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隱含波動樹評價選擇權另一方法

An Alternative Method of Options Pricing by Implied Trees

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摘要

本文提出了一個固定機率-隨機波動度的隱含波動二元樹的建構方法。此方法改善了先前其他學者曾提出方法的缺點。相較於 Derman-Kani 隱含波動二元樹與 Li 隱含波動二元樹，以此方法建構隱含波動樹時，具有相當的穩定性。在 Derman-Kani 隱含波動二元樹中有不良機率的問題，亦即在二元樹建構的同時，會出現機率大於 1 或小於 0 的狀況；在 Li 隱含波動二元樹中，雖改良了不良機率發生的情形，但當隱含波動微笑曲線陡峭時，在建構樹的過程中，股價仍會發生違反無套利原則的狀況。然而，本文所提出的新方法，不僅改善了上述二者的缺點，在二元樹的建構概念上相當的簡單易懂，選擇權評價的結果也相當穩定。

關鍵字: 波動度微笑曲線、波動度面、隱含波動度樹、二元樹

Abstract

This thesis proposes a constant probability-stochastic volatility implied binomial tree. Our method improves upon some weaknesses of previous works. Compared with the Derman-Kani tree (1994) and the Li tree (2000), our method is considerably more stable. In our method, neither the invalid transition probability problem occurs, like in the Derman-Kani tree, nor the results of option pricing diverge when the slope of volatility with respect to the strike price is steep, as in the Li tree. Incorporating the known local volatility function, our method constructs the implied binomial tree directly by forward induction. The option value is calculated from the stock prices in the terminal nodes of the tree backward. As a whole, for the proposed constant probability-stochastic volatility implied binomial tree, its construction is direct, and its implementation is straightforward.

Keywords: volatility smile, volatility surface, implied tree, binomial tree

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Chapter 1
Introduction

1.1 Introduction

Since the market crash in October 19th, 1987, numbers of studies have indicated the volatility smile phenomenon in which at-the-money (ATM) options tend to have lower implied volatilities than both in-the-money (ITM) and out-of-the-money (OTM) ones. Several researchers have also examined the existence in various markets, and the patterns exhibit uniquely. Stock options typically have smirk with negative slope with respect to the strike price (see Corrado and Su (1997)), currency options usually show a smile (see Hull (2007)), and commodity options represent a smirk with positive slope with respect to the strike price (see Melick and Thomas (1997)).

Financial economists provide several explanations to the occurrence of volatility smiles, such as Barndorff-Nielsen and Shepherd (1999), London (2004), Ait-Sahalia and Lo (2000), Jackwerth (2000), Bates (2000), and Bollen and Whaley (2004). The leverage effect, the correlation effect, the investor wealth effect, the risk effect, the liquidity effect and the demand-supply of options are all possible explanations (we provide the details in Chapter 2). All of these reasons are sufficient to shape the probability distribution of asset returns, and consequently lead to a common conclusion. The pattern of implied volatility depends on how the return distribution behaves.

On the other hand, the failure of the Black-Scholes (1976) option pricing model is thought of as the structural cause of volatility smile. Its inability to describe the structure of market option prices arises from the model's assumptions of constant volatility and normal distribution of asset returns. In reality, the implied volatilities differ across strike prices and terms to expiration and the probability distribution of asset returns is empirically fat-tailed and skewed rather than normal, which is examined by Derman and Kani (1994). This implies that the market prices of OTM puts and calls are higher than their theoretical values. If the volatility is solved inversely by the Black-Scholes formula given all parameters other than volatility known, the volatility implied from the model would therefore be higher than assumed.

Constant volatility, however, is not suitable for pricing options under such a circumstance. Stochastic volatility models and deterministic volatility models are proposed in which varying volatilities are taken into consideration. The former are represented by Hull & White (1987), Dupire (1992), Heston (1993), Stein and Stein (1991) and Nelson (1991), and the latter are suggested by Derman and Kani (1994), Rubinstein (1994) and Li (2000). Among them, the tree-based implied volatility models, one kind of deterministic model, are easy to understand and implement. By incorporating the market prices of options and forwards, Derman and Kani (1994) and Li (2000) derived algorithms to construct implied binominal trees. These two algorithms are regarded as the extensions of the Cox, Ross, and Rubinstein (CRR) (1979) tree and the Jarrow and Rudd (1982) (JR) tree, respectively, by taking the smile effect into account.

1.2 Motivations and Contributions

The main disadvantage of the Derman-Kani tree is the invalid transition probability problem, in which the transition probability may become greater than one or less than zero. In other words, this allows the existence of arbitrage opportunity. Although there are some ways to override these nodes that break the no-arbitrage principle to continue the construction of the tree, the implied information will be lost by this artificial modification.

Li, on the other hand, derives another implied binomial tree by using constant probability approach, that is, to set the transition probabilities between nodes of the tree as 0.5. This algorithm, in effect, solves the problem of invalid transition probability, and the pricing results are quite stable even after numbers of iteration. However, the stock prices on the nodes may still violate the no-arbitrage principle when the slope of the implied volatility with respect to the strike price is steep. When this happens, the results of option pricing will, in effect, diverge as the number of time partitions increases. Under such a circumstance, the construction of the Li tree is not plausible anymore.

In order to cope with the problems exhibited in the Derman-Kani tree and the Li tree, we propose an alternative method to construct the implied binomial tree. The idea is identical to the Li tree, yet the implementation is quite different. Instead of using a mirror constant probability binomial tree with constant volatility corresponding to the implied volatility for pricing options, our method is to grow the stock prices on the nodes within the tree in the next time step directly from the stock prices on the nodes in the previous time step by multiplying by the up-move or down-move parameters in their binomial stock price settings. These parameters are calculated according to a known local volatility, which is a function of strike price and time to maturity.

Compared with the Derman-Kani tree, our method does not have the invalid transition probability problem because the up and down probabilities are set to 0.5. Contrary to the Li tree, our proposed approach could still work even if the slope of the volatility with respect to the strike price is relatively steep.

1.3 Organization of this Thesis

The structure of this thesis is organized as follows. Chapter 2 introduces notions on the implied volatility surface, the local volatility surface, and the strike structure and the term structure of volatility, exploits economic and structural causes of volatility smile, and reviews volatility modeling approaches, the implied trees proposed by Derman and Kani, and Li in particular. Chapter 3 and Chapter 4 explore the Derman-Kani tree and the Li tree respectively and specify their problems in detail. Chapter 5 is our proposed method. Finally, Chapter 6 concludes.

Chapter 2
Literature Review

This chapter concentrates on the introduction of basic notations about the volatility, including 1) the definitions of implied volatility and local volatility, and the relationship between them, 2) the causes of the strike structure of volatility, and 3) volatility modeling methods. Finally, The last part reviews implied tree based option valuation approaches.

2.1 Implied Volatility Surface

The Black-Scholes model, which is one of the most important instruments for options pricing, assumes that volatilities are constant for options on the same underlying asset with the same expiration date and different exercise prices. However, the volatility backed from the Black-Scholes formula, namely implied volatility, changes depending on strike prices and time to maturities. The relationship between the implied volatility of an option and its exercise price is known as the volatility smile or the strike structure of implied volatility; the pattern of implied volatility across time to expiration is called the term structure of volatility. The combination of the strike structure and the term structure, which is plotted in space, generates a volatility surface. In other words, the implied volatility surface defines implied volatility as a function of strike price and time to maturity.

2.2 Local Volatility Surface

Local volatility, on the other hand, is thought of as the market consensus estimate of instantaneous volatility at a certain future market level of strike and future time. Local volatilities corresponding to different future market levels and times together comprise the local volatility surface.

According to Derman and Kani's (1997) explanation, the relationship between local volatilities and implied volatilities in the option world is analogous to the relationship between forward rates and spot rates in the fixed-income world. In general, the variation of local volatilities is greater than of implied volatilities.

2.3 Causes of Strike Structure of Volatility

Previous studies explained the existence of the strike structure of volatility in twofold, the behavioral causes and the structural causes. The former is concentrated on the financial behaviors of investors, and the latter originates from the violations of the Black-Scholes assumptions.

From economic viewpoint, the leverage effect, the correlation effect, the investor wealth effect, the risk effect, the liquidity effect and the demand-supply of options are all possible attributions to the strike structure of implied volatility. First, Campbell and Kyle (1993), Barndorff-Nielsen and Shepherd (1999) indicated the leverage effect that debt-equity ratio rises as stock prices fall. Thus, this leads to a rise in volatility. Second, London (2004) stated the correlation effect. He argued that stock prices are likely to be highly correlated in down markets so that the diversification effect is reduced, and eventually causes volatility to rise. Third, the investor wealth effect and the risk effect are the two sides of one coin, which are proposed by Ait-Sahalia and Lo (2000) and Jackwerth (2000). As the market falls, investors feel poorer and become more risk averse so that any news brings about greater market reactions and trading volume, which drives volatility to rise. As volatility rises, risk premium required by investors increases, which results in a decline in the market price. In addition, the liquidity effect as a source of volatility smile is claimed by Platen and Schweitzer (1998), Pena, Rubio and Serna (1999) and Bates (2000). Due to low liquidity by market makers who take the other side of those trades selling OTM puts and buying OTM calls, the liquidity premium makes the prices of OTM options become higher. As a consequence, these OTM options are priced higher in implied volatilities compared with ATM ones, which are priced lower in implied volatilities. Finally, from the perspective of demand-supply of options, Rubinstein (1994), Bollen and Whaley (2004) suggested that there is strong demand for OTM puts created by portfolio hedgers and there is strong supply of OTM calls by portfolio writers. The demand-supply of options, therefore, makes the OTM options priced higher than ATM ones in implied volatilities.

From the perspective of structural causes, the violations of the Black-Scholes assumptions are the most profound. The Black-Scholes model theoretically postulates that asset prices follow a lognormal diffusion process under the risk-neutral measure with constant volatility. The logical conclusions are 1) historical volatilities of underlying asset obtained from time series data are constant over time; 2) all options of the same underlying asset with different exercise prices and expiration dates have the same implied volatility; and 3) the risk-neutral probability distribution of underlying asset prices (returns) at some future date is lognormal (normal, respectively). However, these logical conclusions contradict what is observed in the market. Indeed, it has been empirically proved that the volatilities vary at different strike prices across time, and the probability distributions of asset returns are leptokurtic with thick tail rather than normal, as demonstrated by Jackwerth and Rubinstein (1996) and Melick and Thomas (1997). Furthermore, Rubinstein (1994) interpreted the fear of market crash as "crash-o-phobia," that market participants have different views on the distribution of the underlying asset from assumed by the Black-Scholes model. The following table illustrates more detailed explanations to this point.

$\ln S_t$	Fat-tailed instead of normal
At-the-money	Under the fat-tailed distribution, the probability that a call is exercised is 0.5. It is the same as in the normal distribution.
Out-of-the-money call	Under the fat-tailed distribution, the probability that a call is exercised is higher than in the normal distribution so that the implied volatility is higher.
Out-of-the-money put	Under the fat-tailed distribution, the probability that a put is exercised is higher than in the normal distribution so that the implied volatility is higher.

Table 2.1 The causes of higher implied volatilities for OTM options.

2.4 Volatility Modeling

The inadequacy of the Black-Scholes model or other models with constant volatility as the basic assumption drives a vast number of studies to take the strike structure and the term structure effects of volatility into account. Stochastic volatility models and deterministic volatility models are two mainstreams of the volatility modeling.

In stochastic volatility models, the volatility is regarded as an unknown factor, which is directly associated with a certain type of diffusion process, such as Brownian motion or mean-reverting process. The researches in this direction are Hull and White (1987), Stein and Stein (1991), Heston (1993), and Nicolato and Venardos (2003). The models toward this direction are featured by increasing complexity or serial dependence; however, they also have their drawbacks. First, Lim and Zhi (2002) suggested that using merely the underlying assets and riskless bonds is inadequate to hedge the volatility risk directly, and the option valuation is in general no longer preference free. Second, Mahieu and Schotman (1998) pointed out the intractability of the likelihood-function of stochastic volatility models. Since the volatility is an unobserved component and the model is non-Gussian, the likelihood function is only available in the form of a multiple integral. But, exact likelihood-oriented methods require simulations and are computer intensive. Third, the option price relies on several additional parameters whose values have to be estimated. The precision of estimation poses another issue.

Deterministic volatility models, on the other hand, hypothesize that the local volatility of the underlying asset's return is a deterministic function of the asset price and time. In this case, the option valuation based on the Black-Scholes partial differential equation remains possible. These models capture the exact characteristics of the strike structure and the term structure of volatility by finding the local volatility function of the underlying asset. This sort of method is feasible in case the following conditions are meant. In the framework of deterministic volatility model, 1) markets are regarded as dynamically complete and 2) options are redundant assets that can be replicated by combinations of other assets. Accordingly, options can be priced by the no-arbitrage principle without restoring to general equilibrium models.

On the whole, there are three categories of deterministic volatility models: 1) the constant elasticity of variance model (CEV) such as Cox and Ross (1976), Jarrow & Rudd (1987); 2) the implied tree approach such as Dupire (1994), Derman and Kani (1994), Rubinstein (1994), and Li (2000); and 3) the kernel approach such as Ait-Sahalia and Lo (2000). The CEV model does not take into account the characteristics of skewness and kurtosis for asset returns and the volatility smile for option prices. In addition, the kernel approach has to construct a non-parametric estimator based on the state-price density, and to perform the Monte Carlo simulation. Yet, the severe finite sample bias, indicated by Pritsker (1997), is the primary shortcoming of the kernel approach. In contrary to the CEV models and the kernel approach, the implied tree based models are relatively simple to implement without posting a structural form for the volatility function. Option valuation is carried out through constructing the implied tree by feeding market observed data.

2.5 Implied Trees

The fundamental concept of the implied tree approaches is to build a binomial or trinomial tree that can fit currently traded derivatives prices whether exactly or in some ways, and the tree can further be adapted to price any other derivatives on the same underlying asset with the same or earlier maturity.

Jackwerth (1999) divided the existing approaches to construct implied trees into three classes. First, implied binomial trees are constructed by only using backward induction. These trees fit either the volatility smile, e.g. Rubinstein (1994), or both the volatility smile and the time dimension of implied volatilities such as Jackwerth (1997). The second class builds the implied binomial trees by using backward and forward induction simultaneously, Derman and Kani (1994), Barle and Cakici (1998), and Brown and Toft (1999), for instance. Additionally, the third class constructs implied trinomial trees with both backward and forward inductions. Dupire (1994), Derma, Kani and Chriss (1996) are examples. The unique feature of the third class is that the complete state space of implied trinomial trees is fixed in advance, and, therefore, only the transition probabilities have to be specified.

On the other hand, concerned with the requirement of extracting risk-neutral distribution of asset returns in the process of building the implied tree, the literature takes two directions In the methods of Dupire (1994), Rubinstein (1994), Jackwerth (1997) and Brown and Toft (1999), the risk-neutral distribution of asset returns have to be extracted in the process of constructing the implied tree. On the contrary, approaches of Derman and Kani (1994), Barle and Cakici (1998), Derma, Kani and Chriss (1996), and Li (2000) construct the implied tree directly by feeding market prices and other required variables.

Li (2000) pointed out some advantages of pricing options by implied trees: 1) It is a preference-free approach. The market price of volatility risk tends need not to be specified. 2) All contingent claims priced based on the model are consistent with the market. This could be used for pricing exotic options and over-the-counter options using standard options as inputs for the implied model. 3) No assumption is made on the form of local volatility function, which is determined to be consistent with the market.

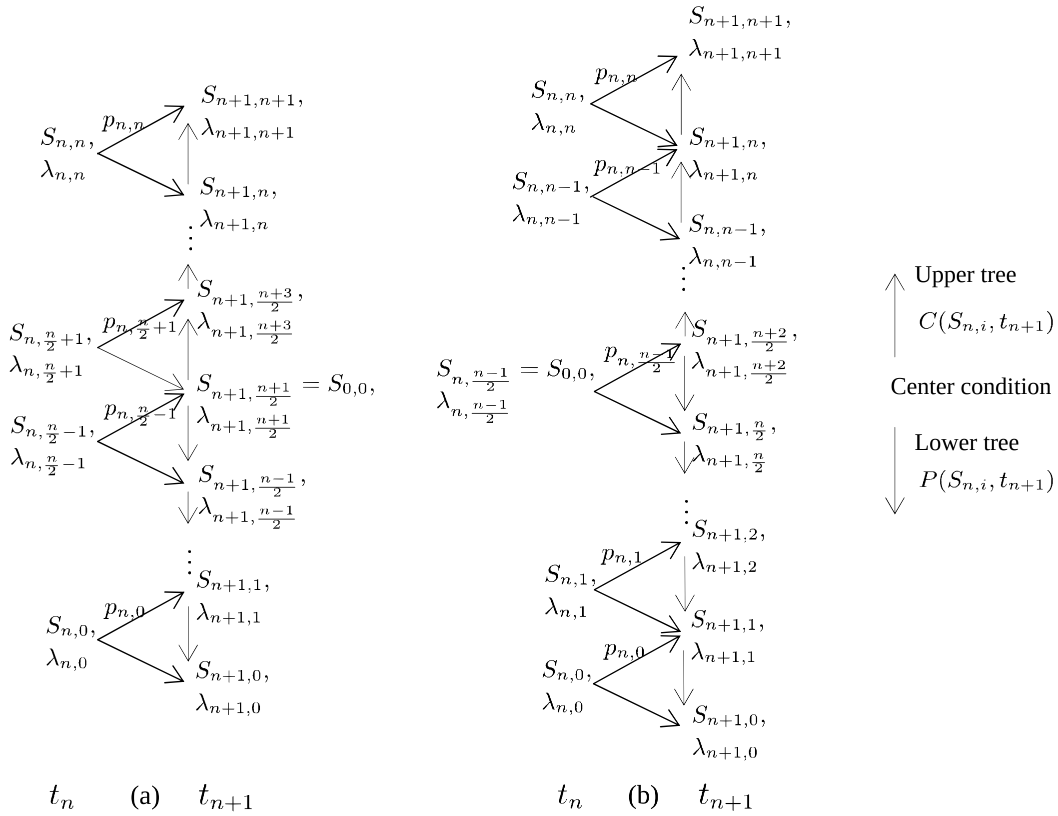
Chapter 3
The Derman-Kani Tree

Derman and Kani were the first authors who proposed the implied binomial tree. Their tree captures the feature of volatility smile by incorporating market observations. However, the primary drawback is that the invalid transition probability problem occurs even with a small number of time partitions when growing the trees. This further leads to the violation of the no-arbitrage principle. Although the nodes with stock prices which violate the no-arbitrage principle could be replaced, such kind of manipulation makes the original idea of catching market information less meaningful. In other words, the market information is lost in the procedure of replacing these stock prices.

This chapter concentrates on the analysis of problems happen in the Derman-Kani tree. First, the Derman-Kani algorithm is described, including the assumptions, settings and the implied tree construction method. Then, the occurrence of invalid transition probabilities between nodes with stock prices in the process of evolving the implied tree is illustrated numerically. Finally, the disadvantage of replacing those stock prices that violate the no-arbitrage principle is indicated specifically.

3.1 The Derman-Kani Algorithm

The Derman-Kani method constructs the implied binomial tree by forward induction initially and then calculates option value by backward induction. The implied binomial tree is built with uniformly spaced time partitions, Δt apart. The root of the tree begins at $t = 0$ with the current spot price, and the stock prices in the nodes in future time steps evolve from the central nodes up and down by using all market prices of calls and puts, respectively. All transition probabilities, Arrow-Debreu prices and the stock prices are also calculated in each time step. After the implied tree is constructed, the option value is calculated from the stock prices in the terminal nodes of the tree by backward induction. Figure 3.1 shows the time step n with n implied binomial tree nodes and their already known stock prices $S_{n,i}$.



Notation:

r: Known forward riskless interest rate

$S_{n,i}$: Stock price at node (n,i) at time step n, node i

$f_{n,i}$: Known forward price at node (n,i) at time step n, node i

$\lambda_{n,i}$: Known Arrow-Debreu price at node (n,i)

$p_{n,i}$: Unknown risk-neutral transition probability from node (n,i) to node $(n+1, i+1)$

Figure 3.1 Inductive procedure in the Derman-Kani tree. The implied binomial tree is built from the central node up and down. The stock price for the central node is derived from stock prices in the previous time step. The upper part of the tree grows from the central node up one by one by using market call prices; and the lower part of the tree evolves from the central node one after one through using market put prices. Inductive procedure is performed for stock prices when *n* is (a) odd and (b) even.

For known stock price $S_{n,i}$ at time step n, there are two possibilities to grow the tree: either going up to $S_{n+1,i+1}$ with probability $p_{n,i}$, or going down to $S_{n+1,i}$ with probability $1-p_{n,i}$. Therefore, there are $2n+3$ parameters of transiting from time step n to time step $n+1$, that is, $n+2$ stock prices, $S_{n+1,i}$, and n transition probabilities, p_i. In addition, using the risk-neutrality of the implied binomial tree, the expected value, one period later, of the stock price at any node, is its known forward price, which could be expressed as

$$f_{n,i} = p_{n,i}S_{n,i+1} + (1-p_{n,i})S_i \tag{3.1}$$

Consequently, there are totally $2n+2$ equations for $2n+3$ parameters. In turn, the transition probability could be stated in terms of forward prices and stock prices as

$$p_{n,i} = \frac{f_{n,i} - S_{n+1,i}}{S_{n+1,i+1} - S_{n+1,i}} \tag{3.2}$$

Let $C(S_{n,i}, t_{n+1})$ and $P(S_{n,i}, t_{n+1})$ denote the known market values of European call and put prices, respectively, with strike $S_{n,i}$ maturing at t_{n+1}. The value in a binomial context that assumes constant volatility, with strike X and maturity t_{n+1} is given as

$$\begin{aligned} &C(K, t_{n+1}) \\ &\quad = e^{-r\Delta t}\textstyle\sum_{j=0}^{n}\{\lambda_{n,j}p_{n,j} + \lambda_{n,j+1}(1-p_{n,j+1})\}\max(S_{n+1,j} - X, 0) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &P(K, t_{n+1}) \\ &\quad = e^{-r\Delta t}\textstyle\sum_{j=0}^{n}\{\lambda_{n,j}p_{n,j} + \lambda_{n,j+1}(1-p_{n,j+1})\}\max(X - S_{n+1,j}, 0) \end{aligned} \tag{3.4}$$

where $\lambda_{n,i}$ is the Arrow-Debreu price, i.e., the price today of a security that pays unity at period n, in state i, and zero elsewhere. Their values are:

$$\lambda_{0,0} = 1$$

$$e^{r\triangle t}\lambda_{n+1,i} = \begin{cases} p_{n,n}\lambda_{n,n} & \text{for } i = n+1 \\ p_{n,i-1}\lambda_{n,i-1} + (1-p_{n,i})\lambda_{n,i} & \text{for } 1 \le i \le n \\ (1-p_{n,0})\lambda_{n,0} & \text{for } i = 0 \end{cases} \tag{3.5}$$

Consider the portion of the tree that extends from the central node upward. Because both forward prices and call prices are known, $S_{n+1,i+1}$ could be solved from Eqs. (3.1) and (3.3), and is expressed in terms of $S_{n+1,i}$ as

$$S_{n+1,i+1} = \frac{S_{n+1,i}[e^{r\Delta t}C(S_{n,i}, t_{n+1}) - \sum_i] - \lambda_{n,i}S_{n,i}(f_{n,i} - S_{n+1,i})}{[e^{r\Delta t}C(S_{n,i}, t_{n,i}) - \sum_i] - \lambda_{n,i}(f_{n,i} - S_{n+1,i})} \tag{3.6}$$

where

$$\textstyle\sum = \sum_{j=i+1}^{n}\lambda_{n,j}(f_{n,j} - S_{n,i}) \tag{3.7}$$

In the same way, by solving Eqs. (3.1) and (3.4), the lower part of the tree could be derived iteratively from the central node downward:

$$S_{n+1,i} = \frac{S_{n+1,i+1}[e^{r\Delta t}P(S_{n,i}, t_{n+1}) - \sum_i] + \lambda_{n,i}S_{n,i}(f_{n,i} - S_{n+1,i+1})}{[e^{r\Delta t}P(S_{n,i}, t_{n+1}) - \sum_i] + \lambda_{n,i}(f_{n,i} - S_{n+1,i+1})} \tag{3.8}$$

where

$$\sum = \sum_{j=0}^{i-1} \lambda_{n,j}(S_{n,i} - f_{n,j}) \tag{3.9}$$

The final degree of freedom is assigned to the central condition. If the number of nodes at a given time step is odd, choose the central node's stock price to be equal to the spot price today. If the number is even, then make the average of the logarithm of the two central stock prices in the nodes equal the logarithm of today's spot price. Accordingly, for $i = \frac{n}{2}$, $\ln S_{0,0} = \frac{1}{2}(\ln S_{n+1,\frac{n}{2}+1} + \ln S_{n+1,\frac{n}{2}})$. The central condition implies that $S_{n+1,\frac{n}{2}} = S_{n,i}^2 / S_{n+1,\frac{n}{2}+1}$, where $S_{n,i} = S_{0,0}$. Using this condition in Eq. (3.6), the upper stock price of the two central nodes for even level is:

$$S_{n+1,\frac{n}{2}+1} = \frac{S_{0,0}[e^{r\Delta t}C(S_{0,0}, t_{n+1}) + \lambda_{n,i}S_{0,0} - \sum_i]}{\lambda_{n,i}f_{n,i} - e^{r\Delta t}C(S_{0,0}, t_{n+1}) + \sum_i} \tag{3.10}$$

After the initial node with stock price is calculated, all the stock prices in the nodes above for it $\frac{n}{2} + 2 \le i \le n + 1$ can be calculated one by one using Eq. (3.6) and all the stock prices in the nodes below for it $0 \le i \le \frac{n}{2} + 1$ can be calculated iteratively by using Eq. (3.7).

3.2 Invalid Transition Probabilities

Although the Derman-Kani tree is valid in principle, it is not robust in practice. Unfortunately, the transition probability is pretty sensitive to stock prices. This is easily observed from Eq. (3.2). Both of the numerator and the denominator will become very small as the number of time partitions increases, i.e., when Δt is small. As a result, this leads to large swings in transition probabilities, which is likely to go beyond 1 or below 0. In theory, the transition probabilities, $p_{n,i}$, at any node in the implied tree must lie between 0 and 1. If $p_{n,i} > 1$, the stock price $S_{n+1,i+1}$ will fall below the forward price $f_{n,i}$. Similarly, if $p_{n,i} < 0$, the stock price $S_{n+1,i}$ will be located above the forward price $f_{n,i}$. Either of these conditions causes riskless arbitrage.

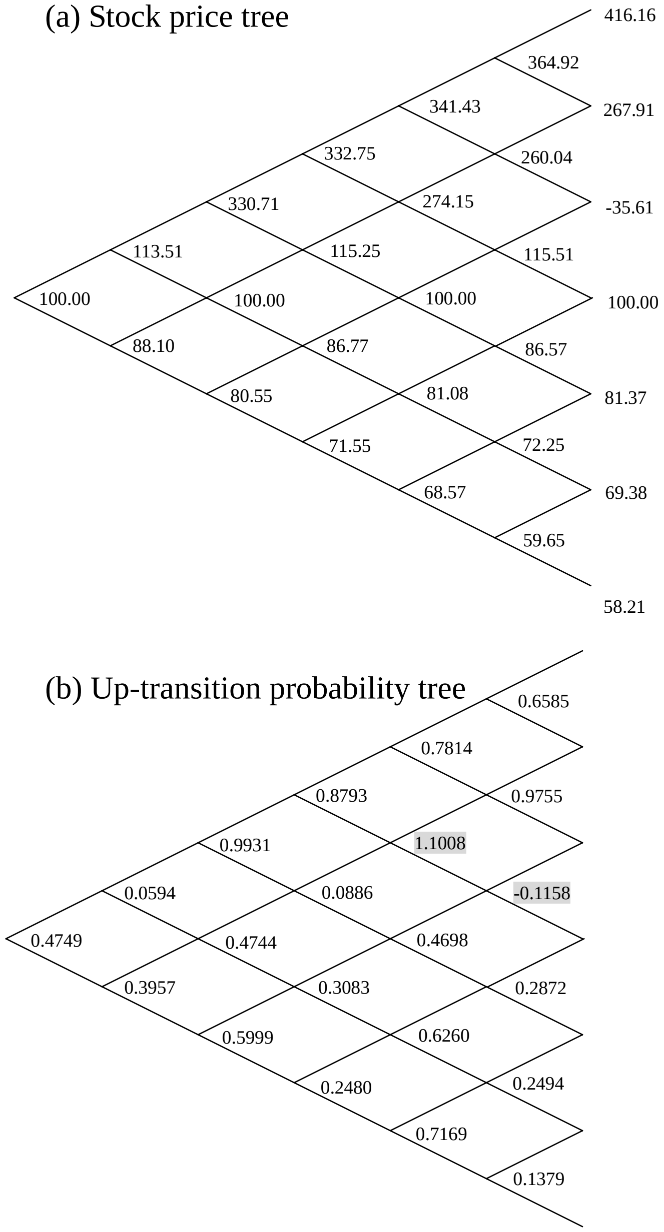


Figure 3.2 Derman-Kani implied binomial tree for stock prices and up-transition probabilities. The implied volatility function adapts Eq. (4.13). The parameters *a*, *b* and *c* are 0.3, -3 and 0.25, respectively. Therefore, the slope of implied volatility with respect to strike is negative, i.e. it is a volatility skew. In addition, the values used are: $S_0 = 100, \quad r = 0.2,\ q = 0,\ t = 0.5$. Figure (a) and (b) are the Derman-Kani implied binomial tree for stock prices and up-transition probability, respectively. The words in shade are the invalid transition probabilities.

The invalid transition probability problem exists even if with only few time partitions of construction. Figure 3.2 illustrates the occurrence of invalid transition probability problem in the Derman-Kani tree for a certain implied volatility smile with time horizon of 0.5 and six time partitions.

3.3 Replacement of Nodes that Violate the No-Arbitrage Principle

As the illustration in the previous section, the transition probabilities between nodes of the tree may become greater than 1 or less than 0 beyond a certain time step. This could happen even within just few time partitions of construction if the slope of the implied volatility with respect to the strike price is steep. To avoid the occurrence of invalid transition probability, each newly determined stock price is required to be within the range:

$$f_{n-1,i} \leq S_{n,i+1} \leq f_{n-1,i+1} \tag{3.11}$$

As a mater of fact, invalid transition probability indicates an arbitrage opportunity. Appendix A indeed proves 1) one invalid transition probability indicates an arbitrage opportunity, and 2) if the stock price is within the range as indicated in Eq. (3.11), it is equivalent to ensuring valid transition probabilities.

If the transition probabilities are invalid, the stock prices which violate the no-arbitrage principle have to be overridden in order to continue the building of the tree. One idea of ensuring the transition probabilities to be valid, i.e., ensuring these stock prices not to violate the no-arbitrage principle, is to maintain the logarithm spacing between adjacent nodes equal to that of previous level. Yet, stock prices may still not be in the range as indicated in the inequality (3.11) after replacement. To avoid it, a choice of for $S_{n,i+1}$ at any point between $f_{n-1,i}$ and $f_{n-1,i+1}$ is sufficient. Simply choose the average of the two forwards.

However, there is one weakness in the procedure of replacing these stock prices that violate the no-arbitrage principle. Once the stock prices are replaced, the market information is lost. The tree with manual manipulation catches less market implied information than desired. Due to modification of stock prices, the final results of some option pricing do not match the option prices well.

Chapter 4
Problems with the Li Tree

In order to deal with the invalid transition probability problem, Li proposed another algorithm for constructing the implied binomial tree, in which the transition probability is constant. As the author demonstrated, for a quite general process with known local volatilities, the constant probability method indeed gives rise to a recombining binomial tree. Compared with the Derman-Kani tree, the Li tree is, in general, considerably more stable and also easier to use. Nevertheless, the stock prices in the nodes may still violate the no-arbitrage principle beyond some steps of iteration and the option pricing result is unstable, or even diverges as the number of time partitions increases when the slope of the implied volatility with respect to the strike price is steep. As a result, the construction of the Li tree is not plausible anymore in this situation.

4.1 The Li Algorithm

First, assume the stock price movement follows the following stochastic differential equation,

$$dS_t/S_t = rdt + \sigma(S_t, t)dZ_t \tag{4.1}$$

where S_t is the stock price at time t, r is the risk-free interest rate, and dZ_t is the Wiener process. Second, assume (1) the local volatility function σ is continuous and non-negative and (2) the local rate r exists. The constant probability tree is constructed as follows:

$$\begin{cases} S_{n+1,i+1} = S_{n,i}[1 + r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] \\ S_{n+1,i} = S_{n,i}[1 + r\Delta t - \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] \\ p_{n,i} = 1/2 \end{cases} \tag{4.2}$$

where $S_{n+1,i+1}$ stands for the $(i+1)^{th}$ stock price at time step $n+1$, moving upward from the i^{th} stock price, $S_{n,i}$, at time step n. $S_{n+1,i}$ represents the i^{th} stock price at time step $n+1$, moving downward from the i^{th} stock price, $S_{n,i}$, at time step n. $p_{n,i}$ is the probability that stock price move upward from $S_{n,i}$ to $S_{n+1,i+1}$.

Since the local volatility is stochastic, the stock prices tend to be different after two time steps of movement in different paths, i.e., the stock price, $S_{n+2,i}^{UD}$, moves upward and than downward is not likely to be the same as the stock price, $S_{n+2,i}^{DU}$, moves downward and than upward. In other words, the tree will explode.

By applying the Taylor's expansion for small Δt, $S_{n+2,i}^{UD}$ and $S_{n+2,i}^{DU}$ are expressed in terms of $S_{n,i}$ as (for detailed derivation, please see Section 5.3.)

$$\begin{cases} S_{n+2,i}^{UD} = S_{n,i} + S_{n,i}[r - \frac{1}{2}\sigma(S_{n,i},t_n)^2 - \frac{1}{2}S_{n,i}\sigma(S_{n,i},t_n)\frac{\partial\sigma(S_{n,i},t_n)}{\partial S_{n,i}}]2\Delta t \\ \qquad\qquad + A(S_{n,i},t_n)(\Delta t)^{3/2} + \mathcal{O}((\Delta t)^2) \\ S_{n+2,i}^{DU} = S_{n,i} + S_{n,i}[r - \frac{1}{2}\sigma(S_{n,i},t_n)^2 - \frac{1}{2}S_{n,i}\sigma(S_{n,i},t_n)\frac{\partial\sigma(S_{n,i},t_n)}{\partial S_{n,i}}]2\Delta t \\ \qquad\qquad - A(S_{n,i},t_n)(\Delta t)^{3/2} + \mathcal{O}((\Delta t)^2) \end{cases} \tag{4.3}$$

where

$$A(S_{n,i},t_n) = \\ S_{n,i}[\frac{\partial\sigma(S_{n,i},t_n)}{\partial t} + (r+\sigma^2)S_{n,i}\frac{\partial\sigma(S_{n,i},t_n)}{\partial S_{n,i}} + \frac{1}{2}S_{n,i}^2\sigma^2(S_{n,i},t_n)\frac{\partial^2\sigma(S_{n,i},t_n)}{\partial S_{n,i}}]$$

Theoretically, when $\Delta t \to 0$, the third term proportional to $(\Delta t)^{3/2}$ in Eq. (4.3) is much smaller than the second term, so it can be neglected. In practice, however, the tree will not recombine because errors by the third term tend to accumulate in the process of forward induction. But, if the stock price in time step $n+2$ is taken as $S_{n+2} = \frac{1}{2}(S_{n+2}^{UD} + S_{n+2}^{DU})$, the errors will be canceled out. As a consequence, the recombined tree for time step $n+1$ is constructed as

$$\begin{cases} S_{n+1,i} = \frac{1}{2}\{S_{n,i}[1 + r\Delta t + \sigma(S_{n,i},t_n)\sqrt{\Delta t}] + \\ \qquad\qquad S_{n,i+1}[1 + r\Delta t - \sigma(S_{n,i+1},t_n)\sqrt{\Delta t}]\} \quad \text{for } i \neq 0, n \\ S_{n+1,0} = S_{n,0}[1 + r\Delta t - \sigma(S_{n,0},t_n)\sqrt{\Delta t}] \\ S_{n+1,n} = S_{n,n}[1 + r\Delta t + \sigma(S_{n,n},t_n)\sqrt{\Delta t}] \\ p_{n,i} = \frac{1}{2} \end{cases} \tag{4.4}$$

Although Eq. (4.4) could be used directly to find stock prices, this is not a good strategy in practice. Since the Arrow-Debreu price $\lambda_{n,i}$ becomes very small for a large number of time steps, using Eq. (4.4) directly results in great inaccuracy.

Fortunately, Arrow-Debreu prices are known for the tree with constant probabilities. The Arrow-Debreu price $\lambda_{n,i}$ on i^{th} node at the n^{th} time step is given by

$$\lambda_{n,i} = \frac{1}{2^n}e^{-rn\Delta t}\frac{n!}{i!(n-i)!} \tag{4.5}$$

In addition, the forward price satisfies the following risk-neutral condition:

$$f_{n-1,i} = p_{n-1,i}S_{n,i+1} + (1 - p_{n-1,i})S_{n,i} \tag{4.6}$$

where $f_{n-1,i}$ is the forward price at i^{th} node in time step $n-1$. Therefore, there are n equations for the forward contracts.

Let $C(X, t_n)$ denote the price of the European call option maturing at time t_n with strike price X. It is supposed to be known either by interpolation form option pieces in the market or by the Black-Scholes formula using the interpolated implied volatility smile. Using the tree approach, the call price is expressed as

$$\begin{aligned} &C(K, t_n) \\ &\quad = e^{-r\Delta t} \textstyle\sum_{j=0}^{n} \{\lambda_{n-1,j} p_{n-1,j} + \lambda_{n-1,j+1}(1 - p_{n-1,j+1})\} \max(S_{n,j} - X, 0) \end{aligned} \tag{4.7}$$

To find the stock prices by Eq. (4.7), one can use the Black-Scholes formula for the left-hand side of Eq. (4.7). However, this is not stable because of the small Arrow-Debreu prices problem for large n. Instead, a "mirror" constant probability binomial tree is used, in which constant volatility corresponds to the implied volatility for pricing options. The strike prices are chosen to be equal to the forward prices at the previous time step $n-1$, $f_{n-1,i} = S_{n-1,i} e^{r\Delta t}$ for $i = 0, 1, \ldots, n-1$. There are n independent options. Using the mirror tree for the left-hand side of Eq. (4.7) while the implied tree is used for the right-hand side so that Eq. (4.7) can be recast in the form

$$\begin{aligned} \textstyle\sum_{j=0}^{n} (\tilde{S}_{n,j} - f_{n-1,i})^{+} \lambda_{n,j} = \textstyle\sum_{j=i+1}^{n} (S_{n,j} - f_{n-1,i})^{+} \lambda_{n,j}, \\ i = 0, 1, \ldots, n-1 \end{aligned} \tag{4.8}$$

where $\tilde{S}_{n,j}$ are the stock prices in the mirror tree with constant volatility corresponds to the implied volatility $\sigma(f_{n-1,i}, t_n)$ and $[x]^{+} = \max(x, 0)$. The parameters $\lambda_{n-1,i}$ are the effective Arrow-Debreu prices, which are:

$$\lambda_{n,i} = \begin{cases} p_{n-1,i-1}\lambda_{n-1,i-1} + (1 - p_{n-1,i})\lambda_{n-1,i}, & \text{for } 0 < i < n, \\ p_{n-1,i-1}\lambda_{n-1,i-1}, & \text{for } i = n \\ (1 - p_{n-1,0})\lambda_{n-1,0}, & \text{for } i = 0 \end{cases} \tag{4.9}$$

The common factor, $2^{-(n)} e^{-r(n)\Delta t}$, in $\lambda_{n,i}$ is extremely small for large n, can be canceled on both sides of Eq. (4.8). So, the equation could be re-written iteratively:

$$\begin{aligned} S_{n,n-l} = f_{n-1,n-l-1} & \\ - \textstyle\sum_{j=n-l+1}^{n} (S_{n,j} - \bar{f}_{n-1,n-l-1}) \theta_{j,n-1} & \\ + \textstyle\sum_{j=1}^{n} (\tilde{S}_{n,j} - f_{n-1,n-l-1})^{+} \theta_{j,n-l} & \qquad \text{for } l = 0, 1, \ldots, n \end{aligned} \tag{4.10}$$

where

$$\theta_{j,k} = \frac{k!(n-k)!}{j!(n-j)!} \tag{4.11}$$

Using Eq. (4.10), $S_{n,n}$ is found firstly for $l=0$, then $S_{n,n-1}$ for $l=1$ and all other stock prices sequentially.

To reduce the calculation error, it is better to derive a similar formula using put options for the lower part of the tree. In case of put option, the equation is:

$$\begin{aligned} S_{n,m} = f_{n-1,m} & \\ + \textstyle\sum_{j=0}^{m-1}(f_{n-1,m} - S_{n,j})\theta_{j,m} & \\ - \textstyle\sum_{j=0}^{n}(f_{n-1,m} - \tilde{S}_{n,j})^{+}\theta_{j,m} & \quad \text{for } m = 0, 1, ..., n \end{aligned} \tag{4.12}$$

By applying Eqs. (4.10) and (4.12), the implied binomial tree can be constructed iteratively and, therefore, it is used for option pricing.

4.2 Problem of Stock Prices which Violate the No-Arbitrage Principle

Compared with the Derman-Kani tree and its extensions, the invalid transition probability problem will not occur in the Li tree and the result of option pricing is considerably stable. However, there are still some problems in the Li algorithm. If the slope of implied volatility with respect to the strike price is steep, 1) the stock prices might still violate the no-arbitrage principle in the process of growing the implied tree, and 2) the option pricing results tend to diverge as the number of time partitions increases. The following section demonstrates how they could happen.

Assume the implied volatility behaves as the implied volatility function[1]:

$$f(S_t, t) = c + a\{1 - \tanh[b(S_t - X)/S_0]\} \tag{4.13}$$

where a, b and c are constant. a controls the range of volatility, b adjusts the change rate of volatility around $S_t = X$, and c determines the magnitude of volatility. The shape of volatility smile can be controlled by changing the parameters. Table 4.1 shows how to assign parameters to desired shape of volatility smile.

[1] This is an assumed implied volatility function. The purpose is to stimulate different shapes of volatility smile.

Desired shape of volatility smile	Parameters setting
Constant volatility	Set $a=0$ or $b=0$ and $c=0$
Volatility skew with positive slope with respect to the strike price	Set$a>0$, $b>0$ and $c>0$
Volatility skew with negative slope with respect to the strike price	Set $a>0$, $b<0$ and $c<0$
Volatility smile symmetrically	Set $a>0$, $b>0$ and $c>0$ for $S_t<X$
	Set $a>0$, $b<0$ and $c>0$ for $S_t>X$

Table 4.1 The volatility function, parameters and shape of volatility smile.

Figure 4.1 shows the results of option pricing under different shapes of volatility smile. When the volatility smile is smooth and positively sloped with respect to the strike price, the option pricing results converge as the number of time partitions increases. As illustrated in Panel (a), with smooth slope of implied volatility smile, the call price converges to around 10.30 when the number of time partitions is greater than 22, and the variation of the pricing results becomes smaller and smaller.

In contrast, when the volatility smile is steep and positively sloped with respect to the strike price, the option pricing results diverge as the number of time partitions increases. As demonstrated in panel (b) of Figure 4.2, the call prices are stable when the number of time partitions is equal to or smaller than 11, whereas the pricing results become relatively unstable and diverge when the number of time partitions is greater than 12.

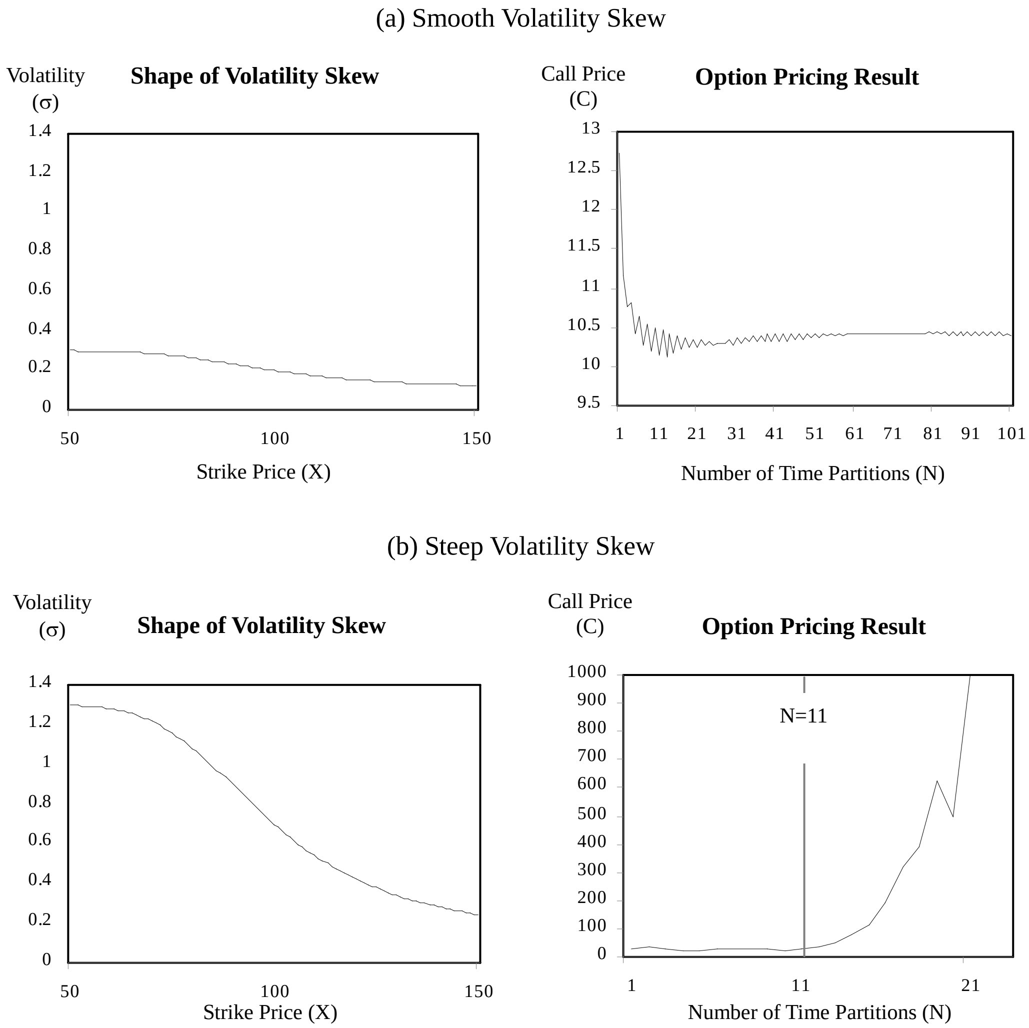


Figure 4.1 The shape of volatility skew and option pricing results by using the Li tree. The option pricing result of the Li tree has a lot to do with the shape of implied volatility. In Panel (a), the implied volatility with respect to the strike price is smooth, the call price is stable as the number of time partitions increases. Whereas, in Panel (b), the implied volatility with respect to the strike price is steep, the call price diverges as the number of time partitions increases. The implied volatility function applies Eq. (4.13). Settings of required parameters are: $a = 0.1$, $b = -0.3$, $c = 0.25$, $S_0 = 100$, $X = 100$, $r = 0.2$, $q = 0$ and $t = 0.5$ for case (a) and $a = 0.6$, $b = -0.3$, $c = 0.25$, $S_0 = 100$, $X = 100$, $r = 0.2$, $q = 0$ and $t = 0.5$ for case (b)

Looking into the stock prices in the nodes within the tree, it is found that the problems that occur in the Derman-Kani tree remains with the Li-tree when the slope of volatility with respect to the strike price is steep. First, stock prices violate the no-arbitrage principle if they are not in the range of the following inequality:[2]

$$f_{n-1,i} \le S_{n,i+1} \le f_{n-1,i+1} \tag{4.14}$$

As illustrated in Figure 4.2, the values in shade are all stock prices that violate the no-arbitrage principle.

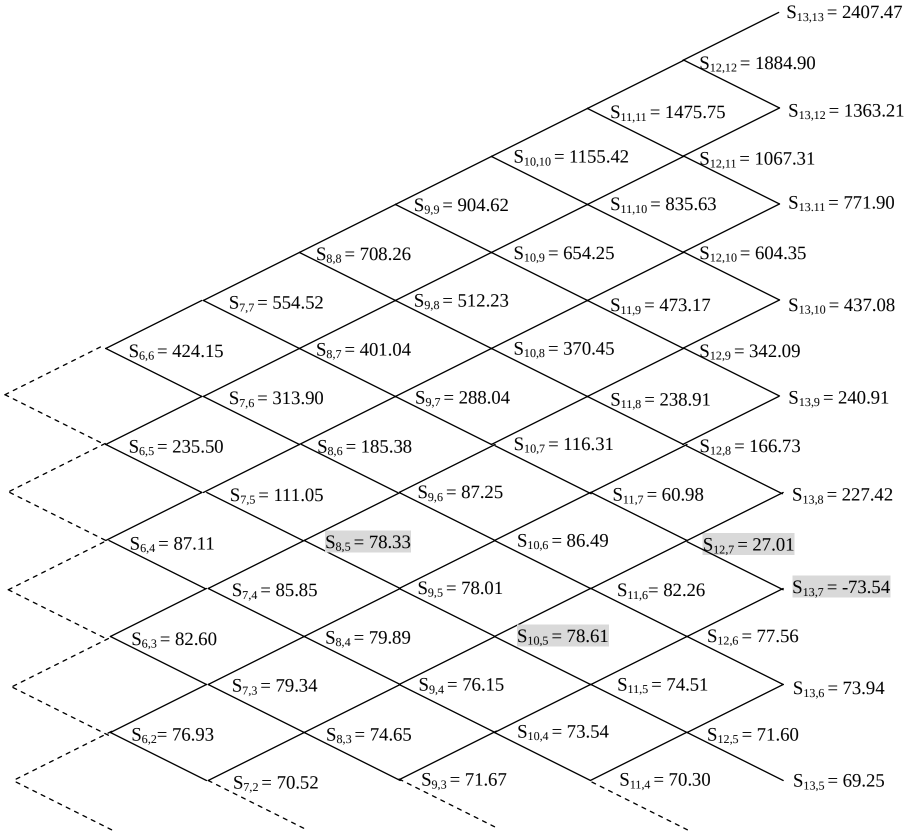


Figure 4.2 Nodes of stock prices in the Li tree. When the slope of implied volatility with respect to strike is steep, the nodes with stock prices in the Li tree may still violate the no-arbitrage principle. For these values in shade are the stock prices that violate the no-arbitrage principle. The volatility function applies Eq. (4.13). Settings for parameters are: $a = 0.6$, $b = -0.3$, $c = 0.25$, $S_0 = 100$, $X = 100$, $r = 0.2$, $q = 0$, $t = 0.5$ and $n = 13$.

Looking back to Eqs. (4.10), the divergence of the option pricing results is likely to be attributed to the term $(\tilde{S}_{n,j} - F_{n-1,n-l-1})^{+}\theta_{j,n-l}$. On the one hand, when the

[2] Proof is provided in Appendix B.

volatility is large, the difference between stock price in the mirror tree and forward price will be large. This difference is huge especially when the slope of implied volatility with respect to the strike price is steep. The forward price calculated from stock price in the previous time step is small in relative to the stock price, $\tilde{S}_{n,j}$, which is obtained from the mirror tree, because the volatility is greater than it was in the previous time step. On the other hand, $\theta_{j,k}$ is big as the number of time partitions, n, is large. For example, suppose it is now calculating the node 12 at time step 13 and $j = 10$, $\theta_{10,12}$, by Eq. (4.11), is therefore 22. As these two components are large, the stock price will be large as well.

In conclusion, although the Li tree is in general more stable than the Derman-Kani tree, and the invalid transition probability problem does not exist in the Li tree, the Li tree still faces some problems when the slope of implied volatility with respect to the strike price is steep. In this case, for one thing, the option pricing results do not converge as the number of time partitions increases. For the other thing, some stock prices in the nodes within the tree violate the no-arbitrage principle.

Chapter 5
An Alternative Method in Constructing Implied Tree

The settings and basic idea of our method are identical to the Li tree, yet the implementation is quite different. Rather than deriving an algorithm by using the mirror trees, which have constant probability and constant volatility, our approach constructs the implied binomial tree directly based on known local volatilities. However, whether the tree will recombine or not under stochastic volatility setting is essential. Indeed, assuming the stock price movement follows the geometric Brownian motion, and the local volatility function is continuous and non-negative, the tree which is constructed by our method will be proved to recombine. Besides, the recombining tree does eventually converge to the desired process.

In our method, because the transition probabilities are set to be constant (in fact, 0.5 for stock prices to move either upward or downward), the invalid transition probability problem will not occur like in the Derman-Kani tree. On the other hand, the stock prices in the nodes are growing within the tree directly from the previous time step by multiplying the up-move or down-move parameters. It is easy to implement.

This Chapter firstly describes the settings for constructing an implied binomial tree, and proves the recombining property. Then, several scenarios are provided to see if the option pricing by applying our method will finally converge.

5.1 Building a Recombining Binomial Tree

Building a recombining tree is important because the nodes of one time step increase linearly with the number of time partitions in a recombining tree. On the other hand, if the tree does not recombine, the nodes of the tree will grow largely to the n^{th} power of 2 in time step n. In other words, the exponential increase of nodes in the non-recombining tree makes the binomial construction impractical as n goes large.

In the tree with constant volatility, the nodes will recombine after several steps of growing. Yet, if the volatility is stochastic, the tree is not likely to recombine. Looking at Eq. (5.1), the stock prices in time step $n+1$ grow from the stock prices in the previous time step, n.

$$\begin{cases} S_{n+1,i+1} = S_{n,i} e^{\sigma(S_{n,i},t_n)\sqrt{\Delta t}} \\ S_{n+1,i} = S_{n,i} e^{\sigma(S_{n,i},t_n)\sqrt{\Delta t}} \\ p_{n,i} = 1/2 \end{cases} \tag{5.1}$$

where $S_{n,i}$ represents the i^{th} stock price in time step n, $S_{n+1,i+1}$ is the stock price moving upward from $S_{n,1}$ and $S_{n+1,i}$ is stock price moving downward from $S_{n,1}$. Applying the Taylor expansion gives rise to the following equation:

$$\begin{cases} S_{n+1,i+1} = S_{n,i}[1 + r\Delta t + \sigma(S_{n,i},t_n)\sqrt{\Delta t}] \\ S_{+1,i} = S_{n,i}[1 + r\Delta t - \sigma(S_{n,i},t_n)\sqrt{\Delta t}] \\ p_{n,i} = 1/2 \end{cases} \tag{5.2}$$

It is obvious that the tree tends not to recombine after some steps of evolving by Eq. (5.2). As illustrated in Figure 5.1, different paths of movement lead to different results. The stock price moves upward first and then downward differs from the stock price moves downward first and then upward. That is, $S^{UD}_{n+2,i}$ is not equal to $S^{DU}_{n+2,i}$ because the parameter $\sigma(S_{n,i},t_n)$ is stochastic at different nodes and different time steps.

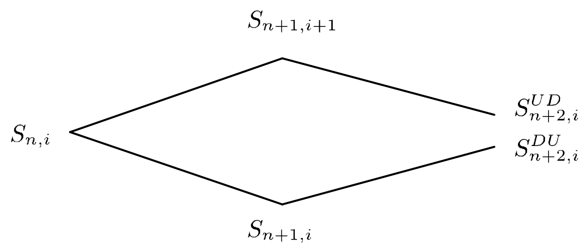


Figure 5.1 Non-recombining under stochastic volatility. $S_{n,i}$ represents the i^{th} stock price in time step n. S^{UD}_{n+2} goes in the path from $S_{n,i}$ upward to $S_{n+1,i+1}$ and than from $S_{n+1,i+1}$ downward to S^{UD}_{n+2}, whereas S^{DU}_{n+2} goes in the path form $S_{n,i}$ downward to $S_{n+1,i}$ and than from $S_{n+1,i}$ upward to S^{DU}_{n+2}.

If the tree does not recombine, there will be a big problem when discounting the stock prices in the terminal nodes since an exponential number of nodes in the non-recombining tree consumes too much time in computing. However, the subsequent section shows that our proposed tree will recombine under some assumptions and settings.

5.2 Assumptions and Settings

Before constructing the recombining tree, this section discusses the conditions under which the implied binomial tree could be constructed by our method.

Assumption 1 *The stock price satisfies the following stochastic differential equation*

$$dS_t = rS_t dt + \sigma(S_t, t)S_t dZt \tag{5.3}$$

where S_t *is the stock price at time* t*, r is the instantaneous interest rate,* $\sigma(S_t, t)$ *is the local volatility function of stock price and time, and* dZ_t *follows the Wiener process with mean of* 0 *and variance of* t*.*

Assumption 2 *The instantaneous interest rate r and the local volatility function* $\sigma(S_t, t)$ *are continuous and non-negative.*

Assumption 3 *A solution to the stochastic integral equation*

$$S_t = S_0 + \int_0^t S_u r du + \int_0^t S_u \sigma(S_{u,}, u) dZ_u \tag{5.4}$$

exists with probability 1 for $0 < t < \infty$ *and is unique in distribution.*

Assumption 4 $\dfrac{\partial\sigma}{\partial t}, \dfrac{\partial\sigma}{\partial S}, \dfrac{\partial^2\sigma}{\partial S^2}$ *exist and*

$$A(S_i, t_i) \equiv S_i[\frac{\partial\sigma}{\partial t} + (r+\sigma^2)S_i\frac{\partial\sigma}{\partial S_i} + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2}{\partial S_i}] \tag{5.5}$$

is well defined for $(S, t) \in [0, \infty) \times [0, T]$

Given that all the above conditions hold, our tree could be constructed.

5.3 Building a Constant Probability-Stochastic Volatility Recombining Tree

According to Eq. (5.2), the stock price movement after two time steps is

$$\begin{cases} S_{n+2,i}^{UD} = S_{n,i}[1 + r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] \\ \qquad\qquad [1 + r\Delta t - \sigma(S_{n+1,i+1}, t_{n+1})\sqrt{\Delta t}] \\ S_{n+2,i}^{DU} = S_{n,i}[1 + r\Delta t - \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] \\ \qquad\qquad [1 + r\Delta t + \sigma(S_{n+1,i+1}, t_n)\sqrt{\Delta t}] \end{cases} \tag{5.6}$$

$S_{n+2,i}^{UD}$ moves in the path from $S_{n,i}$ upward to $S_{n+1,i+1}$ and then from $S_{n+1,i+1}$ downward to $S_{n+2,i}^{UD}$, whereas $S_{n+2,i}^{DU}$ moves in the path from $S_{n,i}$ downward to

$S_{n+1,i}$ and then from $S_{n+1,i}$ upward to $S_{n+2,i}^{DU}$. It is apparent that as one stock price grows in different paths, the stock price varies depending on which path it takes, that is, $S_{n+2,i}^{UD}$ and $S_{n+2,i}^{DU}$ are not likely to be the same since the local volatility is stochastic. Nevertheless, given the settings and assumptions mentioned in Section 5.2, the tree will recombine. Therefore, the following section will prove that $S_{n+2,i}^{UD} = S_{n+2,i}^{DU}$.

The stock price, $S_{n+2,i}^{DU}$, that moves upward and then downward is firstly discussed. We get started with the volatility.

By Ito's Lemma,

$$\begin{aligned}\sigma(S_{n+1,i}, t_{n+1}) = \sigma(S_{n,i}, t_n) \\ &+ \frac{\partial \sigma}{\partial S_{n,i}}(S_{n+1,i+1} - S_{n,i}) \\ &+ \frac{\partial \sigma}{\partial t}\Delta t \\ &+ \frac{1}{2}\frac{\partial^2 \sigma}{\partial S_{n,i}^2}(S_{n+1,i+1} - S_{n,i})^2\end{aligned} \tag{5.7}$$

And,

$$\begin{aligned}S_{n+1,i+1} - S_{n,i} &= S_{n,i}(1 + r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}) - S_{n,i} \\ &= S_{n,i}(r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t})\end{aligned} \tag{5.8}$$

Thus,

$$\begin{aligned}&(S_{n+1,i+1} - S_{n,i})^2 \\ &\quad = S_{n,i}^2[r^2(\Delta t)^2 + \sigma^2(S_{n,i}, t_n)\Delta t + 2r\sigma(S_{n,i}, t_n)(\Delta t)^{3/2}]\end{aligned} \tag{5.9}$$

When $\Delta t \to 0$, the terms $(\Delta t)^{3/2}$ and $(\Delta t)^2$ are much smaller, so they can be neglected. (This policy will be adapted throughout the induction process.) Hence,

$$(S_{n+1,i+1} - S_{n,i})^2 = S_{n,i}^2[\sigma^2(S_{n,i}, t_n)\Delta t] \tag{5.10}$$

By replacing Eqs. (5.9) and (5.10) into Eq. (5.7), Eq. (5.7) is written as

$$\begin{aligned}\sigma(S_{n+1,i+1}, t_{n+1}) = \sigma(S_{n,i}, t_n) \\ &+ \frac{\partial \sigma}{\partial S_{n,i}}S_{n,i}(r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}) \\ &+ \frac{\partial \sigma}{\partial t}\Delta t \\ &+ \frac{1}{2}\frac{\partial^2 \sigma}{\partial S^2}S_{n,i}^2[\sigma^2(S_{n,i}, t_n)\Delta t]\end{aligned} \tag{5.11}$$

As a result, by replacing $\sigma(S_{n+1,i+1}, t_{n+1})$ as in Eq. (5.11) into Eq. (5.6), Eq. (5.6) is therefore written as

$$\begin{aligned} S^{UD}_{n+2,i} &= S_{n,i}[1 + r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] \\ &\qquad [1+r\Delta t-\sigma(S_{n,i}, t_n)\sqrt{\Delta t}-\frac{\partial\sigma}{\partial S_{n,i}}S_{n,i}\sigma(S_{n,i}, t_n)\Delta t] \\ &= S_{n,i} + S_{n,i}[r - \frac{1}{2}\sigma(S_{n,i}, t_n)^2 - \frac{1}{2}S_{n,i}\sigma(S_{n,i}, t_n)\frac{\partial\sigma}{\partial S_{n,i}}]2\Delta t \\ &\qquad - A(S_{n,i}, t_n)(\Delta t)^{3/2} + \mathcal{O}((\Delta t)^2) \end{aligned} \tag{5.12}$$

where

$$A(S_{n,i}, t_n) = S_i[\frac{\partial\sigma}{\partial t} + (r + \sigma^2)S_{n,i}\frac{\partial\sigma}{\partial S_{n,i}} + \frac{1}{2}S^2_{n,i}\sigma^2\frac{\partial^2\sigma}{\partial S_{n,i}}] \tag{5.13}$$

In the same way

$$\begin{aligned} S^{DU}_{n+2,i} &= S_{n,i} + S_{n,i}[r - \frac{1}{2}\sigma(S_{n,i}, t_n)^2 - \frac{1}{2}S_{n,i}\sigma(S_{n,i}, t_n)\frac{\partial\sigma}{\partial S_{n,i}}]2\Delta t \\ &\qquad + A(S_{n,i}, t_n)(\Delta t)^{3/2} + \mathcal{O}((\Delta t)^2) \end{aligned} \tag{5.14}$$

In Eqs. (5.12) and (5.14), the terms $S_{n,i}$ and Δt are the same. Moreover, the terms of order higher than or equal to $(\Delta t)^2$ could be safely neglected since $(\Delta t)^2$ is relatively small as the number of time partitions, n, is large. The third terms, however, attribute to the difference of $S^{UD}_{n+2,i}$ and $S^{DU}_{n+2,i}$.

If $A(S_{n,i}, t_n) = 0$, then under **Assumption 2** and **3**, the tree built by Eq. (5.2) will recombine. In other words, $S^{UD}_{n+2,i}$ is the same as $S^{DU}_{n+2,i}$. Nevertheless, if $A(S_{n,i}, t_n)$ is set to be 0, the errors caused by the third term are apt to accumulate in the process of forward construction of the tree. Fortunately, the third terms in Eqs. (5.12) and (5.14) are identical except for their sign. If the stock price in time step $n + 2$ is taken as $S_{n+2,i} = \frac{1}{2}(S^{UD}_{n+2,i} + S^{DU}_{n+2,i})$, the error will be canceled out. As a result, the recombining tree for time step $n + 1$ is constructed as

$$\begin{cases} S_{n+1,i} = \frac{1}{2}\{S_{n,i}[1 + r\Delta t + \sigma(S_{n,i}, t_n)\sqrt{\Delta t}] + \\ \qquad S_{n,i+1}[1 + r\Delta t - \sigma(S_{n,i+1}, t_n)\sqrt{\Delta t}]\} \quad \text{for } i \neq 0, n \\ S_{n+1,0} = S_{n,0}[1 + r\Delta t - \sigma(S_{n,0}, t_n)\sqrt{\Delta t}] \\ S_{n+1,n} = S_{n,n}[1 + r\Delta t + \sigma(S_{n,n}, t_n)\sqrt{\Delta t}] \\ p_{n,i} = \frac{1}{2} \end{cases} \tag{5.15}$$

After the recombining tree is constructed by forward induction, the option price is calculated backward inductively from the stock prices in the terminal nodes of the tree.

5.4 Numerical Illustration

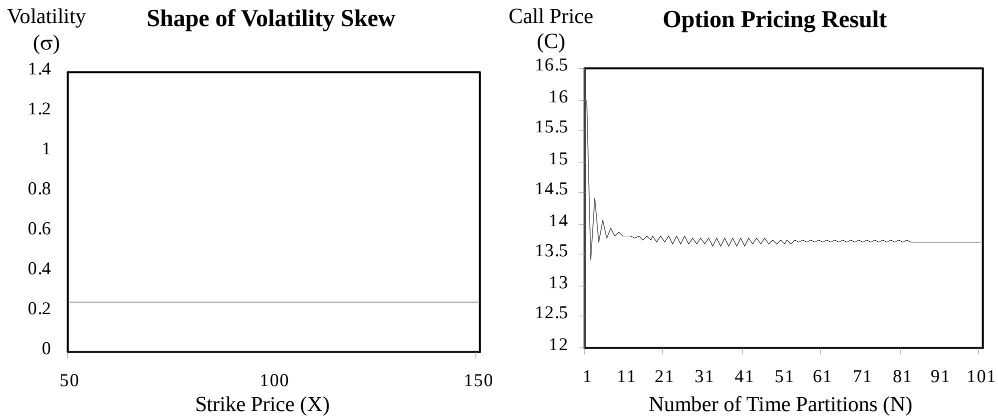
In order to check the convergence and option pricing results of our method, several scenarios are considered. The shapes of volatility smile are different in each of these scenarios. Equation (4.13) is used as the deterministic volatility function. As mentioned in Chapter 4, the shape of volatility smile is determined by providing different parameters in the volatility function. Hence, Table 5.1 provides seven scenarios that are applied to simulate various shapes of volatility smile. In addition, the initial stock price, S_0, and strike price, X, are both set as 100, risk-less rate, r, is 0.2, and dividend yield, q, is 0.

Figure 5.2 is the option pricing results in seven different scenarios according to the descriptions in Table 5.1. Different scenarios provide different shapes of local volatility smile, varying in shape, direction of slope with respect to the strike price, and steepness of slope with respect to the strike price. In a variety of situations, our method all performs well that the option prices will eventually converge as the number of time partitions increases.

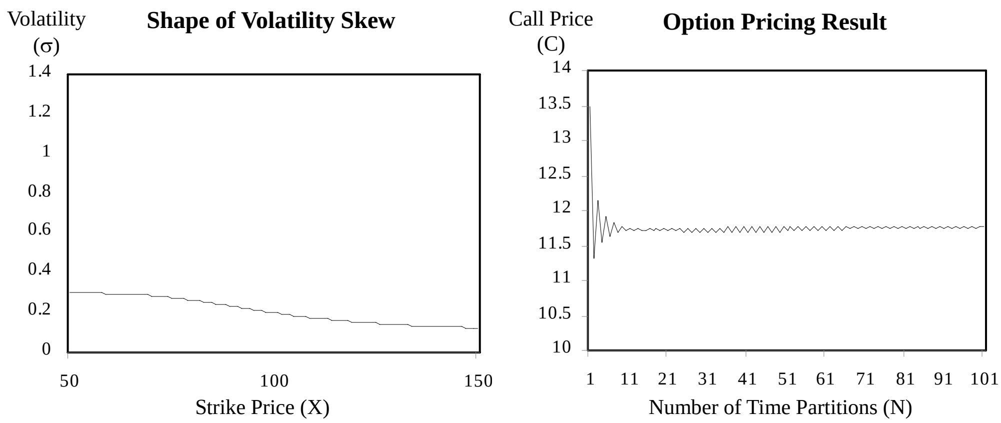
Desired shape of volatility smile		**Parameters setting**		
Case description		***a***	***b***	***c***
Scenario 1	Constant volatility	0.0	0.0	0.25
Scenario 2	1) Volatility skew 2) Negative slope with respect to strike 3) The slope is smooth	0.1	–3.0	0.1
Scenario 3	1) Volatility skew 2) Negative slope with respect to strike 3) The slope is steep	0.6	–3.0	0.1
Scenario 4	1) Volatility skew 2) Positive slope with respect to strike 3) The slope is smooth	0.1	3.0	0.1
Scenario 5	1) Volatility skew 2) Positive slope with respect to strike 3) The slope is steep	0.6	3.0	0.1
Scenario 6	1) Volatility Smile 2) The slope is smooth	0.1	3.0 for $S_t > X$ –3.0 for $S_t < X$	0.1
Scenario 7	1) Volatility Smile 2) The slope is smooth	0.6	3.0 for $S_t > X$ –3.0 for $S_t < X$	0.1

Table 5.1 Different scenarios of volatility smile.

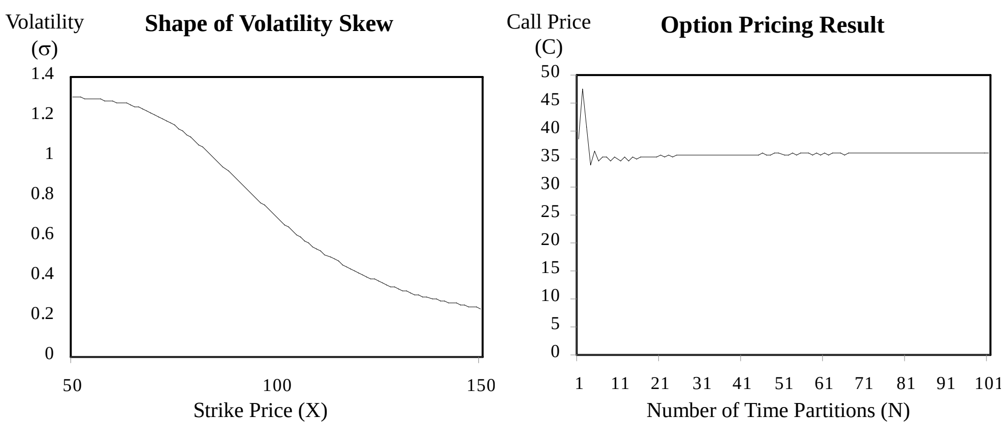
(a) Scenario 1



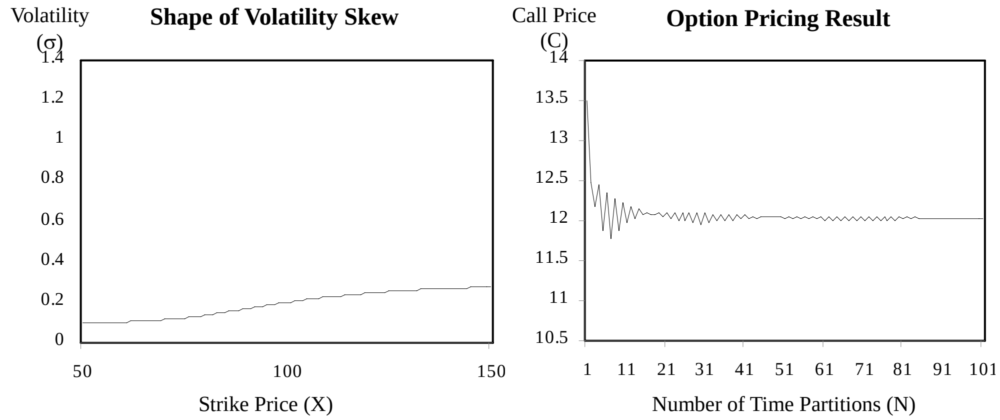
(b) Scenario 2



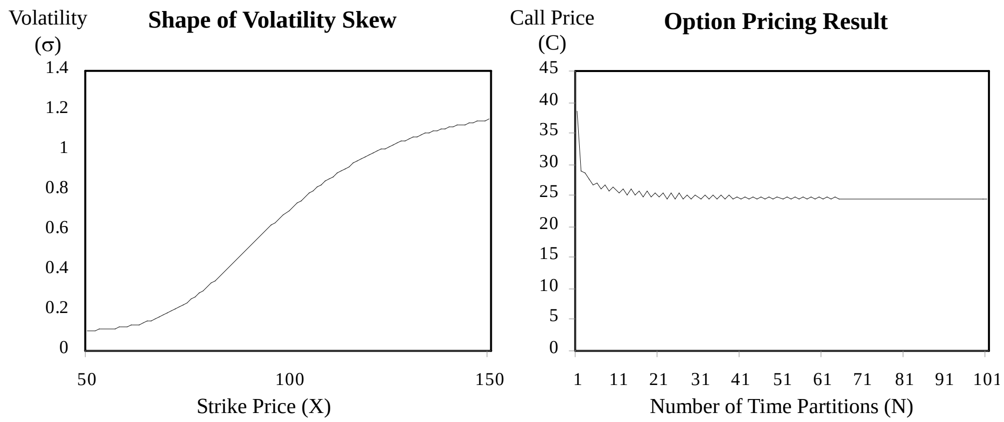
(c) Scenario 3



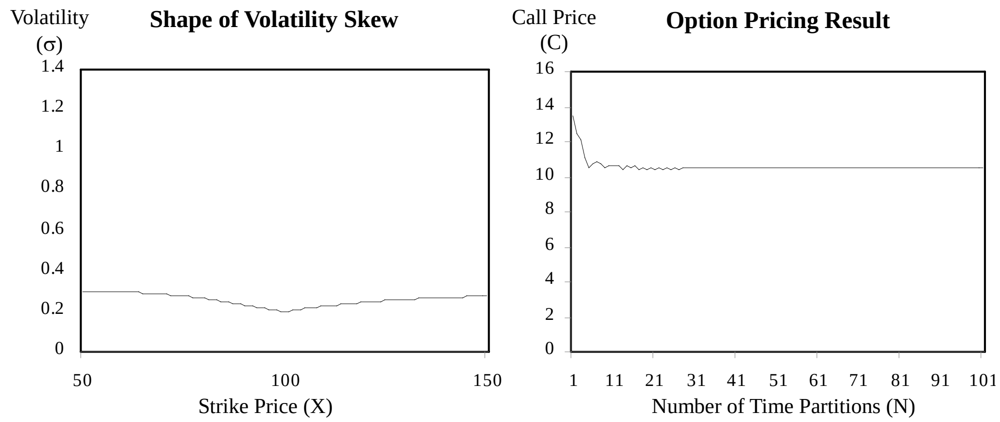
(d) Scenario 4



(e) Scenario 5



(f) Scenario 6



(g) Scenario 7

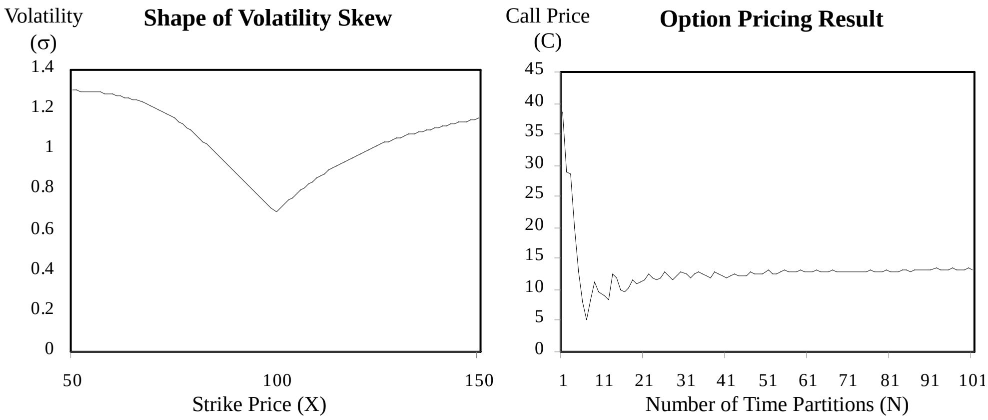


Figure 5.2 Option pricing results in seven different scenarios. Options are priced by our method in different scenarios. Different scenarios provide different shapes of local volatility smile, varying in shapes, direction of slopes with respect to the strike price, and steepness of slopes with respect to the strike price. Under various situations, our method all performs well that the option prices will eventually converge as the number of time partitions increases. The local volatility function applies Eq. (4.13). Settings of parameters are shown as Table 5.1

Altogether, our tree solves the invalid transition probability problem that occurs in the Dreman-Kani tree, and deals with the option pricing instability problem that happens in the Li tree. According to the results in seven scenarios discussed above, the alternative method that we proposed is quite stable in all kinds of situations.

Chapter 6
Conclusions

We proposed an alternative method that constructs a constant probability-stochastic volatility recombining implied binominal tree. As mentioned in Chapter 5, the tree is considered to be recombined under some general assumptions with stochastic volatility. In addition, the numerical illustration shows that stock prices in the nodes within the tree as well as the results of option pricing are considerably stable.

Our method for constructing the implied binomial tree addresses several weaknesses of the Derman-Kani tree and the Li tree. First, in the Derman-Kani tree, the invalid transition probability problem commonly occurs. For those transition probabilities less than 0 or greater than 1, arbitrage opportunity is allowed. The violation of the no-arbitrage principle contradicts the generally accepted theoretical assumptions in the pricing process. Although there are some ways to modify these stock prices that are invalid, the manual manipulation causes much loss in capturing market information and the resulting option prices do not equal market observations. In contrast, in our method, the transition probability is set to be constant, i.e., there is a 50% chance for the stock price to go either up or down. Under such a setting, the invalid transition probability problem will never occur.

Second, even though the Li tree is more stable than the Derman-Kani tree, and the invalid transition probability problem is also mitigated, there is still some room for enhancement. As the slope of volatility with respect to the strike price is steep, the Li algorithm also faces the problem that some stock prices on the tree will violate the no-arbitrage principle, and thus impacting on the option prices. Additionally, the results of option pricing do not converge in the Li tree when the slope of volatility with respect to the strike price is steep. In contract, our method constructs the binomial tree directly from known local volatilities, and the results of option pricing converge as desired and are more stable than the Li algorithm.

In conclusion, compared with the Derman-Kani tree and the Li tree, our method is much better. The constant probability-stochastic volatility recombining implied binominal tree, on the one hand, ensures the no-arbitrage opportunity not happen in every step of growing the tree. It is also very simple, direct and easy to understand.

Appendix A

This Appendix is intended to prove that the invalid transition probability in the Derman-Kani tree indicates the existence of arbitrage opportunities. Also, to avoid the occurrence of invalid transition probability, each newly determined stock price is required to be within the range as indicated in Eq. (3.11).

The up-move transition probability in the Derman-Kani tree is defined as:

$$p_{n-1,i} = \frac{f_{n-1,i} - S_{n,i}}{S_{n,i+1} - S_{n,i}} \tag{A.1}$$

where $p_{n,i}$ is the up-move transition probability at i^{th} node at time n, $f_{n,i}$ is the i^{th} forward price at time n, and $S_{n,i}$ is the i^{th} stock price at time n.

There are two cases for the transition probability to be invalid, which are 1) the transition probability is greater than 1, and 2) the transition probability is less than 0. We are going to discuss them separately.

Case 1 $p_{n,i} > 1$

From Eq. (A.1),

$$\begin{aligned} p_{n-1,i} &= \frac{f_{n-1,i} - S_{n,i}}{S_{n,i+1} - S_{n,i}} > 1 \\ \Rightarrow f_{n-1,i} - S_{n,i} &> S_{n,i+1} - S_{n,i} \\ \Rightarrow f_{n-1,i} &> S_{n,i+1} \end{aligned} \tag{A.2}$$

In addition, since $S_{n,i+1}$ is the up-move from $S_{n-1,i}$, and $S_{n,i}$ is the down-move from $S_{n-1,i}$, $S_{n,i+1}$ is bound to be greater than or equal to $S_{n,i}$. That is, the following condition must be hold

$$S_{n,i+1} \geq S_{n,i}. \tag{A.3}$$

Combining with Inequities (A.2) and (A.3) gives rise to the following inequality:

$$f_{n-1,i} > S_{n,i+1} \geq S_{n,i} \tag{A.4}$$

If the relationship between stock prices and forward prices holds as inequality (A.4), one can make a riskless arbitrage by taking a short position on the forward at time $(n-1)$ and to buy the stocks for settlement at time n. The net payoff is 0 at time $(n-1)$, but the net payoff is either $f_{n-1,i} - S_{n,i+1}$ or $f_{n-1,i} - S_n$ at time n.

Both of these two payoffs are greater than 0, whereas the initial cost is 0. These indicate arbitrage opportunities. Therefore, if the inequality (A.4) holds, arbitrage opportunities exist.

Case 2 $p_{n,i} < 0$

From (A.1),

$$\begin{aligned} & p_{n-1,i} < 0 \\ \Rightarrow\ & 1 - p_{n-1,i} > 1 \\ & 1 - \frac{f_{n-1,i} - S_{n,i}}{S_{n,i+1} - S_{n,i}} > 1 \\ \Rightarrow\ & S_{n,i+1} - f_{n-1,i} > S_{n,i+1} - S_{n,i} \\ \Rightarrow\ & f_{n-1,i} < S_{n,i} \end{aligned} \tag{A.5}$$

Also, since $S_{n,i+1}$ is the up-move from $S_{n-1,i}$, and $S_{n,i}$ is the down-move from $S_{n-1,i}$, $S_{n,i}$ is bound to be less than or equal to $S_{n,i+1}$. That is, the following condition must be hold

$$S_{n,i} \leq S_{n,i+1}. \tag{A.6}$$

Combining with Inequities (A.5) and (A.6) gives rise to the following inequality:

$$f_{n-1,i} < S_{n,i} \leq S_{n,i+1} \tag{A.7}$$

If the relationship between stock prices and forward prices holds as inequality (A.7), one can make a riskless arbitrage by taking a long position on the forward at time $(n-1)$ and to settle the forward contract to buy the stocks and sell the stock immediately at time n. The net payoff is 0 at time $(n-1)$, but the net payoff is either $S_{n,i+1} - f_{n-1,i}$ or $S_n - f_{n-1,i}$ at time n. Both of these two payoffs are greater than 0, whereas the initial cost is 0. Therefore, if the inequality (A.7) holds, arbitrage opportunities exist.

From Case 1 and Case 2, if the probability is greater than 1 or less than 0, there are arbitrage opportunities. Therefore, the forward price has to be within the range as the following inequality to ensure no arbitrage opportunities:

$$S_{n,i} \leq f_{n-1,i} \leq S_{n,i+1} \qquad \text{for } i = 1 \ldots n-1 \tag{A.8}$$

Looking at $i = j$ and $i = j+1$ in inequality (A.8) simultaneously, there are two inequalities. They are:

$$\begin{cases} S_{n,j} \le f_{n-1,j} \le S_{n,j+1} \\ S_{n,j+1} \le f_{n-1,j+1} \le S_{n,j+2} \end{cases} \tag{A.9}$$

Combining these two inequities into one gives rise to the following inequality:

$$S_{n,j} \le f_{n-1,j} \le S_{n,j+1} \le f_{n-1,j+1} \le S_{n,j+2} \tag{A.10}$$

To rule out arbitrage opportunities, this inequality must also hold. Because the forward prices on time $n-1$ are known, if $S_{n,j+1}$ is less than $f_{n-1,j}$ or greater than $f_{n-1,j+1}$, the opportunities exist.

In conclusion, invalid transition probabilities indicate arbitrage opportunities. To ensure arbitrage opportunities not occur, the stock prices in the nodes within the implied tree have to be within the following range:

$$f_{n-1,j} \le S_{n,j+1} \le f_{n-1,j+1} \tag{A.11}$$

We have now proved that the invalid transition probability in the Derman-Kani tree indicates the existence of arbitrage opportunities. The following is to prove inequality (A.11) must hold for the transition probability to be valid.

Looking at $j = i-1$ and $j = i$ in inequalities (A.11) simultaneously, there are two inequalities, which are:

$$\begin{cases} f_{n-1,i-1} \le S_{n,i} \le f_{n-1,i} \\ f_{n-1,i} \le S_{n,i+1} \le f_{n-1,i+1} \end{cases} \tag{A.12}$$

Combining these two inequities into one gives rise to the following inequality:

$$f_{n-1,i-1} \le S_{n,i} \le f_{n-1,i} \le S_{n,i+1} \le f_{n,i+1} \tag{A.13}$$

If the first inequality does not hold, the transition probability, $p_{n-1,i}$ will be less than 0; if the second inequality does not hold, the transition probability, $p_{n-1,i}$ will be greater than 1. The transition probabilities in both cases are invalid. The proof is as follows.

Case 1 $S_{n,i} > f_{n-1,i}$

If $S_{n,i} > f_{n-1,i}$, then

$$f_{n-1,i} - S_{n,i} < 0 \tag{A.14}$$

Also, $S_{n,i} \le S_{n,i+1}$ must hold in the tree, i.e., $S_{n,i+1} - S_{n,i} > 0$. Therefore, the up-move transition probability, $p_{n-1,i}$, is less than 0. That is

$$p_{n-1,i} = \frac{f_{n-1,i} - S_{n,i}}{S_{n,i+1} - S_{n,i}} < 0 \qquad \text{(A.15)}$$

This is an invalid transition probability.

Case 2 $f_{n-1,i} > S_{n,i+1}$

If $f_{n-1,i} > S_{n,i+1}$, then

$$f_{n-1,i} - S_{n,i+1} > 0 \qquad \text{(A.16)}$$

Subtracting S_{n-1} from both f_{n-1} and $S_{n,i+1}$, inequality (A.16) becomes

$$(f_{n-1,i} - S_{n,i}) - (S_{n,i+1} - S_{n,i}) > 0 \qquad \text{(A.17)}$$

By arranging items in inequality (A.17), (A.17) therefore becomes

$$\frac{f_{n-1,i} - S_{n,i}}{S_{n,i+1} - S_{n,i}} = p_{n-1,i} > 1 \qquad \text{(A.18)}$$

It is invalid for a transition probability to be greater than 1.

According to both cases discussed above, if the inequality (A.13) does not hold, i.e., arbitrage opportunity exists; the transition probability will be invalid.

Appendix B

This section is to prove that if the stock prices in the nodes within the Li tree are out of a specific range as indicated in inequality (4.14), arbitrage opportunities exist.

To rule out arbitrage opportunities, the stock prices have to be within the range:

$$f_{n-1,j} \leq S_{n,j+1} \leq f_{n-1,j+1} \tag{B.1}$$

where $f_{n,i}$ is the i^{th} forward price at time n, and $S_{n,i}$ is the i^{th} stock price at time n. In opposite to (B.1), if the relationship between the forward price and stock price is either $f_{n-1,j} > S_{n,j+1}$ or $S_{n,j+1} > f_{n-1,j+1}$, then arbitrage opportunities exist. Let's discuss these two cases separately.

Case 1 $f_{n-1,j} > S_{n,j+1}$

Suppose the relationship between the forward price and stock price is

$$f_{n-1,j} > S_{n,j+1} \tag{B.2}$$

Also, since $S_{n,i+1}$ is the up-move form$S_{n-1,i}$, and $S_{n,i}$ is the down-move from $S_{n-1,i}$, $S_{n,i+1}$ is bound to be greater than or equal to $S_{n,i}$. That is, the following condition must hold

$$S_{n,i+1} \geq S_{n,i}. \tag{B.3}$$

Combining with inequities (B.2) and (B.3) gives rise to the following inequality:

$$f_{n-1,i} > S_{n,i+1} \geq S_{n,i} \tag{B.4}$$

If the relationship between stock prices and forward prices holds as inequality (B.4), one can make a riskless arbitrage by taking a short position on the forward at time $(n-1)$ and to buy the stocks for settlement at time n. The net payoff is 0 at time $(n-1)$, but the net payoff is either $f_{n-1,i} - S_{n,i+1}$ or $f_{n-1,i} - S_n$ at time n. Both of these two payoffs are greater than 0, whereas the initial cost is zero. These indicate arbitrage opportunities. Therefore, if inequality (B.4) holds, arbitrage opportunities exist.

Case 2 $S_{n,j+1} > f_{n-1,j+1}$

Suppose the relationship between the forward price and stock price is

$$S_{n,j+1} > f_{n-1,j+1} \tag{B.5}$$

Also, since $S_{n,i+1}$ is the up-move from $S_{n-1,i}$, and $S_{n,i}$ is the down-move from $S_{n-1,i}$, $S_{n,i}$ is bound to be less than or equal to $S_{n,i+1}$. That is, the following condition must hold

$$S_{n,i} \leq S_{n,i+1}. \tag{B.6}$$

Combining with inequities (B.5) and (B.6), gives rise to the following:

$$f_{n-1,i} < S_{n,i} \leq S_{n,i+1} \tag{B.7}$$

If the relationship between stock prices and forward prices holds as inequality (B.7), one can make a riskless arbitrage by taking a long position on the forward at time $(n-1)$ and to settle the forward contract to buy the stocks and sell the stocks immediately at time n. The net payoff is 0 at time $(n-1)$, but the net payoff is either $S_{n,i+1} - f_{n-1,i}$ or $S_n - f_{n-1,i}$ at time n. Both of these two payoffs are greater than 0. Therefore, if inequality (A.7) holds, arbitrage opportunities exist.

From Case 1 and Case 2, if the probability is greater than 1 or less than 0, there is an arbitrage opportunity. Therefore, the forward price has to be within the range as the following inequality to rule out arbitrage opportunities.

$$S_{n,i} \leq f_{n-1,i} \leq S_{n,i+1} \qquad \text{for } i = 1 \ldots n-1 \tag{B.8}$$

Looking at $i = j$ and $i = j+1$ in inequality (B.8) simultaneously, there are two inequities. They are:

$$\begin{cases} S_{n,j} \leq f_{n-1,j} \leq S_{n,j+1} \\ S_{n,j+1} \leq f_{n-1,j+1} \leq S_{n,j+2} \end{cases} \tag{B.9}$$

Combining these two inequities into one gives rise to the following:

$$S_{n,j} \leq f_{n-1,j} \leq S_{n,j+1} \leq f_{n-1,j+1} \leq S_{n,j+2} \tag{B.10}$$

To rule out arbitrage opportunities, this must also hold. Because the forward prices on time $n-1$ are known, if $S_{n,j+1}$ is less than $f_{n-1,j}$ or greater than $f_{n-1,j+1}$, arbitrage opportunities exist.

In conclusion, invalid transition probabilities indicate arbitrage opportunities. To ensure arbitrage opportunities not happen, the stock prices in the nodes within the implied tree have to be within the following range:

$$f_{n-1,j} \leq S_{n,j+1} \leq f_{n-1,j+1} \tag{B.11}$$

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