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評價信用衍生性商品之動態違約系統的模型建構

Pricing Portfolio Credit Derivatives Using a Simplified Dynamic Model

林冠志

Kuan-Chi Lin

指導教授：呂育道 博士

Advisor: Yuh-Dauh Lyuu, Ph.D.

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摘要：

本論文主要是在介紹評價信用衍生性金融商品的動態模型，並應用之來評價標準化的擔保債權憑證（CDO）商品，iTraxx Europe index。此動態模型相對於之前市場上常用的 copula 靜態模型，應用的範圍更廣，其可以評價更多的新奇型信用衍生性商品。本文引用了 Hull and White (2007)提出的簡單動態模型的概念，修改其中幾項模型參數並探討出較合適的評價過程，使此模型更具經濟上的意義。除了直接的解析解，我們還探討了二元樹與蒙地卡羅模擬的方法，使評價過程更具多元性。其中我們補足了此模型轉成二元樹模型的數學證明，使此模型的架構更加完整。最後，我們發現此模型帶入市場資料後所求得的商品分券價值亦相當接近於市場價值。

Abstract

This thesis investigates dynamic methods for pricing portfolio credit derivatives, especially the standardized market for CDO: iTraxx Europe index. Compared with previous static models, i.e., the copula functions, the dynamic models are applicable to much more exotic portfolio credit derivatives. This thesis uses the concept of the dynamic model from Hull and White (2007). But we modify it by adjusting some parameters. We also find a better way for calibration to give the model more economic sense. The iTraxx Europe index can also be valued analytically using our model. Besides the analytic method, we consider the binomial tree and Monte Carlo method to make pricing more flexible. Finally, the revised dynamic model captures the advantages of the original one and also provides a good fit to CDO quotes.

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Chapter 1

Introduction

1.1 Background and Literature Review

The standard market model for pricing portfolio credit derivatives is the copula model, which assumes a simple one factor model for a company's time to default. While there are several types of copula function models, Li (2000) introduces the one-factor Gaussian copula model for the case of two companies and Gregory and Laurent (2005) extend the one-factor model to the case of N companies. Many alternatives to the Gaussian copula such as the t-copula, the double-t copula, the Clayton copula, the Archimedian copula, the Marshall Olkin copula have been suggested. However, these approaches are problematic for two main reasons. First, there is no dynamic consistency, and, second, there is no theoretical basis for the choice of any particular dependence structure. These copula models which lack of dynamics are called static models, because they do not describe how the default environment evolves. Thus, they do not allow us to price, for example, forward starting credit products or options on CDO tranches.

Stimulated by the perspective of an emerging market of the exotic credit products mentioned above, there has been much research on developing a dynamic model that fits market data and tracks the evolution of the credit risk of a portfolio, including Albanese et al. (2006), Bennani (2006), Brigo et al. (2007), Di Graziano and Rogers (2006), Hull and White (2007), Schönbucher(2006), Sidenius (2006), among others. Among these, the dynamic models can be categorized into three categories: structural model, top-down model and reduced-form model.

The most basic version of the structural model is similar to the Gaussian copula model. Albanese et al (2006) propose a rating transition model within the structural framework where the distance to default of each single obligor is represented by a Markov chain. Structural models have the advantage that they have sound economic fundament.

The top-down model is a dynamic model which involves the development of a model for the evolution of the losses on a portfolio. It considers the frameworks for modeling quantities directly related to the loss distribution of a pool of names. In other words, it models directly the cumulative portfolio loss process. This approach is pursued in Bennani (2006), Sidenius, Piterbarg and Andersen (2006) and Schönbucher

(2006). Bennani (2006) proposes a model of the instantaneous loss as a percentage of the remaining notional principal. Sidenius et al (2006) use concepts from the Heath, Jarrow, and Morton (1992) interest-rate model to suggest a complex general no-arbitrage approach to modeling the probability that the loss at a future time will be less than some level. Sidenius et al (2006) model the dynamics of portfolio loss distributions in the absence of information about default times. Schönbucher (2006) models the evolution of the loss distribution from the transition rates of an auxiliary time-inhomogeneous Markov chain which reproduces the desired transition probability distribution. Stochastic evolution of the cumulative loss process is then obtained by using the transition rates with stochastic dynamics.

The reduced-form approach for developing a dynamic model is to specify correlated diffusion processes for the hazard rates of the underlying companies. Di Graziano and Rogers (2005) present a new approach to default correlation modeling, where defaults of different names are driven by a common continuous-time Markov process. Individual default probabilities and default correlations can then be calculated in closed form. Brigo et al. (2007) consider a dynamical model for the number of defaults of a pool of names. The model is based on the notion of generalized Poisson process, allowing for more than one default in small time intervals. Hull and White (2007) develop a model that is both reduced-from and top-down. It is easy to implement and easy to calibrate to market data. Under the model the hazard rate of a company has a deterministic drift with periodic impulses. The impulse size plays a similar role to default correlation in the Gaussian copula model. Additionally, Brigo et al. (2007), Di Graziano and Rogers (2006) and Hull and White (2007) all have given examples of calibration to CDO tranche quotes with a high degree of precision.

This thesis modifies the model of Hull and White (2007). The objective here is to specify the procedure from model set-up to calibration more completely. In particular, we adjust some parameters to give the model more economic sense, and propose a calibration method where index tranches quotes are matched as closely as possible.

1.2 Structures of the Thesis

The remainder of this thesis is organized as follows. Chapter 2 reviews basic concepts and pricing technologies of CDO and index tranche as well as the calibration to index spread. In chapter 3 we present three algorithms for pricing under our dynamic model: analytical method, binomial tree and Monte Carlo method. Chapter 4 discusses parameters and numerical data from calibration. Finally Chapter 5 summarizes our results and points to future research.

Chapter 2

A Primer on CDO and Index Tranche Pricing

2.1 Introduction to an Index Tranche

The two most actively traded CDS indexes are the Dow Jones CDX NA IG index and the iTraxx Europe index. The former includes 125 North American investment grade companies. The latter includes 125 European investment-grade companies. Index tranches of these CDS indexes are CDO tranches whose underlying portfolio is composed of the 125 companies in the CDS indexes. For both index tranches, each company is equally weighted. They have standardized documentation and use standard attachment and detachment points. They are sliced into five tranches: equity tranche, junior mezzanine tranche, senior mezzanine tranche, junior senior tranche, and super senior tranche. The standard tranche structure in terms of attach point-detachment point pairs of the Dow Jones CDX NA IG is 0-3%, 3-7%, 7-10%, 10-15%, and 15-30%. As for the iTraxx Europe, it is 0-3%, 3-6%, 6-9%, 9-12%, and 12-22%. For both indexes, they are quoted on maturities of three, five, seven and ten years.

The premium of the equity tranche is paid differently from the nonequity tranches. It includes two parts. The first is the upfront percentage payment as a percentage of the notional, and the second is the fixed 500 basis points premium per annum. The market quote is the upfront percentage payment. For the nonequity tranches, the premium includes only the second part. Their market quotes are the premium in basis points, paid quarterly in arrears to purchase protection from defaults. Thus for pricing these index tranches, we should notice the different quote conventions for the various tranches.

	Attach point	Detachment point	Maturities			
			3 yr	5 yr	7 yr	10 yr
Index			77.00	101.00	104.00	106.00
Tranche	0%	3%	n/a	32.00	39.00	43.00
	3%	6%	n/a	395.00	485.00	n/a
	6%	9%	n/a	245.00	280.00	310.00
	9%	12%	n/a	n/a	180.00	n/a
	12%	22%	n/a	n/a	90.00	100.00

Table 2.1 iTraxx CDO tranche quotes in basis points of Series 9 on April 2, 2008.

The market also quotes the index spreads which are the average of the CDS spreads of the companies in the portfolio pool for each maturity. For example, the Index row in the Table 2.1 shows the index spreads. Before pricing the index tranches we would like to use the index spreads to calibrate the implied default probability of each company under the assumption of homogeneity of the model to be described in Chapter 3. The following two sections review the concepts we use for calibrating and pricing index tranches.

2.2 Extract Implied Default Probability from CDS Spread

In this section, we present the method for calibrating the hazard rate from the index spread of an index tranche. The reduced-form model used here is the time-inhomogeneous Poisson process with time varying intensity $\lambda(t)$ and cumulated hazard function $\Gamma(t)=\int_0^t \lambda(u)du$ detailed in Appendix A. For calibration we will take the hazard rate to be deterministic and piecewise constant: $\lambda(t)=\lambda_i$ for $t\in[T_{i-1},T_i)$, where T_i are the relevant maturities. Let $\beta(t)$ be the index of the first T_i after t; for example, if $t=6.25$, then $\beta(t)=7$. The cumulated hazard function is

$$\Gamma(t)=\int_0^t \lambda(u)du=\sum_{i=1}^{\beta(t)-1}(T_i-T_{i-1})\lambda_i+\left(t-T_{\beta(t)-1}\right)\lambda_{\beta(t)}. \tag{2.1}$$

A typical CDS contract usually specifies two potential cash flow streams: a default leg and a premium leg. On the default leg side, the protection seller makes one payment only if the reference credit defaults. The amount of a contingent payment is usually the notional amount multiplied by the recovery rate. On the premium leg side, the buyer of protection makes a series of fixed, periodic payments of CDS premium until the maturity or the reference credit defaults. For a breakeven spread, the net present value of both legs must equal zero.

The payment on CDS is assumed to be quarterly in arrears. We also assume a constant recovery rate, R, and deterministic interest rates. Let d_i denote the riskless discount factor from 0 to t_i, S_{CDS} be the spread for a CDS contract, T be the maturity year, τ be the time point when credit event occurring and Q is the probability of a credit event occurring under the risk-neutral world. The present value of default leg of a CDS is

$$(1-R)\sum_{i=1}^{4T} d_i Q\left(\tau\in[t_{i-1},t_i]\right)=(1-R)\sum_{i=1}^{4T} d_i\left(\exp\left(-\Gamma(t_{i-1})\right)-\exp\left(-\Gamma(t_i)\right)\right)$$

(see Eq. (A.1) in Appendix A). With similar computation, the present value of the premium leg is

$$S_{CDS}\sum_{i=1}^{4T} d_i(t_i - t_{i-1})Q(\tau > t_i) + S_{CDS}\sum_{i=1}^{4T} d_i \frac{(t_i - t_{i-1})}{2} Q(\tau \in [t_{i-1}, t_i])$$
$$= S_{CDS}\sum_{i=1}^{4T} d_i(t_i - t_{i-1})\exp(-\Gamma(t_i)) + S_{CDS}\sum_{i=1}^{4T} d_i \frac{(t_i - t_{i-1})}{2}(\exp(-\Gamma(t_{i-1})) - \exp(-\Gamma(t_i))).$$

With a piecewise constant $\lambda(t)$, we obtain the breakeven spread

$$S_{CDS} = \frac{(1-R)\sum_{i=1}^{4T} d_i(\exp(-\Gamma(t_{i-1})) - \exp(-\Gamma(t_i)))}{S_{CDS}\sum_{i=1}^{4T} d_i(t_i - t_{i-1})\exp(-\Gamma(t_i)) + S_{CDS}\sum_{i=1}^{4T} d_i \frac{(t_i - t_{i-1})}{2}(\exp(-\Gamma(t_{i-1})) - \exp(-\Gamma(t_i)))}. \quad (2.2)$$

From the market data, we have fair CDS quotes, i.e., the index spreads for $T = 3,5,7,10$. We then use the fair quotes as the breakeven spread to calibrate the intensity parameters. Chapter 4 will present the numerical examples based on the market data on April 2, 2008.

2.3 Valuation of a CDO

Before explaining the model we give some assumptions and a general pricing equation for CDO. We assume homogeneity so that all companies have the same notional value and same default probabilities. Assume there are N companies in the underlying portfolio of a CDO contract with total notional principal P. Let a and b be the tranche attachment and detachment points, respectively. The protection seller for a tranche of a CDO provides protection against losses on the portfolio that are in the range $a \times P$ to $b \times P$ for the life of the instrument. The protection buyer pays a certain number of basis points on the outstanding notional principal of the tranche. The tranche principal equals $(b-a)\times P$ initially and declines as losses in the range $a \times P$ to $b \times P$ are experienced. That is, if the pool losses over $[0,T]$ are less than $a \times P$, the seller does not suffer any loss; otherwise, the seller absorbs the losses up to the tranche size $(b-a)\times P$. In return for the protection, the buyer pays periodic premiums at payment dates $t_1 < t_2 < \cdots < t_m = T$, where $t_i > T_0 = 0$ for $i = 1,2,\ldots,m.$ The payments on CDO are also assumed to be quarterly in arrears. From the assumption of homogeneity, denote the tranche principal at time t when there have been n defaults by

$$W(n,t) = \begin{cases} (b-a)\times P & n < \frac{a \times N}{1-R} \\ b \times P - n \times \frac{P}{N} \times (1-R) \quad \text{if} & \frac{a\times N}{1-R} < n < \frac{b \times N}{1-R} \\ 0 & n > \frac{b\times N}{1-R} \end{cases} \quad (2.3)$$

If we derive the expected tranche principal, $\mathrm{E}[W_i]$, at time t_i, then we can value a CDO tranche at time zero according to the following procedure. Let S_{CDO} be the spread for a CDO contract. In general, valuation of a CDO tranche balances the expectation of the present values of the premium payments against the payoff from effective tranche losses, i.e., the premium leg against the default leg such that

$$\mathrm{E}\left[\sum_{i=1}^{4T} S_{CDO}\left(t_i - t_{i-1}\right)W_i d_i\right] = \mathrm{E}\left[\sum_{i=1}^{4T} (W_i - W_{i-1}) d_i\right].$$

The right-hand side, the premium leg, is the calculation of the expected tranche principal at specified times, which equals the expected tranche principal at payments dates multiplied by the spread S_{CDO}. The expected payoff, the default leg, between two payments dates equals the reduction in the expected tranche principal between those times.

The breakeven spread s is therefore given by

$$S_{CDO} = \frac{\mathrm{E}\left[\sum_{i=1}^{4T} (W_i - W_{i-1}) d_i\right]}{\mathrm{E}\left[\sum_{i=1}^{4T} \left(t_i - t_{i-1}\right)W_i d_i\right]} = \frac{\sum_{i=1}^{4T} (\mathrm{E}[W_i] - \mathrm{E}[W_{i-1}]) d_i}{\sum_{i=1}^{4T} \left(t_i - t_{i-1}\right)\mathrm{E}[W_i] d_i}. \qquad (2.4)$$

Alternately, if the spread is set, the value of the CDO is the difference between the two legs:

$$V_{CDO} = \sum_{i=1}^{4T} \left(t_i - t_{i-1}\right)\mathrm{E}[W_i] d_i - \sum_{i=1}^{4T} (\mathrm{E}[W_i] - \mathrm{E}[W_{i-1}]) d_i$$

Therefore, the problem is reduced to the computation of the expected tranche principal, $\mathrm{E}[W_t]$, at time t.

Chapter 3

The Dynamic Model and Its Implementations

In this chapter, we review the model of Hull and White (2007). Additionally we modify some parameters and the way of calibration to give the original model more economic sense. The first implementation of the model is the analytical method. However, in order to price other exotic portfolio credit products which must be priced by a backward recursion algorithm, we discretize the model on a binomial tree. Typically, these are products with embedded options, such as tranche options or leveraged super seniors. Finally, we present a Monte Carlo simulation method for pricing.

3.1 The Dynamic Model

3.1.1 Model Review

The main model idea of this thesis is that we start by assuming that there exists a process for hazard rate which drives the common dynamics of the credits in the portfolio. Periodically there are economic shocks to the default environment. When a shock occurs each company has a nonzero probability of default. As a result the economic shocks are accompanied with defaults at the same time. It is these shocks and their sizes that create the default correlation. Empirical evidence suggests that default correlations increase when hazard rates are high. So the default correlation is positively related to the default rate. Thus, in a risk-neutral world, Hull and White (2007) construct the model of hazard rate, X, to be one that has a deterministic drift and periodic impulses:

$$X(t) = M(t) + \sum_{j=1}^{N(t)} H_j$$

The number of economic shocks, $N(t)$, is a jump process with intensity λ and jump size

$$H_j = H_0 e^{j\beta}, \tag{3.1}$$

where H_0 and β are positive constants, and j means the j^{th} economic shock.

Hull and White (2007) first present a simplified version of the model for calibration. This is a one-parameter model where the drift of the hazard rate is zero and the jump size is constant for any economic shock; in other words, $M(t)=0$ and $\beta=0$. The jump intensity $\lambda(t)$ is time-dependent, and it is extracted from index spreads. Thus that leaves only one free parameter, the implied jump size H_0, which is calibrated to quotes of index tranches.

The calibration for the original one, the three-parameter version of the model, is done in different way. The drift and the jump size are both nonzero, but the jump intensity λ is now assumed to be constant. The drift of the hazard rate, $M(t)$, is determined to match index spreads, and the three parameters λ, H_0 and β are calibrated to quotes of index tranches.

3.1.2 Model Modification

Although the three-parameter version of the model presented by Hull and White (2007) is designed to provide a good fit to all spreads of all maturities, the calibration methods for the jump intensity of this version and the simplified one are not consistent. The three-parameter version's assumption of constant jump intensity does not make more economic sense than the simplified version's assumption of time-dependent jump intensity. As mentioned in section 2.2, we would like to use the index spread to calibrate the implied jump intensity function of the model. The jump intensity function is based on the Poisson process which is used to model either rare events or discretely countable events. If we use this information, then we can say the default is accompanied with economic shocks which create the default correlation. This is consistent with our model's philosophy. Furthermore the intensity we calibrated is time-varying, which makes more economic sense. Additionally, we consider the drift term to be submerged by H_0, which is the jump size when there are no economic shocks. Based on the above considerations, our dynamic model is

$$X(t)=\sum_{j=0}^{N(t)} H_j \tag{3.2}$$

Our model also can be presented as two versions, the one-parameter vision and the two-parameter version. Both versions' jump intensity will be calibrated to index spreads and their other free parameters will be calibrated to tranche quotes.

3.2 Three Implementations of the Model

3.2.1 Analytical Method

We assume homogeneity for the model so that all companies have the same default probabilities and the default probabilities of companies are independent of one another. Let $S(t)$ to be the cumulative probability of survival by time t conditional on a particular hazard rate path between time 0 and time t. The transformation of $S(t)$ from X is defined by $S(t)=\exp(-X(t))$. If $S(t)$ is known, the probability distribution of the number of defaults up to time t can be calculated. If there are N companies in the portfolio, then the probability that n of them will default by time t is

$$\frac{N!}{n!(N-n)!}(1-S(t))^n S(t)^{N-n}. \qquad (3.3)$$

Now, we proceed to price the index tranches on the dynamic model using the analytical method. From the end of section 2.3, what we need to do is to calculate the expected tranche principal, $\mathrm{E}[W_i]$. The probability of J jumps between time zero and time t is

$$P(J,t)=\frac{\Gamma(t)^J \exp(-\Gamma(t))}{J!}, \qquad (3.4)$$

(see Eq. (2.1) in section 2.2). The value of S in our model at time t if there have been J jumps is

$$S(J,t)=\exp\left[-\sum_{j=0}^{J} H_j\right]. \qquad (3.5)$$

The probability of n defaults in the portfolio by time t conditional on J jumps can be calculated from Eq. (3.2); we denote that value by $\Phi(n,t|J)$. The expected principal on the tranche at time t conditional on J jumps is

$$E(t|J)=\sum_{n=0}^{N}\Phi(n,t|J)W(n,t), \qquad (3.6)$$

(see Eq. (2.3) in section 2.3). Finally, the unconditional expected tranche principal, $\mathrm{E}[W_t]$, at time t is therefore

$$E(W_t)=\sum_{J=0}^{\infty}P(J,t)E(t|J). \qquad (3.7)$$

Finally, with this equation, the tranche spread can be recovered from Eq. (2.4) at the end of section 2.3. Therefore, the index tranches can be valued analytically using our dynamic model.

3.2.2 Binomial Tree Method

In this section, we build a binomial tree for our dynamic model. Under the binomial tree, we can price the exotic portfolio credit products which must be priced by a backward recursion algorithm. Typically, these are products with embedded options, such as tranche options or leveraged super seniors.

To construct the tree, the life of the model is divided into a number of short time intervals. For every fixed positive integer m, partition the trading interval $[0,T]$ into m subintervals of length $h_m = \frac{T}{m}$. The subscript denotes dependence on the particular value of m chosen. Denote the time corresponding to the end of the i^{th} subinterval by τ_i and let $\tau_0 = 0$. The τ_i are chosen so that there are nodes on each payment date, in other words, for each k, $\tau_i = t_k$ for some i. In practice this is achieved by creating v equal time steps between each payment date for some integer v. Let T be the maturity year, m is equal to $v \times 4T$, because the payments on credit portfolio products are assumed to be quarterly in arrears.

Now consider the jump component of Eq. (3.2), the impulses of the hazard rate. For the Poisson jump component, the probability of a single jump occurring in an interval of length h_m is equal to $\lambda h_m + o(h_m)$. The probability of multiple jumps in the same interval equals $o(h_m)$, where the symbol $o(h_m)$ represents any function such that $\lim_{h \to 0}\left[\frac{o(h_m)}{h_m}\right] = 0$. Therefore, we assume that the probability of a jump during each time interval is equal to λh_m. We also assume that multiple jumps at any discrete date cannot occur. This leads to a tree with the geometry shown in Figure 3.1. In this figure, $H_j = H_0 e^{j\beta}$ is the size of the j^{th} jump. The probability on the upper and lower branches emanating from a node at time τ_i are $\lambda_i h_m$ and $1 - \lambda_i h_m$, respectively, where $\lambda_i = \lambda(\tau_i)$, which is calibrated from the index spread.

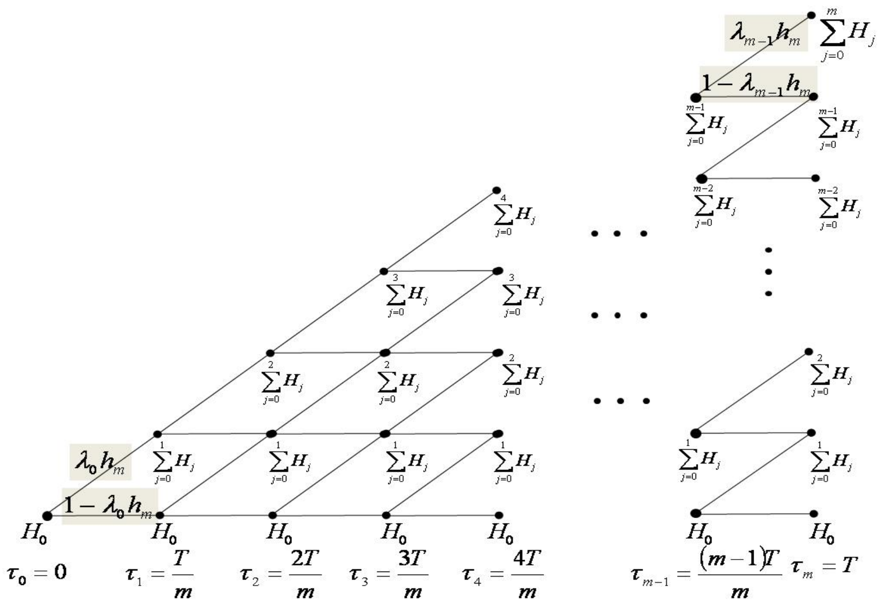

Figure 3.1 The Binomial Tree for Hazard Rate.

It is reasonable for us to fit the process of hazard rate by this binomial tree, because if $N = m \to \infty$ and $P = \lambda h_m \to 0$ in such a way that $NP = \lambda T$ is constant, then the binomial distribution converges to the Poisson distribution with mean λT. In Appendix B, we prove that the expectation of hazard rate under this binomial tree method does indeed converge to the value calculated under the above analytical method when the partition is dense enough, i.e., $m \to \infty$.

Denote the j^{th} node at time τ_i by (i, j). Let S_{ij} and W_{ij} be the cumulative survival probability and expected tranche principal at node (i, j), respectively. The value of S_{ij} is equal to $\exp(-X_{ij})$, which will be used to calculate the value of $\Phi(n, \tau_i | J)$, the probability of *n* defaults by time τ_i. Thus W_{ij} can then be calculated from Eqs. (2.3) and (3.5).

Let PL_{ij} and DL_{ij} be the premium leg and the default leg which we defined analogously in section 2.3, respectively. Sometimes τ_i correspond to payment dates and others do not. Let δ_i be the day count factor defined as follows. For calculating PL_{ij}, if τ_i is a payment date so that $\tau_i = t_k$, then δ_i equals the accrual fraction $t_k - t_{k-1}$. Otherwise when τ_i is not a payment date, $\delta_i = 0$. We use backward recursion algorithm to calculate the breakeven spread of an index tranche. At the final nodes $PL_{ij} = \delta_i W_{ij}$ and $DL_{ij} = 0$. At earlier nodes, they can be calculated by working backward through the tree using

$$PL_{ij} = \left[\lambda_i h_m PL_{i+1,j+1} + (1 - \lambda_i h_m) PL_{i+1,j}\right] \times \frac{d_{i+1}}{d_i} + \delta_i W_{ij}$$

$$DL_{ij} = \left[\lambda_i h_m DL_{i+1,j+1} + (1 - \lambda_i h_m) DL_{i+1,j}\right] \times \frac{d_{i+1}}{d_i}$$

$$+ \left[\lambda_i h_m \left(W_{ij} - W_{i+1,j+1}\right) + (1 - \lambda_i h_m)\left(W_{ij} - W_{i+1,j}\right)\right] \times \frac{d_{i+1}}{d_i}.$$

Note that using this binomial tree method, we calculate W_{ij}, the conditional expected tranche principal given the J^{th} jump at time τ_i rather than the expected tranche principal, $\mathrm{E}\left[W_{\tau_i}\right]$, at time τ_i, which converges to Eq. (3.7) of the analytical method.

Finally, the breakeven spread of a index tranche is $\frac{DL_{00}}{PL_{00}}$, which converges to the breakeven spread calculated by analytical method theoretically.

3.2.3 Monte Carlo Simulation Method

Another approach for valuation is to use Monte Carlo simulation to evaluate the number of economic shocks between payment dates, and then calculate the hazard rate and the cumulative survival probability under that circumstance. The simulation algorithm for Poisson distribution is quite standard. It makes use of a relationship between the Poisson distribution and the exponential distribution.

We generate exponential distributed samples from a uniformly distributed variable. If y is an exponentially-distributed random variable with mean $1/\lambda$, then the number of samples needed to sum up to, but not beyond the period between two payment dates, 0.25, is Poisson distributed with mean λ.

First, we calibrate the intensity function from the index spread. Second, we generate samples from a Poisson distribution with the corresponding intensities at each payment day by the algorithm above. Then we calculate the hazard rate and the cumulative survival probability from Eqs. (3.1) and (3.5), respectively, where the jump number J is the value we sampled. The remaining procedures to calculate the breakeven spread of index tranche are the same as the analytical method. Finally, we repeat this simulation for one million times and calculate the average value of the breakeven spread of index tranche, which also converges to the breakeven spread calculated by analytical method theoretically.

Chapter 4

Calibration and Numerical Results

Our dynamic model is calibrated to the market quotes in Table 2.1 for iTraxx Europe Index of Series 9 observed on April 2, 2008. All calibrations assume recovery rate, $R = 40\%$, and the risk-free interest rate $r = 5\%$. At the beginning of this chapter, we calibrate the jump intensity function and then use this data and analytic method to calibrate other parameters in the two versions of our dynamic model. At the end, we use all the data from calibration to compare the other two pricing methods, binomial tree and Monte Carlo method with the analytic method.

Firstly, we use the model of CDS spread as Eq. (2.2) mentioned in section 2.2 to calibrate the jump intensity $\lambda(t)$ to the index spread in Table 2.1. The calibration result is shown in Table 4.2 and the piecewise constant intensity function is presented in Figure 2.1. From the calibrated intensity graphs, the market perceives the second interval as the most risky, because the intensity is highest in that period.

Maturities (yr)	Maturity (date)	Index spread (bps)	Intensity
3	2011/4/2	77.00	1.2833%
5	2013/4/2	101.00	2.3937%
7	2015/4/2	104.00	1.8934%
10	2018/4/2	106.00	1.8775%

Table 4.2 Piecewise constant intensity calibrated to index spread on April 2, 2008.

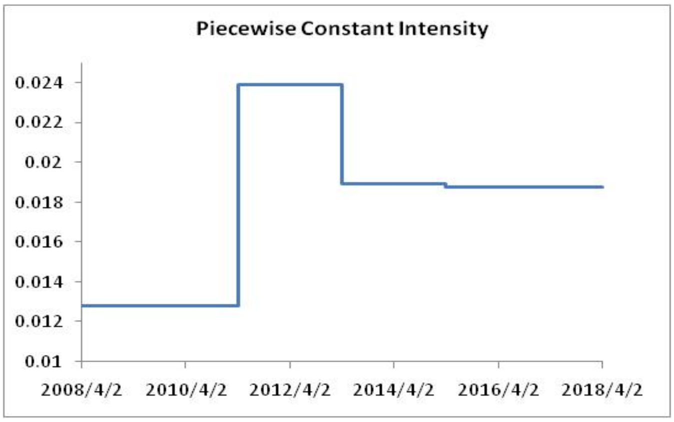

Figure 2.1 Piecewise constant intensity function calibrated to iTraxx index spreads on April 2, 2008.

Second, we calibrate the jump sizes implied from the iTraxx market quotes in Table 2.1 using our one-parameter model. Additionally, we calibrate the tranche correlations implied from the same quotes using the Gaussian copula model. The results of the jump size and the tranche correlation are given in Table 4.3 and Table 4.4, respectively. Additionally, Figure 4.2 plots Table 4.3 and Table 4.4.

a	*b*	Maturities			
		3 yr	5 yr	7 yr	10 yr
0%	3%	n/a	0.021624	0.025900	0.028973
3%	6%	n/a	0.046711	0.052679	n/a
6%	9%	n/a	0.080631	0.086129	0.091677
9%	12%	n/a	n/a	0.116501	n/a
12%	22%	n/a	n/a	0.161174	0.166269

Table 4.3 Implied jump sizes using our one-parameter model for available tranches of iTraxx on April 2, 2008.

a	*b*	Maturities			
		3 yr	5 yr	7 yr	10 yr
0%	3%	n/a	0.520072	0.492428	0.491335
3%	6%	n/a	0.859860	0.764007	n/a
6%	9%	n/a	0.044982	0.928230	0.885820
9%	12%	n/a	n/a	0.062800	n/a
12%	22%	n/a	n/a	0.180656	0.103481

Table 4.4 Implied tranche correlations using the Gaussian copula model for available tranches of iTraxx on April 2, 2008.

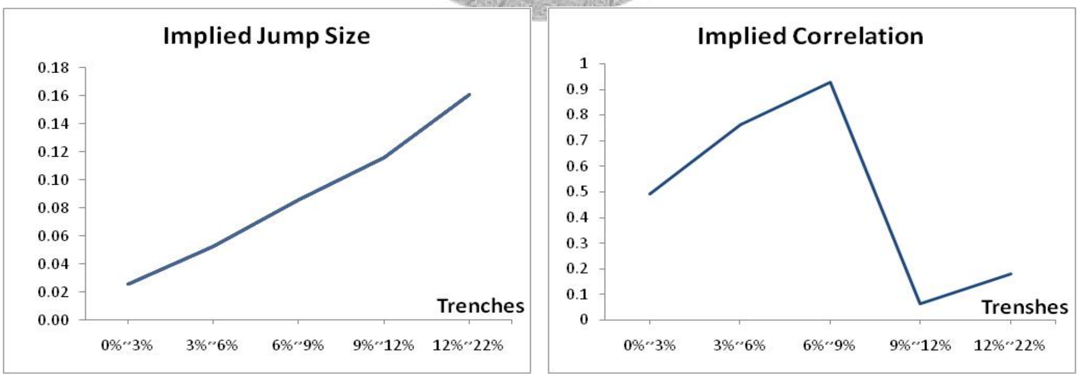

Figure 4.2 Implied jump size of our one-parameter model (left) compared with the implied tranche correlation of the Gaussian copula model (right) from the 7-year tranches quotes of iTraxx on April 2, 2008.

As mentioned in section 3.1.1, the implied jump size creates the default correlation which is positively related to the default rate. Thus the jump size approaches zero when the default correlation approaches zero. As the jump size becomes large, the default correlation approaches one. Figure 4.2 compares the jump sizes with the tranche correlations. It can be seen that the two exhibit different patterns. On April 2, 2008, a correlation structure that resembles a wave for implied tranche correlations rather than steeply upward sloping skew for implied jump sizes. Figure 4.3 compares the jump sizes with the base correlations. The pattern of implied jump size is similar to base correlation which is much smoother and more stable. Additionally, the advantage of calculating an implied jump size rather than an implied copula correlation is that the jump size is associated with a dynamic model, whereas the copula correlation is associated with a static model. To sum up, the pattern of implied jump size in our one-parameter dynamic model resembles the base correlation of the static model.

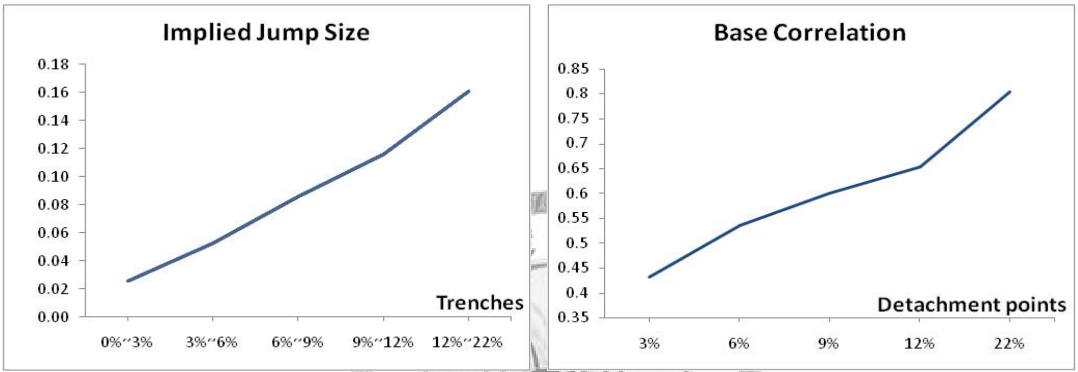

Figure 4.3 Implied jump size of our one-parameter model (left) compared with the base correlation (right) from the 7-year tranches quotes of iTraxx on April 2, 2008.

Third, we use the optimization numerical method to calibrate the parameters H_0 and β of our two-parameter model, simultaneously. That is to minimize the sum of squared differences between market tranche spreads and model tranche spreads. The procedure involves repeatedly

I. Choosing trial values of H_0 and β ;

II. Calculating the sum of squared differences between model spreads and market spreads for all tranches of all maturities available.

This iterative procedure is used to find the values of H_0 and β . Two parameters are used to match as closely as possible available tranche spreads and so there is always a unique optimal solution. We list in Table 4.4 the calibration result and the values of the calibrated parameters.

For the iTraxx data in Table 2.1 the best fit parameter values are $H_0 = 0.046750$ and $\beta = 1.835630$. The pricing errors are shown in Table 4.4. The model fits market data much better than versions of the one-parameter model where the jump size is constant for any economic shock.

a	b	Maturities			
		3 yr	5 yr	7 yr	10 yr
0%	3%	n/a	44.35	36.53	31.50
3%	6%	n/a	65.56	–88.86	n/a
6%	9%	n/a	–74.41	–104.85	–131.83
9%	12%	n/a	n/a	–6.84	n/a
12%	22%	n/a	n/a	3.87	–3.47

Table 4.4 Errors resulting from calibration of our two-parameter model to the iTraxx data on April 2, 2008. (For example, the quote for the 12% to 22% 7-year tranche is 90bps and the model spread is 93.87bps.)

Calibrating to the iTraxx data on April 2, 2008, the values of H_j in our two-parameter model are initially small, but increase fast; i.e, $H_1 = 0.2931$, $H_2 = 1.8373$, and $H_3 = 11.5184$. Thus the survival probability decreases fast when the economic shock increases. The value of H_0 indicates the initial survival probability and the value of β discovers the velocity of decreasing. There is a small probability of low values of S being reached. For the intensity function calibrated from the data on April 2, 2008, the probability that S at the end of 5 years is about 93.21%. This is also consistent with the original model of the results in papers such as Hull and White (2006), which show that it is necessary to assign a very low, but non-zero, probability to a very high hazard rate in a static model in order to fit market quotes.

We try to observe the fluctuation of the parameters from calibrating to all the available iTraxx tranche data between March 27, 2008 and April 2, 2008. This data includes the spreads on 3-, 5-, 7- and 10-year CDO tranches as well as 3- to 10- year index spreads. Among these days, the number of available CDO tranche spreads is between 11 and 15. The jump parameter values are showed in Figure 4.4, where we can infer that the two parameters fluctuate in opposite way.

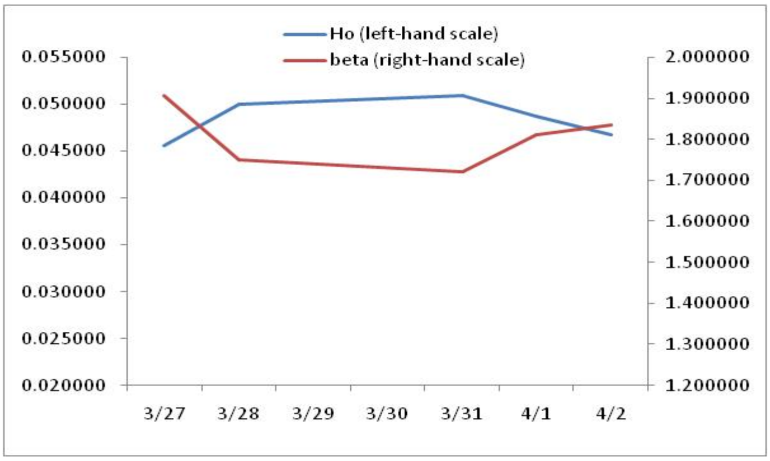

Figure 4.4 Jump parameters H_0 and β calibrated to iTraxx data using our two-parameter model between March 27, 2008 and April 2, 2008.

Compared with the original model of Hull and White (2007), we calibrate the two parameters of our two-parameter model to the data in Table 4.5, which is the iTraxx CDO quotes on January 30, 2007 from Hull and White (2007). In addition, the best fit parameter values of their three-parameter model are $H_0 = 0.00223$, $\beta = 0.9329$ and $\lambda = 0.1310$, and their pricing errors are shown in Table 4.6. The best fit parameter values of our two-parameter model are $H_0 = 0.03569$ and $\beta = 1.42379$, and our pricing errors are shown in Table 4.7. It turns out that our model is not as good as the original model of Hull and White (2007), although our model and the procedure of calibration make more economic sense. Our model's spreads are close to the market spreads only for some tranches, such as the 12%~22% tranche of 5-year and 7-year. For the senior tranche of 3%~6% our model's error is quite large. The model of Hull and White (2007) fits market data well for almost all the tranches quotes.

	Attach point	Detachment point	Maturities			
			3 yr	5 yr	7 yr	10 yr
Index			15.00	23.00	31.00	42.00
Tranche	0%	3%	n/a	10.25	24.25	39.30
	3%	6%	n/a	42.00	106.00	316.00
	6%	9%	n/a	12.00	31.50	82.00
	9%	12%	n/a	5.50	14.50	38.25
	12%	22%	n/a	2.00	5.00	13.75

Table 4.5 iTraxx CDO tranche quotes in basis points on January 30, 2007.

a	*b*	Maturities			
		3 yr	5 yr	7 yr	10 yr
0%	3%	n/a	1.34	2.75	4.32
3%	6%	n/a	0.37	3.12	–1.37
6%	9%	n/a	–0.54	–2.69	–1.92
9%	12%	n/a	–1.01	–1.55	–0.12
12%	22%	n/a	–0.47	–0.21	1.28

Table 4.6 Errors in basis points resulting from calibration of three-parameter model of Hull and White (2007) to the iTraxx data in Table 4.5 on January 30, 2007.

a	*b*	Maturities			
		3 yr	5 yr	7 yr	10 yr
0%	3%	n/a	–48.44	–32.48	–15.12
3%	6%	n/a	–85.96	–13.11	194.89
6%	9%	n/a	–21.99	–14.25	20.01
9%	12%	n/a	–10.65	–7.45	8.00
12%	22%	n/a	0.92	3.12	10.09

Table 4.7 Errors in basis points resulting from calibration of our two-parameter model to the iTraxx data in Table 4.5 on January 30, 2007.

Finally, we use the jump parameters calibrated to the iTraxx market quotes on April 2, 2008 to compare the results of model spreads generated by the analytical method with those obtained by the binomial tree method and the Monte Carlo simulation. The results are presented in Table 4.8.

Model spreads (bps)	Tranche of 6% ~ 9% on April 2, 2008		
	5 yr	7 yr	10 yr
Analytical method	170.59	175.15	178.17
Binomial tree method	169.95	174.96	178.02
Monte Carlo method	172.05	176.73	180.06

Table 4.8 Model spreads for 6%~9% tranche for different methods on April 2, 2008.

For the binomial tree method, we choose proper size of m to let the spreads converge to the analytical one as closely as possible. They turn out to be 60, 196 and 360 for the maturity of 5, 7 and 10 year, respectively. For the Monte Carlo method, we simulate the number of economic shocks for one million times and calculate the average value. The model spreads obtained by the binomial tree method and the

Monte Carlo simulation are both close to the values generated by the analytical method, it corresponds to the fact that it should converge theoretically as we mentioned in section 3.2.2 and 3.2.3. However, Monte Carlo simulation is time-consuming, especially when the intensity is very small which causes low probability of economic shocks.

Chapter 5

Conclusion and Future Work

This thesis presents a revised dynamic model based on Hull and White (2007). Their pricing algorithms are also revised to valuate portfolio credit derivatives. Our justification for the modification is that our model and way of calibration make more economic sense. We also use the result that the binomial distribution converges to Poisson distribution for a mathematical proof to support the procedure of representing the model in the form of a binomial tree. Although the numerical result for matching market quotes is not as well as the original model of Hull and White (2007), it captures the other specificities and advantages. For example, our model is a dynamic model which can be represented as a binomial tree. It is simple and easy to implement.

This thesis highlights further research in the future. The first is to use the tree algorithm to price exotic credit portfolio derivatives such as tranche options, forward tranches, leveraged super-senior etc. The second is to develop an efficient Monte Carlo simulation whose implementation of the model can price more strongly path-dependent credit derivatives. Finally, we consider it worthwhile to develop the revised dynamic model further to make it provide a good fit to CDO quotes of all maturities.

Appendix A

Poisson Process

Poisson processes are usually used to model either rare events or discretely countable events. Both properties let Poisson processes apply to the rare and discrete default event. A Poisson process with intensity $\lambda > 0$ is a non-decreasing, integer-values process with initial value $N(0) = 0$ whose increments are independent and satisfy, for all $0 \le t < T$

$$P[N(T) - N(t) = n] = \frac{1}{n!}(T-t)^n \lambda^n \exp[-(T-t)\lambda].$$

Here are some further properties:

1. The Poisson process has no memory. The probability of n jumps in $[t, t+s]$ is independent of $N(t)$ and the history of N before t.
2. Let $\tau_1, \tau_2, \ldots, \tau_m, \ldots$ be the first, second etc. jump times of N. Then $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$, in other words, the times between any jump and the subsequent one, are exponentially distributed.
3. Two or more jumps at exactly the same time have probability zero.

In order to reach a more realistic shape of the spread curve we must allow the default intensity to change over time. We consider time-varying intensity $\lambda(t)$, which is assumed to be a positive and piecewise right-continuous function. Define

$$\Gamma(t) = \int_0^t \lambda(u) du,$$

the cumulated intensity. A time inhomogeneous Poisson process with intensity function $\lambda(t) > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy

$$P[N(T) - N(t) = n] = \frac{1}{n!}[\Gamma(T) - \Gamma(t)]^n \exp\{-[\Gamma(T) - \Gamma(t)]\}.$$

It properties are similar to the properties of a homogeneous Poisson process.

If M_t is a Poisson process with intensity one, then a time-inhomogeneous Poisson process N_t with intensity $\lambda(t)$ is defined as $N_t = M_{\Gamma(t)}$. So a time inhomogeneous Poisson process is just a time-changed Poisson process with intensity one. From $N_t = M_{\Gamma(t)}$ we have obviously that N jumps the first time at τ if and only if M jumps the first time at $\Gamma(\tau)$. Since we know that M is a Poisson process with intensity one for which the first jump time is exponentially distributed, we have $\Gamma(\tau) = \xi \sim$ exponential(1), in other words, $\tau = \Gamma^{-1}(\xi)$, with ξ is a standard exponential random variable. Also, we have easily

$$\begin{aligned} P[s < \tau < t] &= P[\Gamma(s) < \Gamma(\tau) < \Gamma(t)] \\ &= P[\Gamma(s) < \xi < \Gamma(t)] \\ &= P[\xi > \Gamma(s)] - P[\xi > \Gamma(t)] \\ &= \exp(-\Gamma(s)) - \exp(-\Gamma(t)), \end{aligned} \tag{A.1}$$

which is the probability of the first jump to occur between *s* and *t*.

In particular, the formula

$$P(\tau > t) = \exp(-\Gamma(t)) = \exp\left(-\int_0^t \lambda(u) du\right) \tag{A.2}$$

tells us the probability of the first jump to occur after *t*. Eqs. (A.1) and (A.2) are applicable to calculate the spread of credit derivatives by Eq. (2.2), the breakeven spread of a CDS contract at the end of section 2.2.

Appendix B

Expectation of Hazard Rate

We check whether the expectation of hazard rate from binomial tree is approach to the value we calculate from analytical method when the partition is dense enough.

Let $N(t)$ be the Poisson process with intensity λ. The jump size, H, defined in the dynamic model is the function of $N(t)$, that is $H(N(t)) = H_0 e^{\beta N(t)}$, where H_0 and β are positive constants. Then the hazard rate, $X(t)$, is equal to $\sum_{N(t)=0}^{N(t)} H(N(t))$.

From analytical method,

$$
\begin{aligned}
\mathrm{E}[X(t)] &= \sum_{k=0}^{\infty} \mathrm{E}\left[\sum_{N(t)=0}^{k} H_0 e^{\beta N(t)} \middle| N(t)=k \right] \mathrm{P}(N(t)=k) \\
&= \sum_{k=0}^{\infty} \left(\sum_{N(t)=0}^{k} H_0 e^{\beta N(t)} \right) \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \sum_{k=0}^{\infty} \frac{H_0\left(e^{(k+1)\beta}-1\right)}{e^{\beta}-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
&= \frac{H_0 e^{-\lambda t}}{e^{\beta}-1} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \left(e^{(k+1)\beta}-1\right) = \frac{H_0 e^{-\lambda t}}{e^{\beta}-1} \left[e^{\beta} \sum_{k=0}^{\infty} \frac{\left(e^{\beta}\lambda t\right)^k}{k!} - \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \right] \\
&= \frac{H_0 e^{-\lambda t}}{e^{\beta}-1} \left[e^{\beta} e^{e^{\beta}\lambda t} - e^{\lambda e} \right] = \frac{H_0}{e^{\beta}-1} \left[\exp\left(\beta + \lambda t\left(e^{\beta}-1\right)\right) - 1 \right]
\end{aligned}
$$

From fitting binomial tree, we use parameters defined in section 3.2 except that λ is constant for simplifying the proof as above.

$$
\begin{aligned}
&\lim_{m\to\infty} \sum_{k=0}^{m} \binom{m}{k} (\lambda h_m)^k (1-\lambda h_m)^{m-k} \sum_{N(t)=0}^{k} H(N(t)) \\
&= \lim_{m\to\infty} \sum_{k=0}^{m} \binom{m}{k} \left(\lambda \frac{t}{m}\right)^k \left(1-\lambda \frac{t}{m}\right)^{m-k} \frac{H_0\left(e^{(k+1)\beta}-1\right)}{e^{\beta}-1} \\
&= \frac{H_0}{e^{\beta}-1} \lim_{m\to\infty} \left[e^{\beta} \sum_{k=0}^{m} \binom{m}{k} \left(e^{\beta}\lambda \frac{t}{m}\right)^k \left(1-\lambda \frac{t}{m}\right)^{n-k} - \sum_{k=0}^{m} \binom{m}{k} \left(\lambda \frac{t}{m}\right)^k \left(1-\lambda \frac{t}{m}\right)^{n-k} \right] \\
&= \frac{H_0}{e^{\beta}-1} \lim_{m\to\infty} \left\{ e^{\beta} \left[e^{\beta}\lambda \frac{t}{m} + 1 - \lambda \frac{t}{m} \right]^m - \left[\lambda \frac{t}{m} + 1 - \lambda \frac{t}{m} \right]^m \right\} \\
&= \frac{H_0}{e^{\beta}-1} \lim_{m\to\infty} \left\{ e^{\beta} \left[1 + \frac{\lambda t\left(e^{\beta}-1\right)}{m} \right]^m - 1^m \right\} \\
&= \frac{H_0}{e^{\beta}-1} \left[e^{\beta} \exp\left(\lambda t\left(e^{\beta}-1\right)\right) - 1 \right] = \frac{H_0}{e^{\beta}-1} \left[\exp\left(\beta + \lambda t\left(e^{\beta}-1\right)\right) - 1 \right]
\end{aligned}
$$

The limit of the expectation of hazard rate from fitting binomial tree agrees with the value from analytical method.

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