}\left(F_{n,j}-S_{n+1,i+1}\right) \\ F_{n,i}=p_{n,i}S_{n+1,i+2}+\left(1-p_{n,i}-q_{n,i}\right)S_{n+1,i+1}+q_{n,i}S_{n+1,i} \end{cases}, \tag{33}$$

where $p_{n,i}$ is the up transition probability and $q_{n,i}$ is the down probability at each node.

According to the result of the equations, we obtain the formulas for the up transition probability and down transition probability at each node above the center node. The transition probabilities are presented as follows

$$p_{n,i} = \frac{e^{r\Delta t} C\left(S_{n+1,i+1}; T_{n+1}\right) - \sum_{j=i+1}^{2n} \lambda_{n,j}\left(F_{n,j} - S_{n+1,i+1}\right)}{\lambda_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right)} . \tag{34}$$

$$q_{n,i} = \frac{F_{n,i} - p_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right) - S_{n+1,i+1}}{S_{n+1,i} - S_{n+1,i+1}} . \tag{35}$$

To solve the transition probabilities at each node below the center node is similar to the way we solve for the transition probabilities at each node above the center node. We use forward stock prices and put option prices to solve them. And then we obtain the formula for the up transition probability and down probability at each node below the center node. They are presented as follows

$$q_{n,i} = \frac{e^{r\Delta t} P\left(S_{n+1,i+1}; T_{n+1}\right) - \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n+1,i+1} - F_{n,j}\right)}{\lambda_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right)} . \tag{36}$$

$$p_{n,i} = \frac{F_{n,i} + q_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right) - S_{n+1,i+1}}{S_{n+1,i+2} - S_{n+1,i+1}} . \tag{37}$$

In the implied trinomial trees, there is still a problem of negative transition probabilities. If we obtain negative transition probabilities, we will try to overwrite them. There are a lot of ways to overwrite negative transition probability. One way we use is to choose the value of middle transition probability to be zero, in other words, we try to reducing the sub-tree to be a binomial tree. By doing this, we will obtain the formula for the overwriting value of the transition probabilities.

If $S_{n+1,i+1} < F_{n,i} < S_{n+1,i+2}$

$$p_{n,i} = \frac{1}{2}\left(\frac{F_{n,i} - S_{n+1,i+1}}{S_{n+1,i+2} - S_{n+1,i+1}} + \frac{F_{n,i} - S_{n+1,i}}{S_{n+1,i+2} - S_{n+1,i}}\right) \tag{38}$$

$$q_{n,i} = \frac{1}{2}\left(\frac{S_{n+1,i+2} - F_{n,i}}{S_{n+1,i+2} - S_{n+1,i}}\right) \quad . \tag{39}$$

If $S_{n+1,i} < F_{n,i} < S_{n+1,i+1}$

$$p_{n,i} = \frac{1}{2}\left(\frac{F_{n,i} - S_{n+1,i}}{S_{n+1,i+2} - S_{n+1,i}}\right) \tag{40}$$

$$q_{n,i} = \frac{1}{2}\left(\frac{S_{n+1,i+2} - F_{n,i}}{S_{n+1,i+2} - S_{n+1,i}} + \frac{S_{n+1,i+1} - F_{n,i}}{S_{n+1,i+1} - S_{n+1,i}}\right) \quad . \tag{41}$$

After getting the transition probabilities, we can use these transition probabilities to calculate the implied local volatility for corresponding node in the tree. The implied local volatility is given by

$$\sigma_{n,i} = \left[\frac{p_{n,i}\left(S_{n+1,i+2} - F_0\right)^2 + \left(1 - p_{n,i} - q_{n,i}\right)\left(S_{n+1,i+1} - F_0\right)^2 + q_{n,i}\left(S_{n+1,i} - F_0\right)^2}{F_0^2 \Delta t}\right]^{\frac{1}{2}} \quad , \tag{42}$$

where

$$F_0 = p_{n,i} S_{n+1,i+2} + \left(1 - p_{n,i} - q_{n,i}\right) S_{n+1,i+1} + q_{n,i} S_{n+1,i} \quad .$$

4. 3 Pricing Volatility and Variance Swaps

In this section, we try to use the implied trinomial tree to price volatility and variance swaps. After constructing the implied trinomial tree, we have already obtained the implied local volatilities or variances for each corresponding node. Now we use these implied local volatilities and variances to price volatility and variance swaps.

We try to price variance swaps first. Because, by definition, we know that volatilities are the square roots of variances, and that variances have the property of the additivity that makes variance much easier to price than volatilities.

By definition, the variance for a given price history is the continuous integral and given by $V = \frac{1}{T}\int_0^T \sigma^2(t,\ldots)dt$, where T is the whole life of the contract and $\sigma^2(t,\cdots)$ is the local variance for every time period. And then now we try to present it by the discrete form, that is

$$\begin{aligned} V &= \frac{1}{T}\sum_0^T \sigma^2(t,\ldots)\Delta t \\ &= \frac{\Delta t}{T}\sum_0^T \sigma^2(t,\ldots) \quad . \end{aligned} \tag{43}$$

According to the implied trinomial tree, we have already obtained implied local variances for each node. We try to find out the expected local variances for each time. Then use the definition by Eq. (8) to calculate the fair delivery price for the variance swap contract, see Figure 4. So we obtain

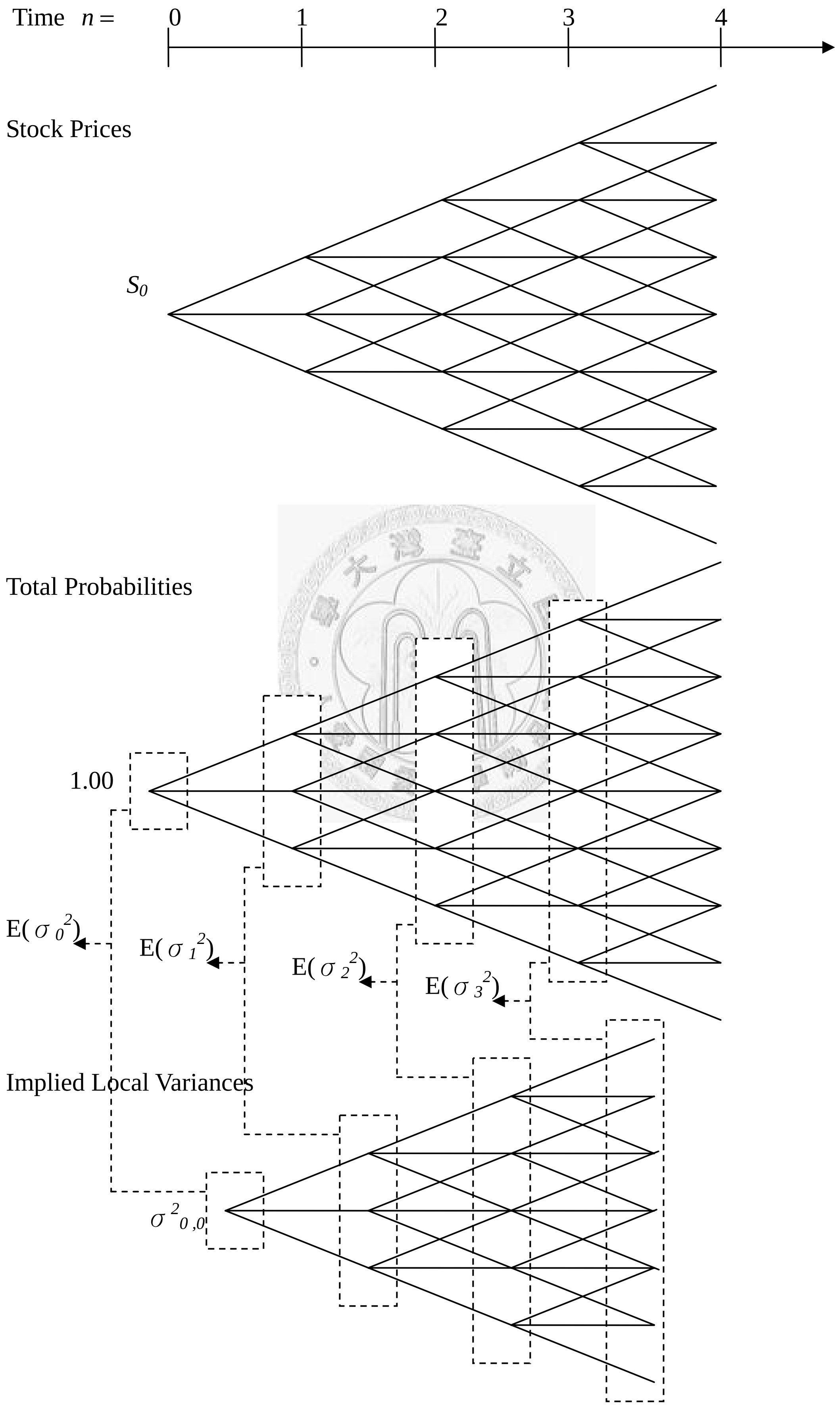
$$K_{\text{var}} = \frac{\Delta t}{T} \sum_{i=0}^{n} E\left(\sigma_i^2\right) \quad . \tag{44}$$

After calculating the variance for the whole life of the swap contract, we now try to calculate the volatility. Because local volatilities are the square root of local variances, they do not have the property of additivity. We have to calculate the local variances first, find out the variance for the whole life of the swap contract, and then take the square root of it to approximate the volatility for the whole life of the swap contract. Therefore,

$$K_{vol} = \sqrt{\frac{\Delta t}{T} \sum_{t=0}^{T} E\left(\sigma_t^2\right)} \quad , \tag{45}$$

where K_{vol} is the fair delivery price for the volatility swap contract.

Figure 4. The Process of Calculating the Fair Delivery Price of a Variance Swap.



4. 4 An Example

In Demeterfi, Derman, Kamal, and Zou (1999), they have given a detailed example of variance swap. We try to use the same assumption and data of this example, and use our pricing method which is using implied trinomial trees to price volatility and variance swap.

First of all, we have to introduce the basic assumption and pricing result of the example in Demeterfi, Derman, Kamal, and Zou (1999). When we want to price a swap on the realized variance of the daily returns of some equity index, the fair delivery price is determined by the cost of the replication. If we could buy options with strike prices from zero to infinity, we would get the fair delivery price of variance swap by the formula Eq. (18)

$$K_{\text{var}} = \frac{2}{T}\left(rT - \left(\frac{S_0}{S^*} e^{rT} - 1 \right) - \log\frac{S^*}{S_0} + e^{rT} \int_0^{S^*} \frac{1}{K^2} P(K)\, dK + e^{rT} \int_{S^*}^{\infty} \frac{1}{K^2} C(K)\, dK \right) \quad ,$$

with the choice which S^* is equal to S_0, that is $S^* = S_0$.

Unfortunately, only a small set of discrete option strike prices are available. The formula will have some error. So we try to use some other approximation. Starting with the definition of the fair variance given by Eq. (13)

$$K_{\text{var}} = \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \quad ,$$

and it can be written as

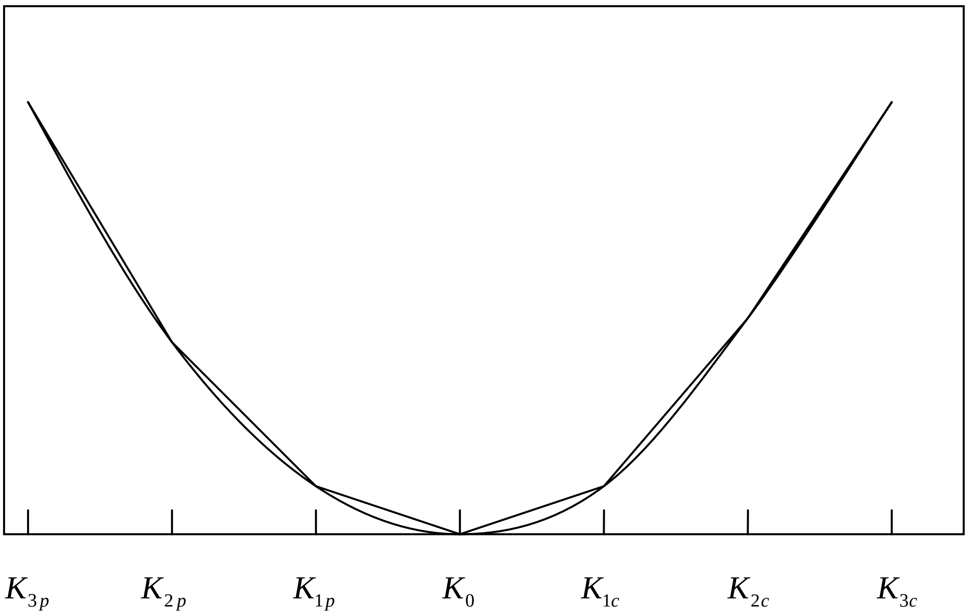
$$\begin{aligned} K_{\text{var}} &= \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S^*}{S^*} - \log\frac{S^*}{S_0} + \frac{S_T - S^*}{S^*} - \log\frac{S_T}{S^*}\right] \\ &= \frac{2}{T}\left[rT - \left(\frac{S_0}{S^*}e^{rT} - 1\right) - \log\frac{S^*}{S_0}\right] + e^{rT}\Pi_{CP} \quad , \end{aligned} \tag{46}$$

where Π_{CP} is the present value of the portfolio of options with payoff at expiration given by

$$f\left(S_T\right) = \frac{2}{T}\left(\frac{S_T - S^*}{S^*} - \log\frac{S_T}{S^*}\right) \quad . \tag{47}$$

Assume that we can trade call options with strike prices K_{ic} such that $K_0 = S^* < K_{1c} < K_{2c} < K_{3c} < \cdots$ and put options with strike prices K_{ip} such that $K_0 = S^* > K_{1p} > K_{2p} > K_{3p} > \cdots$. We can approximate the payoff function with a piece-wise linear function as in Figure 5

Figure 5. Log Payoff and Options Portfolio at Maturity.



The first segment to the right of S^* is equivalent to the payoff of a call option with strike K_0. The number of options is determined by the slope of this segment

$$w_c\left(K_0\right)=\frac{f\left(K_{1c}\right)-f\left(K_0\right)}{K_{1c}-K_0} \quad . \tag{48}$$

Similarly, the second segment looks like a combination of calls with strike K_0 and K_{1c}. Given that we already have $w_c\left(K_0\right)$ call options with strike K_0, we need to find out $w_c\left(K_{1c}\right)$ call options with strike K_{1c} where

$$w_c\left(K_{1c}\right)=\frac{f\left(K_{2c}\right)-f\left(K_{1c}\right)}{K_{2c}-K_{1c}}-w_c\left(K_0\right) \quad . \tag{49}$$

Continuing this way, we can build the whole payoff curve. In general, the number of call options of strike $K_{n,c}$ is given by

$$w_c\left(K_{n,c}\right)=\frac{f\left(K_{n+1,c}\right)-f\left(K_{n,c}\right)}{K_{n+1,c}-K_{n,c}}-\sum_{i=0}^{n-1} w_c\left(K_{i,c}\right) \quad . \tag{50}$$

The other side of the curve can be build by put options

$$w_p\left(K_{n,p}\right)=\frac{f\left(K_{n+1,p}\right)-f\left(K_{n,p}\right)}{K_{n+1,p}-K_{n,p}}-\sum_{i=0}^{n-1}w_p\left(K_{i,p}\right) \quad . \tag{51}$$

This approximation guarantees that these payoffs will always exceed or match the value of the log contract, but never be worth less. After calculating the weight of each option, Π_{CP} is obtained

$$\Pi_{CP}=\sum_i w\left(K_{ip}\right)P\left(S,K_{ip}\right)+\sum_i w\left(K_{ic}\right)C\left(S,K_{ic}\right) \quad . \tag{52}$$

Assume the initial stock index S_0 is 100, the continuously compounded annual risk-less interest rate r is 5%, the dividend yield q is zero, and the maturity of the variance swap is 3 months $\left(T=0.25\right)$. Also assume that we can buy options with strike prices from 50 to 150, uniformly spaced 5 points apart, and the at-the-money implied volatility is 20% with a skew such that when the strike price decreases for 5 points, the implied volatility will increase 1 volatility point.

After calculating the weight of each option, we obtain the cost of the options portfolio. We calculate the fair delivery price for variance swap by the equation

$$K_{\text{var}} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S^*} e^{rT} - 1 \right) - \log \frac{S^*}{S_0} \right] + e^{rT} \Pi_{CP}$$
$$= \frac{2}{T}\left[rT - \left(\frac{S_0}{S^*} e^{rT} - 1 \right) - \log \frac{S^*}{S_0} \right] + e^{rT} \left[\sum_i w\left(K_{ip}\right) P\left(S, K_{ip}\right) + \sum_i w\left(K_{ic}\right) C\left(S, K_{ic}\right) \right]. \tag{53}$$

The cost of the variance swap is the result of K_{var}, that is $K_{\text{var}} = \left(20.467\%\right)^2$. And then the fair delivery price of the volatility swap is $K_{vol} = 20.467\%$.

After introducing the example in Demeterfi, Derman, Kamal, and Zou (1999), we now use our method to calculate the same example in order to compare the result and see whether our method is available.

Again, we assume the initial stock index S_0 is 100, the strike price K is 100, the continuously compounded annual risk-less interest rate r is 5%, the dividend yield q is zero, and the maturity of the variance swap is 3 months $\left(T = 0.25\right)$, and the at-the-money implied volatility is 20% with a skew such that when the strike price decreases for 5 points, the implied volatility will increase 1 volatility point. To keep our example simple, we choose the state space of our implied trinomial tree to coincide with the CRR type trinomial tree which has 4period. That is, the number of period to the state space n is four. Use the formulas we have explained in the chapter 4 to solve for the transition probabilities and implied variances. The results are given in Figure 6.

After obtaining the total probabilities and implied variances by implied trinomial tree, we use them to calculate the expected implied variance for each time period.

$$
\begin{aligned}
E\left(\sigma_0^2\right) &= p_{0,0}\times\sigma_{0,0}^2 = 1\times 0.0398 \\
E\left(\sigma_1^2\right) &= p_{1,0}\times\sigma_{1,0}^2 + p_{1,1}\times\sigma_{1,1}^2 + p_{1,2}\times\sigma_{1,2}^2 \\
&= 0.2369\times 0.0561 + 0.4996\times 0.0398 + 0.2634\times 0.0340 \\
&= 0.0421 \\
E\left(\sigma_2^2\right) &= p_{2,0}\times\sigma_{2,0}^2 + p_{2,1}\times\sigma_{2,1}^2 + p_{2,2}\times\sigma_{2,2}^2 + p_{2,3}\times\sigma_{2,3}^2 + p_{2,4}\times\sigma_{2,4}^2 \\
&= 0.0814\times 0.0836 + 0.1879\times 0.0552 + 0.3879\times 0.0381 + 0.2827\times 0.0323 \\
&\quad + 0.06\times 0.0251 \\
&= 0.0426 \\
E\left(\sigma_3^2\right) &= p_{3,0}\times\sigma_{3,0}^2 + p_{3,1}\times\sigma_{3,1}^2 + p_{3,2}\times\sigma_{3,2}^2 + p_{3,3}\times\sigma_{3,3}^2 + p_{3,4}\times\sigma_{3,4}^2 + p_{3,5}\times\sigma_{3,5}^2 + p_{3,6}\times\sigma_{3,6}^2 \\
&= 0.0426\times 0.0838 + 0.059\times 0.0888 + 0.1882\times 0.0506 + 0.3224\times 0.038 \\
&\quad + 0.2746\times 0.0315 + 0.1028\times 0.0242 + 0.0104\times 0.0415 \\
&= 0.0421
\end{aligned}
$$

Next, calculate fair delivery price of the variance swap contract by Eq. (44)

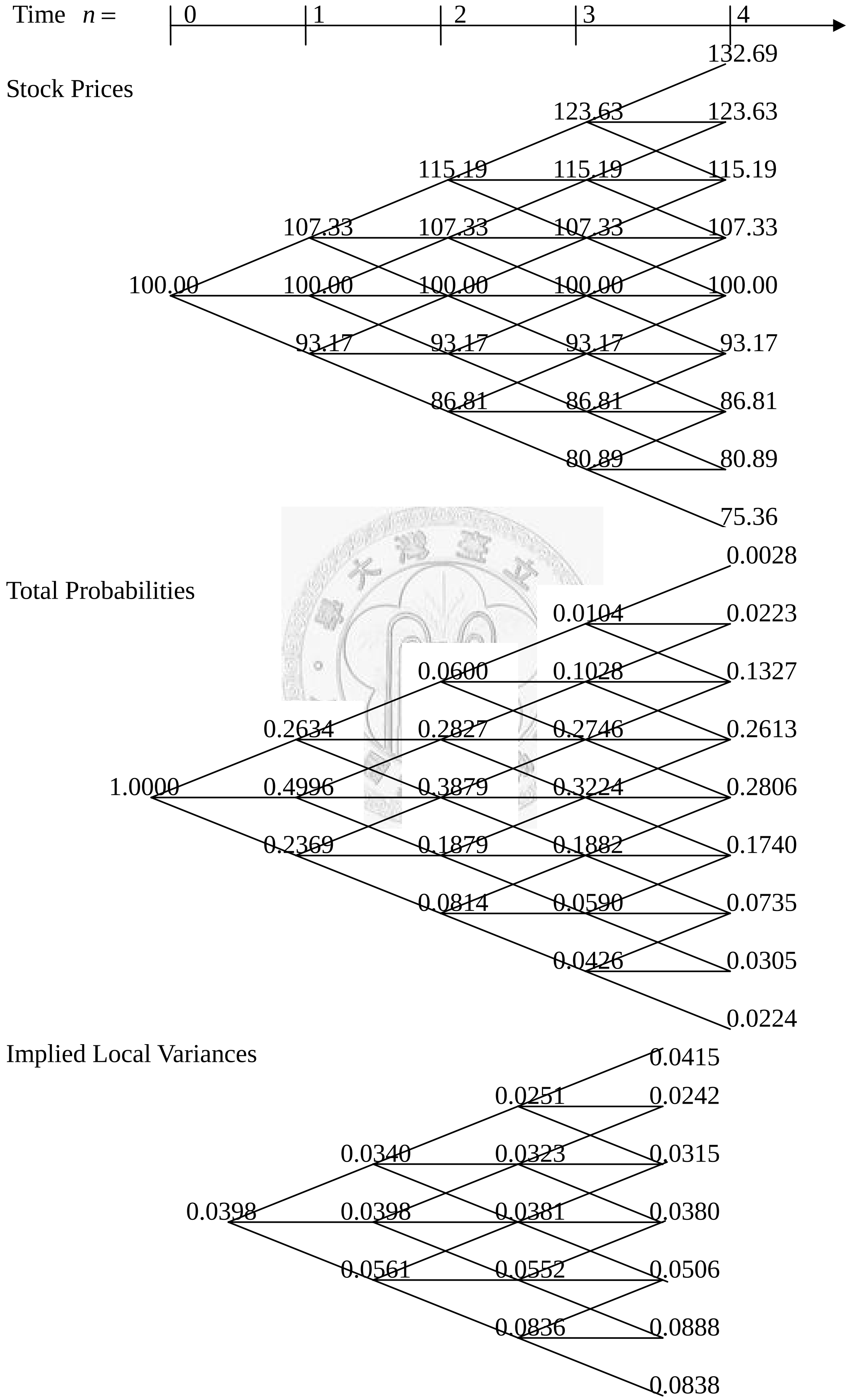
$$K_{\text{var}} = \frac{\Delta t}{T}\sum_{i=0}^{n} E\left(\sigma_i^2\right) = \frac{0.0625}{0.25}\sum_{i=0}^{4} E\left(\sigma_t^2\right) = \frac{0.0625}{0.25}\left(0.0398 + 0.0421 + 0.0426 + 0.0421\right) = 0.0417$$

Finally, take the square root of the variance we find out in order to get the fair delivery price of the volatility swap contract by Eq. (45)

$$K_{vol} = \sqrt{\frac{\Delta t}{T}\sum_{t=0}^{T} E\left(\sigma_t^2\right)} = \sqrt{0.0417} = 0.2041 \quad .$$

Therefore, $K_{\text{var}} = \left(20.41\%\right)^2$ and $K_{\text{v}ol} = 20.41\%$.

Figure 6. The Result of the Example.



Compare results of the general pricing method and our pricing method which use the implied trinomial tree. The results are given by the Table 1. Using our pricing method can arrive at a result a little smaller than the result of the general pricing method. When the number of period becomes large, the result will converge. We have already explained that the fair delivery price may exceed the realized value by using general pricing method. In conclusion, our pricing method is available and more straightforward than the general pricing method.

Table 1. The Assumed Data and the Result of Two Pricing Methods.

Basic Assumption		Pricing by Replication		Pricing by Implied Trinomial Tree		
S_0	100	K_{var}	K_{vol}	n	K_{var}	K_{vol}
K	100	0.0419	0.2047	4	0.0417	0.2041
T	0.25			10	0.0414	0.2034
r	5%			50	0.0408	0.2021
q	0%			100	0.0407	0.2017
σ	20%					
skew	0.2%					

Chapter 5
Strategies and Applications

There are several ways in which volatility and variance swaps can be used. If a trader expected volatility to increase, he could long a volatility or variance swap to take a position on that view, or if his portfolio is naturally short vega, he could use the swap to hedge the loss in option portfolio. In fact, a swap would provide a cleaner vega hedge than offsetting options portfolios.

For example, suppose a trader, whose option portfolio is short volatility with vega position of -\$55,000, suspects an impending market correction and simultaneous increase in volatility sometime over the next six months. Suppose further that six-month implied volatility is 25% and that the trader predicts that volatility will increase by 35 percentage points (from 25% to 60%) over this time frame. If this prediction comes to fruition, the option portfolio could stand to lose \$1,925,000 (\$55,000 × 35) in market value as a result of changes in volatility alone.

To hedge this volatility exposure, the trader could purchase options to neutralized vega. However, the problem with this strategy is that as soon as the underlying index moves, the position is no longer perfectly hedged. In other words, the trader would need to readjust the hedged position every time the market moves. This could be troublesome and expensive in a volatile market.

Rather than hedging with offsetting options, the trader could enter into a long position on volatility or variance swap. Using the volatility swap rate of 25%, a swap having notional value of \$55,000 would hedge the option portfolio. If the actual volatility during this period was 60% as the trader predicted, he would receive \$1,925,000 (\$55,000 × (60% − 25%) × 100) at the end of the swap contract, which could just offset the loss that would have been incurred on the option portfolio. Even if the volatility only rises to *x*% which is smaller than 60%, the trader will receive the amount of (\$55,000 × (*x*% − 25%) × 100) which is smaller than \$1,925,000. In other words, the trader may not offset all his loss. But he will not lose as much if he has not hedged.

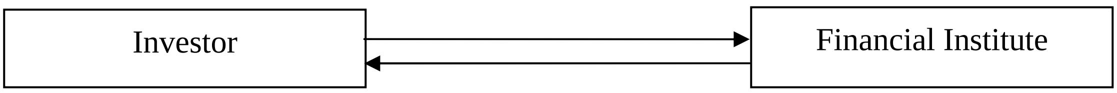
There are other strategies that can be effective as well. Volatility and variance swaps may be used to execute stock index spread trading strategies. In these strategies, a short position of volatility or variance swap on an equity index is hedged by a long position of volatility or variance swap on a different index. This spread trade has a payoff based on the difference between the realized volatilities or variances in these two indices.

For example, suppose a trader shorted a volatility swap on the S&P 500 with a swap rate of 20%, and purchased a volatility swap on the NASDAQ 100 with a swap rate of 25% (see Figure 7). With this strategy, four conditions may happen in the future. The trader is betting that the realized volatility of NASDAQ 100 will exceed the realized volatility of S&P 500 by more than 5 volatility points. The analysis for every possible condition follows.

Figure 7. The Cash Flows of Selling a Volatility Swap on S&P 500 and Buying a Volatility Swap on NASDAQ 100.

Short Volatility Swap on S&P 500

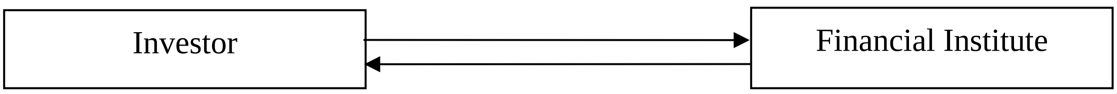
If $\sigma_{S\&P500} > 20\%$, $N \times (\sigma_{S\&P500} - 20\%) \times 100$.



If $\sigma_{S\&P500} < 20\%$, $N \times (20\% - \sigma_{S\&P500}) \times 100$.

Long Volatility Swap on NASDAQ 100

If $\sigma_{NASDAQ100} < 25\%$, $N \times (25\% - \sigma_{NASDAQ100}) \times 100$.



N is the notional amount of the contract.

Condition 1. $\sigma_{S\&P500} > 20\%$ and $\sigma_{NASDAQ100} > 25\%$

When $\sigma_{S\&P500} > 20\%$, the trader who sells a volatility swap on S&P 500 has to pay $N \times (\sigma_{S\&P500} - 20\%) \times 100$ to the financial institute. That is, he will have a cash outflow equal to $N \times (\sigma_{S\&P500} - 20\%) \times 100$. And when $\sigma_{NASDAQ100} > 25\%$, the trader who buys a volatility swap on NASDAQ 100 will receive $N \times (\sigma_{NASDAQ100} - 25\%) \times 100$ from the financial institute. That is, he will have a cash inflow equal to $(\sigma_{NASDAQ100} - 25\%)$. Therefore, if the trader wants to have profit, the volatility of NASDAQ 100 shall exceed the volatility of S&P 500.

Condition 2. $\sigma_{S\&P500} > 20\%$ and $\sigma_{NASDAQ100} < 25\%$

When $\sigma_{S\&P500} > 20\%$, the trader who sells a volatility swap on S&P 500 has to pay $N \times (\sigma_{S\&P500} - 20\%) \times 100$ to the financial institute. That is, he will have a cash outflow equal to $N \times (\sigma_{S\&P500} - 20\%) \times 100$. And when $\sigma_{NASDAQ100} < 25\%$, the trader who buys a volatility swap on NASDAQ 100 has to pay $N \times (25\% - \sigma_{NASDAQ100}) \times 100$ to the financial institute. That is, he will have a cash outflow equal to $N \times (25\% - \sigma_{NASDAQ100}) \times 100$. Therefore, if the trader wants to have profit, the volatility of NASDAQ 100 shall exceed the volatility of S&P 500.

Condition 3. $\sigma_{S\&P500} < 20\%$ and $\sigma_{NASDAQ100} > 25\%$

When $\sigma_{S\&P500} < 20\%$, the trader who sells a volatility swap on S&P 500 will receive $N \times (20\% - \sigma_{S\&P500}) \times 100$ from the financial institute. That is, he will have a cash inflow equal to $N \times (20\% - \sigma_{S\&P500}) \times 100$. And when $\sigma_{NASDAQ100} > 25\%$, the trader who buys a volatility swap on NASDAQ 100 will receive $N \times (\sigma_{NASDAQ100} - 25\%) \times 100$ from the financial institute. That is, he will have a cash inflow equal to $N \times (\sigma_{NASDAQ100} - 25\%) \times 100$. Therefore, if the trader wants to have profit, the volatility of NASDAQ 100 shall exceed the volatility of S&P 500.

Condition 4. $\sigma_{S\&P500} < 20\%$ and $\sigma_{NASDAQ100} < 25\%$

When $\sigma_{S\&P500} < 20\%$, the trader who sells a volatility swap on S&P 500 will receive $N \times (20\% - \sigma_{S\&P500}) \times 100$ from the financial institute. That is, he will have a cash inflow equal to $N \times (20\% - \sigma_{S\&P500}) \times 100$. And when $\sigma_{NASDAQ100} < 25\%$, the trader who buys a volatility swap on NASDAQ 100 has to pay $N \times (25\% - \sigma_{NASDAQ100}) \times 100$ to the financial institute. That is, he will have a cash outflow equal to $N \times (25\% - \sigma_{NASDAQ100}) \times 100$. Therefore, if the trader wants to have profit, the volatility of NASDAQ 100 shall exceed the volatility of S&P 500.

In summary, the trader will benefit when the volatility of NASDAQ 100 exceeds the volatility of S&P 500 by more than 5 volatility points.

Chapter 6
Conclusions

In general, pricing volatility and variance swaps uses the method of replication. In this thesis, we use a methodology that is more direct and intuitive in pricing volatility and variance swaps. And the methodology we choose is the implied tree. We try to use the implied tree to calculate implied local volatilities and variances, and then use these implied local volatilities and variances to price volatility and variance swaps. After using the implied tree to price, we also compare the result of this method to the general pricing method. We find that using this method can obtain the values of volatility and variance swaps similar to the general method.

For further research, we can try to use the modified implied tree to price volatility and variance swaps because the implied tree model we used contains lots of assumptions and includes something that is artificial when we have the negative probabilities in the tree. So, there some inaccuracies remain in our pricing method.

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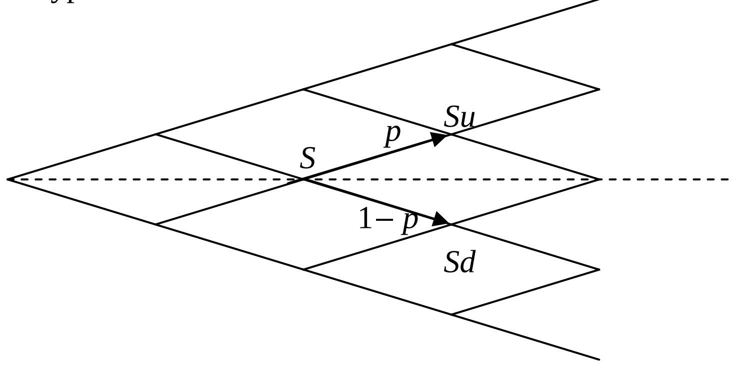
Appendix

A. Constructing Binomial Trees

This appendix provides the constructing method for the constant volatility binomial tree which is first proposed by Cox, Ross, and Rubinstein (1979).

Consider the evaluation of an option on a stock without dividends. Assume that the initial stock price is S_0. Divide the life of the option into time intervals of length Δt. Assume there are two possible states that the stock price at each node may move to when time goes to the next period. For example, the stock price will move from S to Su or Sd. See Figure 8. In general, $u > 1$ and $d < 1$. When the stock price moves from S to Su, it is an up movement. When the stock price moves from S to Sd, it is a down movement. The probability of an up movement will be denoted by p. The probability of a down movement equals $1 - p$.

Figure 8. CRR-Type Binomial Tree.



The risk-neutral valuation principle states that an option can be valued on the assumption that the world is risk neutral. This means that for valuation we can assume that the expected return from all traded securities is the risk-free interest rate, and future cash flows can be valued by discounting their expected values at the risk-free interest rate r. Hence the expected value of the stock price at the end of a time interval of length Δt is $Se^{r\Delta t}$, where S is the stock price at the beginning of the time interval, i.e.,

$$Se^{r\Delta t} = pSu + (1-p)Sd \quad . \tag{A. 1}$$

Eliminate S from both side of this equation to obtain

$$e^{r\Delta t} = pu + (1-p)d \quad . \tag{A. 2}$$

This is the first condition to solve for p, u, and d.

Assume that the stochastic process for stock price which has constant volatility is as follows:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \quad , \tag{A. 3}$$

where μ is the drift term and σ is the volatility. This implies that the variance of the percentage change in the stock price in a small time interval Δt is $\sigma^2 \Delta t$. The variance for a variable X is defined as $E\left(X^2\right) - \left[E\left(X\right)\right]^2$. Then it follows that

$$pu^2 + (1-p)d^2 - \left[pu + (1-p)d\right]^2 = \sigma^2 \Delta t \quad . \tag{A. 4}$$

Substituting the up probability p reduces this equation to

$$e^{r\Delta t}\left(u+d\right)-ud-e^{2r\Delta t}=\sigma^2\Delta t \quad . \qquad \text{(A. 5)}$$

This is the second condition to solve for p, u, and d.

The third condition to solve for p, u, and d by Cox, Ross, and Rubinstein is the assumption that

$$ud=1 \quad . \qquad \text{(A. 6)}$$

Therefore, we obtain

$$p = \frac{e^{r\Delta t}-d}{u-d} \qquad \text{(A. 7)}$$

$$u = e^{\sigma\sqrt{\Delta t}} \qquad \text{(A. 8)}$$

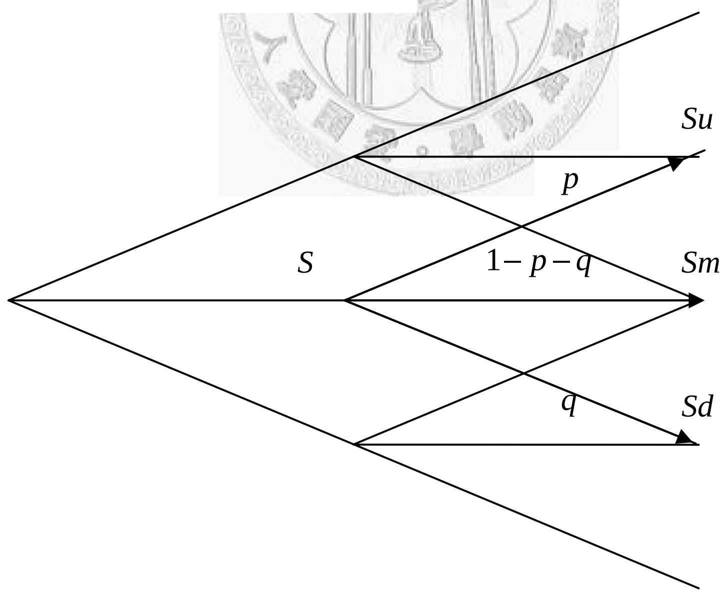
$$d = e^{-\sigma\sqrt{\Delta t}} \quad . \qquad \text{(A. 9)}$$

B. Constructing Trinomial Trees

We have more freedom when constructing constant volatility trinomial tree. In constructing trinomial tree, we can view two steps of binomial tree in combination as a single step of a trinomial tree. The CRR type trinomial tree is constructed analogously to the binomial tree.

Assume there are three possible states that the stock price at each node may move to when time goes to the next period. For example, the stock price will move from S to Su, Sm or Sd. See Figure 9. In general, $u > 1$, $m = 1$, and $d < 1$. When the stock price moves from S to Su, it is an up movement. When the stock price moves from S to Sm, it stays at S and is a middle movement. When the stock price moves from S to Sd, it is a down movement. The probability of an up movement will be denoted by p, the probability of a down movement will be denoted by q, and the probability of a middle movement equals $1 - p - q$.

Figure 9. CRR-Type Trinomial Tree.



As in constructing the binomial tree, we use the risk-neutral valuation principle. So we have that the expected value of the stock price at the end of a time period is

$$Se^{r\Delta t} = pSu + (1-p-q)Sm + qSd \quad . \tag{B. 1}$$

Eliminate S from both side of this equation,

$$e^{r\Delta t} = pu + (1-p-q) + qd \quad . \tag{B. 2}$$

Because we assume that in constructing trinomial tree, we view two steps of binomial tree in combination as a single step of a trinomial tree. Therefore,

$$u = e^{\sigma\sqrt{2\Delta t}} \tag{B. 3}$$

$$d = e^{-\sigma\sqrt{2\Delta t}} \quad . \tag{B. 4}$$

Also similar to constructing the binomial tree, we obtain an equation from calculating the variance. That is,

$$pu^2 + (1-p-q) + qd^2 - \left[pu + (1-p-q) + qd \right]^2 = \sigma^2 \Delta t \quad . \tag{B. 5}$$

Substituting u and d into equations from expected value and variance, we can obtain the up and down probabilities

$$p = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2 \tag{B. 6}$$

$$q = \left(\frac{e^{-\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2 \quad . \tag{B. 7}$$