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控制變異數法在美式選擇權之應用

Variance Reduction Methods for Monte Carlo Valuation of American Options

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摘要

自從 Longstaff and Swartz (2001)提出的最小平方估計法 (least-squares Monte Carlo)，解決了蒙地卡羅模擬法難以用於美式選擇權之訂價的一大缺點。於是，蒙地卡羅模擬法簡單、易懂，且易於應用至多資產商品的特性，使得蒙地卡羅模擬廣泛地被用於選擇權的評價問題上。然而，蒙地卡羅模樣通常需要大量的模擬路徑，才能得到較好的估計；這使得評價變得極為耗時。

本研究即是探討兩種降低變異的方法，希望能藉此提昇蒙地卡羅的模擬效率。這兩種降低變異的方法分別是由 Rasmussen (2005)以及 Duan and Simonato (2001)所提出來的。本研究將之分別應用到美式賣權及極大值買權的評價，結果發現由 Rasmussen (2005)所提出來的方法，皆能有效地降低模擬的變異程度。

Abstract

For many complex options, analytical solutions are not available. In these cases a Monte Carlo simulation is computationally inefficient, the variance reduction method can be used to improve the efficiency of a Monte Carlo simulation.

In this thesis we apply the two variance reduction methods proposed by Rasmussen (2005) and Duan and Simonato (1998) in American option pricing. We find that the variance reduction method proposed by Rasmussen can provide significant improvement of efficiency than Duan and Simonato even the combination of these two methods does not perform better than only using the variance reduction methods proposed by Rasmussen. We also apply this variance reduction method proposed by Rasmussen in the valuation of two-, three- or five max-call options and we find that they can provide significant improvement both on efficiency and accuracy for pricing.

Content

1. Introduction

Closed-form pricing formulas have been derived for many European options under a variety of financial models, the most notable being the Black-Sholes formula under the geometric Brownian motion model. American-type options are options with flexible early exercise features. Examples are American equity and fixed-income options and convertible bonds. These contracts arise in virtually all major financial markets. However, when the option is American-type, the possibility of early exercise should be considered for the determination of the optimal early exercise policy. This often leads to highly complicated calculations.

There is a long and rich history of numerical methods for pricing American-style contingent claims. Among the earliest approaches are the binomial lattice of Cox et al. (1979) and the explicit finite difference scheme of Brennan and Schwartz (1977). These methods work particularly well for American options on a single underlying asset. However, many American-style options have been introduced that depend on multiple underlying assets or state variables. Multidimensional generalizations of the Cox et al. binomial method were proposed in Boyle (1988), Boyle et al. (1989), and others. A related approach involves extensions of the finite difference method to higher dimensions were proposed by Mitchell and Griffiths (2001). Adapting binomial, trinomial, or finite difference methods to higher dimensions works well for options on two or three state variables, but because their computational effort grows exponentially with the number of state variables, these methods are impractical for higher dimensional problems. This is the so-called curse of dimensionality. In contrast, simulation methods do not suffer from this difficulty. This thesis considers adapting the Monte Carlo approach for pricing American options.

The conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte Carlo techniques. Boyle (1977) first proposed Monte Carlo simulation for the pricing of European claims. However, it was not until much later that the possibility of using Monte Carlo simulation for pricing American-style options was suggested by Bossaerts (1989) and Tilley (1993). Tilley ranked simulated stock price from the maximum to minimum and divided them into several groups. He computed the holding value by averaging the discounted payoff within each group and used these holding values to find an exercise boundary. In this method, stock price is the only factor determining whether to exercise or not. Barraquand and Martineau (1995) developed a method which is closely related to that of Tilley (1993) but easier to extend. The idea is to partition the state space of simulated paths into a number of cells in such a way that the payoff from the option is approximately equal across the paths in the particular cell. The probabilities of moving to different cells in the next period conditional on the current cell can then be calculated from the simulated paths. With these probabilities in place, the expected value of keeping the option alive until the next period can be calculated, and a strategy for exercise determined. Broadie and Glasserman (1997) proposed a convergent algorithm based on simulated trees. Their method generated both lower and upper bounds so that valid confidence intervals on the true Bermudan price can be determined. The simulation tree method removes the exponential dependence of the computation time (CPU time) on the problem dimension; however, the CPU time is still exponential in the number of exercise opportunities. A new and somewhat simpler simulation based method to price American options has recently been proposed by Longstaff and Schwartz (2001). The idea is to estimate the conditional expectation of the payoff from keeping the option alive at each possible exercise point from a simple least squares cross-sectional regression using the simulated paths. They show how to price different types of path-dependent options using this least-squares Monte Carlo (LSM) approach.

In these papers, authors introduced numerical methods based on Monte Carlo techniques. The starting point of these methods is to replace the time interval of exercise dates by a fixed finite subset. This amounts to approximating the American option by a Bermudan option, i.e., options with discretely exercisable features not continuous ones. The solution of the discrete optimal stopping problem reduces to an effective implementation of the dynamic programming principle.

Although there are many works about least squares regression methods, most have paid only little attention to the issue of variance reduction. Only a few articles go beyond applying antithetic variates. In this thesis we consider the application of control variates to the valuation of American- or Bermudan-type options. In Broadie and Glasserman (1997), the European option's payoff at expiry is used as a control variate. They found that the control variates work quite well for out-of-the-money options, but are less effective for deep-in-the-money options. They also found that payoff processes of European options and American options are less correlated when American options are deep-in-the-money than when American options are out-of-the-money. The reason is that out-of-the-money American options might have lower probability to be exercised than deep-in-the-money American options. As a result, European options are highly correlated with out-of-the-money American options but less correlated with deep-in-the-money American options. In other words, traditional European option is not a very good control variate for deep-in-the-money options.

Another idea is based on a simple observation that simulated sample paths for the underlying asset price almost always fail to possess the martingale property even though the theoretical model uses the assumption of martingale. The failure to ensure the martingale property has particularly serious consequences in the later time interval when there is more time division. It often requires a very large number of simulation repetitions to dampen these simulation errors. Duan and Simonato (1998) proposed a correction to the standard procedure by ensuring that the simulated sample paths all satisfy the martingale property in each time interval. This correction is referred to as empirical martingale simulation (EMS).

In this thesis we apply the two variance reduction methods proposed by Rasmussen (2005) and Duan and Simonato (1998) in American option pricing. We also compare these two methods and combine them. We find that this new variance reduction method, which combines the above two methods, cannot provide significant improvement of efficiency and accuracy for pricing. Then we also apply variance reduction methods in the valuation of max-call options and we find that they can provide significant improvement on efficiency and accuracy for pricing.

2. The American Option Valuation Problem

The problem of valuing an American option consists of finding an optimal exercise strategy and then valuing the expected discounted payoff from this strategy under the equivalent martingale measure.[1] We let V_r denote the time t solution to this problem, that is,

$$\frac{V_t}{\beta_t} = \sup_{\tau \in \mathcal{T}(t,T)} \mathrm{E}^Q \left[\frac{X_\tau}{\beta_\tau} \mid \mathcal{F}_t \right]$$

where $\{X_t\}_{0 \le t \le T}$ is the payoff process adapted to the filtration and $\mathcal{T}(t,T)$ denotes the set of stopping time τ satisfying

$$t \le \tau \le T$$

We can easily define a lower bound on the American option price at time t denoted by L_t, since for any given exercise strategy or stopping time τ we have

$$\frac{L_t}{\beta_t} = \mathrm{E}^Q \left[\frac{X_\tau}{\beta_\tau} \mid \mathcal{F}_t \right] \le \frac{V_t}{\beta_t}$$

For an upper bound we refer the reader to Theorem 1 of Anderson and Broadie

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[1] In what follows we assume that the financial market is defined for the finite horizon [0, T] on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathrm{P})$. Here the state Ω is the set of all realizations of the financial market, $\mathcal{F}$ is the sigma algebra of events at time T, and P is a probability measure defined on $\mathcal{F}$. The filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ is assumed to be generated by the price processes of the financial market and augmented with the null sets of $\mathcal{F}$, and assuming $\mathcal{F}_T = \mathcal{F}$. We furthermore assume that using the numeraire process $\{\beta_t\}_{0 \le t \le T}$ there exists a measure Q equivalent to P under which all asset prices relative to the numeraire are martingales.

(2004) and leave out the details here.

In the following sections we give numerical results based on the single-asset American put option, using the same combinations of underlying asset prices, time to expiry and volatilities as in Table 1 of Lonstaff and Schwartz (2001). The payoff process of the single-asset put option for strike K is given by

$$X_t = \max(K - S_t, 0)$$

In this case, holders can exercise on the following set of equidistant points only, $t_e = (e/d)T$, for $e = 0,1,2,\ldots,d$ and is hence a Bermudan rather than an American option.

3. Monte Carlo Valuation with Variance Reduction Method

3.1. Control Variates for Monte Carlo Simulation (Rasmussen (2005))

Given a stopping time $\tau \in T\,(t,T)$, we want to determine the following conditional expectation given the information at time t:

$$\frac{L_t}{\beta_t} = E^Q\left[\frac{X_\tau}{\beta_\tau} \,|\, \mathcal{F}_t\right] \tag{1}$$

Using the underlying model to generate N independent paths of the variables determining the payoff process $\{X_t\}_{0 \le t \le T}$ and the numeraire process $\{\beta_t\}_{0 \le t \le T}$, the crude Monte Carlo estimate is

$$\frac{L_t^{(N)}}{\beta_t} = \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i}$$

where X_τ^i / β_τ^i is the discounted payoff from ith path using the exercise strategy given by the stopping time τ.

To reduce the variance of the Monte Carlo estimate of the American option, we can replace the path estimate X_τ^i / β_τ^i with the following path estimate

$$\frac{Z_\tau^i}{\beta_\tau^i} = \frac{X_\tau^i}{\beta_\tau^i} + \theta_t\left(Y^i - E_t^Q\left[Y^i\right]\right) \tag{2}$$

for some appropriately chosen $\mathcal{F}_t$-measurable random variable θ_t, where Y^i is the

ith observation of a random variable for which we can easily compute the time t conditional expectation. The Monte Carlo estimate using control variates is then given by

$$\frac{L_t^{(N)}}{\beta_t} = \frac{1}{N}\sum_{i=1}^{N}\frac{Z_\tau^i}{\beta_\tau^i}$$
$$= \frac{L_t^{(N)}}{\beta_t} + \theta_t\left(Y_t^{(N)} - E_t^Q\left[Y^i\right]\right)$$

where

$$\frac{L_t^{(N)}}{\beta_t} = \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i},\ Y_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N}Y^i$$

By the standard ordinary least-squares theory, the optimal choice of θ_t is

$$\theta_t^* = -\frac{\mathrm{Cov}_t^Q\left[X_\tau/\beta_\tau, Y\right]}{\mathrm{Var}_t^Q\left[Y\right]} \tag{3}$$

which results in the following minimum variance

$$\mathrm{Var}_t^Q\left[\frac{L_t^{(N)\mathrm{CV}}}{\beta_\mathrm{t}}\right] = \frac{1}{N}\mathrm{Var}_t^Q\left[\frac{X_\tau}{\beta_\tau}\right]\left(1-\rho_t^2\left[X_\tau/\beta_\tau, Y\right]\right)$$

where

$$\rho_t\left[\frac{X_\tau}{\beta_\tau}, Y\right] \equiv \frac{\mathrm{Cov}_t^Q\left[X_\tau/\beta_\tau, Y\right]}{\sqrt{\mathrm{Var}_t^Q\left[X_\tau/\beta_\tau\right]\cdot\mathrm{Var}_t^Q\left[Y\right]}}$$

Hence the most effective control variates Y are obtained by having the largest possible correlation, either positive of negative, with the discounted payoff from the

Bermudan option.

A good control variate should have the following two properties: it should be highly correlated with the payoff of the option in question and its conditional expectation should be easy to compute. When looking for a control variate for the Bermudan option, the corresponding European option would be our general choice. Let W_T be the value of a self-financing portfolio at expiry of the Bermudan option. Using the discounted value of this portfolio, the European option control variate Y_T is then defined by

$$Y_T = \frac{W_T}{\beta_T}$$

By construction of the equivalent martingale measure Q, the process $\{Y_t\}_{0 \le t \le T}$ defined by

$$Y_t = E^Q[Y_T \mid \mathcal{F}_t]$$

is a martingale.

The optimal European option control variates is given by

$$Y_T = \frac{X_T}{\beta_T} \tag{4}$$

for which we have

$$Y_t = E^Q\left[\frac{X_T}{\beta_T} \mid \mathcal{F}_t\right] = \frac{C_t}{\beta_t} \tag{5}$$

the discounted European option price. This control variate clearly satisfies the second properties mentioned; however, its correlation with the payoff of the Bermudan option is not high enough, especially for in-the-money options. Therefore, we replace the control variate (4) with the control variates

$$Y_\tau = \frac{W_\tau}{\beta_\tau} = E^Q\left[\frac{W_T}{\beta_T} \mid \mathcal{F}_\tau\right] \tag{6}$$

Thus, rather than sampling the discounted payoff process at expiry of the option, Rasmussen (2005) suggests sampling the discounted value process at the time of exercise of the Bermudan option. The conditional expectation of control variates (4) is

$$E^Q\left[Y_\tau \mid \mathcal{F}_t\right] = Y_t$$

which is identical to the expectation of the control variate (4).

It is intuitively true that European option and American option payoffs are less correlated when the options are in-the-money. From an option holder's viewpoint, the estimate accuracy is of greatest importance for in-the-money options because this is where the critical exercise decisions have to be made. As a result, Rasmussen (2005) proposed a choice of control variates (4) for American and Bermudan options.

He has shown that the application of the control variates to the valuation of American or Bermudan options can be very effective if we sampled these control variates at the exercise time rather than at expiry.

3.2. Empirical Martingale Simulation (Duan and Simonato (1998))

The theoretical works for contingent claim pricing mostly rely on the absence of arbitrage opportunities. The martingale connection to the arbitrage-free price system was first observed by Cox and Ross (1976) and later formalized by Harrison and Kreps (199). For the ease of exposition, we consider a price system consisting of two securities, one risky and the other risk-free. The risky security, a common stock, does not pay dividends and its price, denoted by $S(t)$, has the following dynamics under the risk-neutral probability measure *Q*:

$$s(t) = S_o \exp\left[\int_0^t \left(r - 0.5\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW(s)\right]$$

where r is the continuously compounded return on the risk-free security, $\sigma(s)$ is the instantaneous standard deviation of the asset return and $W(s)$ is a standard Brownian motion under probability measure *Q*. It is easy to verify that the discounted asset price is indeed a *Q*-martingale in that, for any $\tau \geq t \geq 0$,

$$E^Q\left[e^{-r\tau}S(\tau) \mid F_t\right] = e^{-rt}S(t)$$

where $E^Q[\cdot]$ denotes the expectation operator under the risk-neutral measure Q and $\mathcal{F}_t$ the information filtration up to time t.

In a typical Monte Carlo simulation, this martingale property almost always fails in the simulated sample. Valuation often requires a very large number of simulation repetitions to dampen simulation errors. Duan and Simonato (1998) proposed a simple transformation for the original simulation asset process and adjusted them to satisfy the martingale property. Their transformation steps are listed below:

(1) Define the discrete times by $t_0, t_1, t_2, \ldots, t_m$, where t_0 is the current time.

(2) Simulate asset prices at times $t_0, t_1, t_2, \ldots, t_m$ for each simulated path $i = 1,2,\ldots,N$ and define the ith simulated asset price at time t_j by $S^i(t_j)$.

(3) Compute the ith simulated asset returns at time t_j defined as

$$R^i(t_j) = \frac{S^i(t_j)}{S^i(t_{j-1})}, \text{ for all } j = 1,2,\ldots,m; i = 1,2,\ldots,N$$

(4). Set $S^i(t_0) = \widehat{S}^i(t_0) = S_0$ for all $i = 1,2,\ldots,N$ In this equation, $\widehat{S}^i(t_j)$ is the adjusted EMS asset price at the ith sample path and at time t_j and S_0 is the initial asset price at time t_0.

(5). Do a recursive procedure for all $j = 1,2,\ldots,m$; $i = 1,2,\ldots,N$

$$Z^i(t_j) = \widehat{S}^i(t_{j-1}) \cdot R^i(t_j)$$

$$Z^0(t_j) = e^{-rt_j} \frac{1}{N}\sum_{i=1}^{N} Z^i(t_j)$$

$$\widehat{S}^i(t_j) = S_0 \frac{Z^i(t_j)}{Z^0(t_j)}$$

where r is the risk-free interest rate.

Although they had proposed a correction to the standard Monte Carlo simulation procedure and imposed the martingale property on the collection of simulated sample paths, in their paper they only applied this modification within some specific parameters (e.g., maturities less than 1 year) and European-type options. This thesis will apply the modification to American options with longer maturities and different volatility parameters and combine this simulation adjustment to the control variates method proposed by Rasmussen (2005) for American options.

4. The LSM Approach with Control Variates

The main assumption of the LSM approach is that the time t conditional expectation in (1) can be expressed as a linear combination of a countable set of $\mathcal{F}_t$-measurable basis functions as follows:

$$\frac{L_t}{\beta_t} = \sum_{j=1}^{\infty} a_t^j F_t^j \tag{7}$$

The implement this assumption in the LSM approach, (7) can be approximated with a finite sum at a given level. We let L_t^M / β_t denote this approximation when M basis functions are used. We then have

$$\frac{L_t^M}{\beta_t} = \sum_{j=1}^{M} a_t^j F_t^j \tag{8}$$

Using cross-sectional observations of the Monte Carlo generated state variable (such as stock price), the coefficients a_t^j, $j = 1,2,\ldots,M$, are determined by least-squares regression, where the current basis functions F_t^j, $j = 1,2,\ldots,M$, are independent variables and the discounted payoff process X_t/β_t is the dependent variable. As argued by Longstaff and Schwartz (2001), we only need the approximation where the option is in-the-money at time t and include these observations in the regression. With a total of N Monte Carlo generated paths, we let $a_t^{j(N)}$, $j = 1,2,\ldots,M$, denote the time t coefficients determined from the regression using only the in-the-money paths. Hence the approximation used in the implementation denoted by $L_t^{M(N)}/\beta_t$ is given by

$$\frac{L_t^M}{\beta_t} = \sum_{j=1}^{M} a_t^{j(N)} F_t^{\,j} \tag{9}$$

Given the time t value of the state variable summarized by the basis functions $F_t^{\,j}$, j=1, 2, …, M, we exercise the option whenever the time t exercise value exceeds the corresponding conditional expectation. In other words, we will exercise when the following inequality is satisfied

$$\frac{X_t}{\beta_t} \geq \frac{L_t^M}{\beta_t}$$

4.1. The Choice of Basis Functions

For the sake of simplicity and to maintain some financial intuition, we use the following basis functions corresponding to $M = 3$. All functions depend on the current time t and the current stock price S_t, i.e., $F_t^{\,j} = f_j(t, S_t)$, where

$$\begin{aligned} f_0(t,s) &= K \\ f_1(t,s) &= s \\ f_2(t,s) &= C(t,s) \\ f_3(t,s) &= s \cdot C(t,s) \end{aligned}$$

Above, K is the strike price of the Bermudan put option, s is the current asset price and $C(t,s)$ is the time t European put option price expiring at time T and with its other parameters identical to the Bermudan option.

4.2. Accuracy, Stability and Convergence of the LSM Approach

The accuracy of the LSM exercise strategy is solely determined by the accuracy of the following two approximations:

$$\frac{L_t}{\beta_t} \cong \frac{L_t^M}{\beta_t}$$
$$\frac{L_t^M}{\beta_t} \cong \frac{L_t^{M(N)}}{\beta_t}$$

With an infinite computational budget, we should require that $L_t^{M(N)} \to L_t$ as $N \to \infty$ and $M \to \infty$. However, in Clement *et al* (2002), the convergence of $L_t^M \to L_t$ as $M \to \infty$ is established when an orthogonal set of basis functions is used. They also considered the convergence of $L_t^{M(N)} \to L_t$ as $N \to \infty$. Finally, they established a central limit result for the rate of convergence of the LSM algorithm. These results are very important especially from a theoretical point of view. However, from a practical point of view, we are more concerned with the performance given a finite sample of *N* paths and a finite number of basis functions *M*. Hence we focus in the following section on improving the above approximation for a given number of paths and basis functions. The object is to replace (8) with a more accurate approximation.

4.3. Improvement of LSM with Control Variates

In traditional LSM approach, we only choose the discounted payoff X_t/β_t of in-the-money paths to do regression for evaluating the conditional expectation value

$$\frac{L_t}{\beta_t} = E^Q\left[\frac{X_\tau}{\beta_\tau} \mid \mathcal{F}_t\right]$$

as in (1). In this section, we reuse and generalize the concept of control variates as the Monte Carlo variation in section 3.1. We replace the discounted payoff X_t/β_t with the random variable Z_t/β_t :

$$\frac{Z_\tau}{\beta_\tau} = \frac{X_\tau}{\beta_\tau} + \theta_t\left(Y_\tau - E_t^Q[Y_\tau]\right) \tag{10}$$

where Y_τ is the control variates sampled from European option's payoffs at exercise points as in (6), and θ_t is an appropriately chosen $\mathcal{F}_t$-measurable random variable. Rather than a point estimate as in (3), we now use a functional estimate of θ_t^*. By the definition of $\mathrm{Cov}_t^Q[X_t/\beta_t, Y_\tau]$ and $\mathrm{V}ar_t^Q[Y_\tau]$ we get

$$\begin{aligned}\theta_t^* &= -\frac{E_t^Q\left[\left(X_\tau/\beta_\tau\right)\cdot Y_\tau\right] - E_t^Q\left[X_\tau/\beta_\tau\right]\cdot \mathrm{E}_t^Q[Y_\tau]}{E_t^Q[Y_\tau^2] - \left(E_t^Q[Y_\tau]\right)^2} \\ &= -\frac{(LY)_t/\beta_t - \left(L_t/\beta_t\right)\cdot Y_t}{(Y^2)_t - (Y_t)^2}\end{aligned}$$

where L_t/β_t is defined in (1), Y_t is defined in (5), and $(Y^2)_t$ and $(LY)_t/\beta_t$ are defined by

$$\left(Y^2\right)_t \equiv E_t^Q\left[Y_\tau^2\right],$$

$$\frac{(LY)_t}{\beta_t} \equiv E_t^Q\left[\left(\frac{X_t}{\beta_t}\right) \cdot Y_\tau\right]$$

respectively. Generalizing the assumptions of LSM as (7), we assume that the time t conditional expectations Y_t, $\left(Y^2\right)_t$ and $(LY)_t / \beta_t$ can also be expressed as a countable sum of the same set of $\mathcal{F}_t$-measurable basis functions, i.e.,

$$Y_t = \sum_{j=1}^{\infty} b_t^j \cdot F_t^j,$$

$$\left(Y^2\right)_t = \sum_{j=1}^{\infty} c_t^j \cdot F_t^j,$$

$$\frac{(LY)_t}{\beta_t} = \sum_{j=1}^{\infty} d_t^j \cdot F_t^j$$

Again we can approximate the conditional expectations by truncating the sums at the same level M. Let Y_t^M, $\left(Y^2\right)_t^M$ and $(LY)_t^M / \beta_t$ denote the approximations as in (8).

Now we have

$$Y_t^M = \sum_{j=1}^{M} b_t^j \cdot F_t^j,$$

$$\left(Y^2\right)_t^M = \sum_{j=1}^{M} c_t^j \cdot F_t^j,$$

$$\frac{(LY)_t^M}{\beta_t} = \sum_{j=1}^{M} d_t^j \cdot F_t^j$$

We can approximate θ_t^* with θ_t^M given by

$$\theta_t^M = \frac{\left(LY\right)_t^M \big/ \beta_t - \left(L_t^M \big/ \beta_t\right) \cdot Y_t^M}{\left(Y^2\right)_t^M - \left(Y_t M\right)^2}$$

Using the concept of control variates in (10), we can approximate the time t conditional expectation of the discounted payoff from following the strategy τ by $L_t^{M,\mathrm{CV}}$ given by the following expression:

$$\frac{L_t^{M,\mathrm{CV}}}{\beta_t} = \frac{L_t^M}{\beta_t} + \theta_t^M \left(Y_t^M - Y_t\right)$$

5. Numerical Results

5.1. American Put

In this chapter, we use some numerical examples[2] to compare the efficiency and accuracy of three different variance reduction methods. First, we use normalized antithetic paths, i.e., we use traditional moment matching simulation and antithetic method, and denote this method as “AN+MM” in the following. Second, we replicate the simulated paths in the first method and use discounted European options sampling at exercise as control variates and denote this as “CV-at-exercise.” Third, we also replicate the simulated paths in the first method but only adjusting the underlying process to satisfy martingale property and denote this as “EMS” method.

In order to show the efficiency and accuracy of the AN+MM, CV-at-exercise and EMS, we use binomial-based put prices as the benchmark and compare their standard error (S.E.) and mean absolute percentage error (MAPE) to see the efficiency and accuracy of these three methods (AN+MM, CV-at-exercise and EMS). Also we will use ratios of S.E and MAPE between AN+MM and CV-at-exercise or AN+MM and EMS to see whether CV-at-exercise or EMS can improve the efficiency or accuracy of the LSM approach.

From the results of CV-at-exercise method in the second category in Table 1,

[2] These numerical examples are all for single-asset put options, with the same combinations of underlying asset prices, time to expiry and volatilities as in Table 1 of Longstaff and Schwartz (2001).

we find the standard errors in CV-at-exercise are only one tenth of the AN+NN method in the third column of the second category. These improvements are more significant when options are at-the-money ($S = 40$) or out-of-the-money ($S = 42,44$) and with shorter maturity ($t = 1$). This can be confirmed by observing the smaller ratio of S.E of CV divided by S.E of AN+MM listed in the third column of the second category. But from the next two columns, improvement in terms of MAPE is not very significant in CV-at-exercise method except when option is deep-out-of-the-money ($S = 44$). Also from the third column of third category in Table 1, we find that standard errors in the EMS method are not less than the AN+MM method but the MAPE in this method are smaller than AN+MM method when options are in-the-money ($S = 36,38$).

In Table 1, the improvements of standard errors using CV-at-exercise are quite significant. Now we try to combine CV-at-exercise and EMS and evaluate if the resulting method can improve the efficiency or accuracy in pricing Bermudan options compared with the AN+MM method. The results are shown in Table 2.

As before, we use the parameters of Longstaff and Schwartz (2001) in the following numerical examples whose results are listed in Table 2. In our second test, we combine the CV-at-exercise method and the EMS method, denoted by CV+EMS. We also compare the result of CV+EMS with CV-at-exercise and list the results in

Table 2. In order to more clearly compare the results, we find the ratios of their S.E and MAPE and list the results in Table 3. From the results in Table 3, we find only when $\sigma = 0.2$ and options are deep-in-the-money ($S = 36$), can CV+EMS obtain smaller S.E. than CV-at-exercise in all our maturity parameters. In other parameters, the improvement in terms of S.E. in CV+EMS method is not significant. Also, the improvement in terms of MAPE in CV+EMS is not very significant in most parameters of our test except when $\sigma = 0.5$ and options are deep-in-the-money ($S = 36$).

Table 1. This is Monte Carlo valuation of the Bermudan options. We use the binomial tree-based exercise strategy as the benchmark and compare three different variance reduction methods. First, we only use antithetic pair paths and do moment matching method (AN+MM) in the simulation. Second, we replicate the AN+MM and use European options sampling at exercise as control variates (CV-at-exercise). Last, we also replicate the AN+MM and use empirical martingale simulation (EMS) to adjust underlying asset processes. All simulations are based on 2000 pairs of antithetic paths. All options have strike K=40, and the interest rate equals r=0.06. The current stock price S, the volatility σ, and the time to expiry T are given.

S	t	σ	binomial model	AN+MM			CV-at-exercise					EMS				
				price	S.E. (%)	MAPE (%)	price	S.E. (%)	CV/ AN+MM	MAPE (%)	CV/ AN+MM	price	S.E. (%)	EMS/ AN+MM	MAPE (%)	EMS/ AN+MM
36	1	0.2	4.4845	4.4885	2.6433	0.0891	4.4726	0.2777	0.1051	0.2645	2.9693	4.4874	2.6627	1.0074	0.0642	0.7213
		0.4	7.0997	7.1392	4.1982	0.5568	7.0852	0.2724	0.0649	0.2043	0.3670	7.1143	4.9029	1.1678	0.2059	0.3699
	2	0.2	4.8512	4.8498	3.3725	0.0287	4.8474	0.5526	0.1639	0.0790	2.7514	4.8500	3.3443	0.9916	0.025	0.8713
		0.4	8.5310	8.5397	4.6569	0.1019	8.5328	0.6612	0.1420	0.0207	0.2032	8.5383	6.5843	1.4139	0.0857	0.8415
38	1	0.2	3.2529	3.2672	2.2367	0.4402	3.2420	0.2285	0.1022	0.3355	0.7622	3.2586	2.3066	1.0313	0.1747	0.3969
		0.4	6.1805	6.1843	4.2744	0.0610	6.1782	0.2400	0.0561	0.0372	0.6090	6.1816	6.0043	1.4047	0.0185	0.3034
	2	0.2	3.7546	3.7524	2.7642	0.0591	3.7564	0.4651	0.1683	0.0481	0.8138	3.7526	2.9799	1.0780	0.0542	0.9171
		0.4	7.6990	7.6943	4.7164	0.0605	7.6929	0.5717	0.1212	0.0792	1.3098	7.6983	7.0241	1.4893	0.0092	0.1515
40	1	0.2	2.3130	2.3369	2.6482	1.0347	2.2966	0.2157	0.0815	0.7081	0.6844	2.3388	2.8427	1.0735	1.1139	1.0766
		0.4	6.3028	5.3505	4.3863	15.1086	5.2802	0.2161	0.0493	16.2243	1.0738	5.3539	6.1141	1.3939	15.0553	0.9965
	2	0.2	2.8800	2.8947	2.7106	0.5089	2.8532	0.4146	0.1529	0.9298	1.8271	2.8951	3.1117	1.1480	0.5249	1.0315
		0.4	6.9036	6.9458	4.6295	0.6117	6.8654	0.5427	0.1172	0.5527	0.9035	6.9502	7.3925	1.5968	0.6755	1.1042
42	1	0.2	1.6239	1.6400	2.8891	0.9903	1.6163	0.1686	0.0583	0.4685	0.4731	1.6409	3.3076	1.1449	1.0492	1.0594
		0.4	4.6137	4.6182	4.8191	0.0982	4.6128	0.2047	0.0425	0.0192	0.1958	4.6252	6.9900	1.4505	0.2486	2.5310
	2	0.2	2.2249	2.2274	3.0713	0.1132	2.2277	0.4061	0.1322	0.1242	1.0971	2.2291	3.5747	1.1639	0.1908	1.6856
		0.4	6.2670	6.2690	4.8283	0.0312	6.2680	0.4679	0.0969	0.0157	0.5040	6.2682	8.1596	1.6899	0.0199	0.6403
44	1	0.2	1.1212	1.1300	2.6878	0.7858	1.1198	0.1496	0.0557	0.1210	0.1539	1.1309	2.9862	1.1110	0.8653	1.1011
		0.4	3.9616	3.9797	5.6508	0.4576	3.9502	0.1908	0.0338	0.2877	0.6288	3.9657	7.6675	1.3569	0.1034	0.2260
	2	0.2	1.6907	1.7082	3.1561	1.0339	1.6885	0.3569	0.1131	0.1295	0.1253	1.7088	3.6018	1.1412	1.0719	1.0367
		0.4	5.6781	5.6620	4.8480	0.2835	5.6764	0.3953	0.0815	0.0307	0.1081	5.6682	7.9300	1.6357	0.1749	0.6167

Table 2. This is Monte Carlo valuation of the Bermudan options. We use the binomial tree-based exercise strategy as the benchmark and compare two variance reduction methods. First we use CV-at-exercise method. Second, we combine CV-at-exercise and EMS method and denote this as EMS+CV method. All these simulations are based on 2000 pairs of antithetic paths. All options have strike K=40, and the interest rate equal to r=0.06. The current stock price S is given the same value as in Table 1. In order to see the effect of the volatility σ and the time to expiry T , we give additional values to these two parameters besides in Table 1.

		T=0.25							T=0.5						
σ	S	binomial model	CV	S.E (%)	MAPE	CV+ EMS	S.E (%)	MAPE	binomial model	CV	S.E (%)	MAPE	CV+ EMS	S.E (%)	MAPE
0.05	36	4.0000	4.0000	0.0000	0.0000	4.0000	0.0000	0.0000	4.0000	4.0007	0.0052	0.0002	4.0007	0.0052	0.0002
	38	2.0000	2.0000	0.0105	0.0000	2.0000	0.0105	0.0000	2.0000	1.9948	0.0190	0.0026	1.9948	0.0190	0.0026
	40	0.2245	0.2215	0.0549	0.0134	0.2215	0.0546	0.0134	0.2587	0.2541	0.0934	0.0178	0.2541	0.0949	0.0178
	42	0.0019	0.0018	0.0053	0.0618	0.0018	0.0051	0.0666	0.0082	0.0080	0.0141	0.0288	0.0080	0.0141	0.0284
	44	0.0000	0.0000	0.0000	*	0.0000	0.0000	*	0.0001	0.0001	0.0011	0.3318	0.0001	0.0011	0.3318
0.1	36	4.0000	4.0023	0.0171	0.0006	4.0023	0.0171	0.0006	4.0000	4.0018	0.0348	0.0004	4.0018	0.0348	0.0004
	38	2.0000	2.0003	0.7655	0.0001	2.0003	0.7629	0.0002	2.0123	2.0148	0.4546	0.0012	2.0148	0.4477	0.0012
	40	0.5861	0.5819	0.0494	0.0071	0.5819	0.0495	0.0071	0.7372	0.7304	0.1057	0.0092	0.7304	0.1064	0.0092
	42	0.1051	0.1051	0.0177	0.0003	0.1051	0.0197	0.0002	0.2173	0.2178	0.0570	0.0021	0.2177	0.0576	0.0021
	44	0.0104	0.0102	0.0140	0.0219	0.0102	0.0136	0.0208	0.0493	0.0489	0.0273	0.0085	0.0489	0.0262	0.0087
0.2	36	4.0480	4.0472	0.3861	0.0002	4.0472	0.3669	0.0002	4.2105	4.2122	0.3066	0.0004	4.2122	0.2947	0.0004
	38	2.4718	2.4701	0.0814	0.0007	2.4701	0.0785	0.0007	2.8242	2.8212	0.1404	0.0011	2.8212	0.1389	0.0011
	40	1.3527	1.3472	0.0507	0.0041	1.3472	0.0513	0.0041	1.7921	1.7825	0.0964	0.0053	1.7825	0.0959	0.0053
	42	0.6700	0.6685	0.0357	0.0022	0.6685	0.0353	0.0022	1.0928	1.0921	0.0801	0.0007	1.0921	0.0805	0.0007
	44	0.2937	0.2934	0.0273	0.0010	0.2934	0.0289	0.0010	0.6362	0.6361	0.0543	0.0001	0.6361	0.0603	0.0001
0.3	36	4.5098	4.5090	0.1621	0.0002	4.5090	0.1495	0.0002	5.0238	5.0219	0.1863	0.0004	5.0219	0.1737	0.0004
	38	3.1775	3.1732	0.0622	0.0013	3.1732	0.0614	0.0013	3.8403	3.8365	0.1376	0.0010	3.8365	0.1339	0.0010
	40	2.1314	2.1249	0.0468	0.0031	2.1249	0.0477	0.0031	2.8760	2.8646	0.1010	0.0039	2.8647	0.1002	0.0039
	42	1.3800	1.3787	0.0357	0.0009	1.3787	0.0372	0.0009	2.1361	2.1308	0.0836	0.0025	2.1308	0.0831	0.0025
	44	0.8569	0.8564	0.0355	0.0006	0.8563	0.0343	0.0006	1.5595	1.5579	0.0743	0.0010	1.5579	0.0813	0.0010
0.4	36	5.1170	5.1138	0.1156	0.0006	5.1138	0.1038	0.0006	5.9823	5.9777	0.1664	0.0008	5.9777	0.1570	0.0008
	38	3.9069	3.9033	0.0546	0.0009	3.9033	0.0528	0.0009	4.9086	4.9074	0.1267	0.0003	4.9074	0.1256	0.0003
	40	2.9128	2.9056	0.0440	0.0025	2.9056	0.0446	0.0025	3.9653	3.9526	0.0984	0.0032	3.9527	0.1008	0.0032
	42	2.1422	2.1377	0.0360	0.0021	2.1377	0.0357	0.0021	3.2317	3.2285	0.0924	0.0010	3.2286	0.0957	0.0010
	44	1.5469	1.5445	0.0417	0.0016	1.5445	0.0379	0.0015	2.5772	2.5753	0.0859	0.0007	2.5753	0.0878	0.0007
0.5	36	5.8042	5.8021	0.0719	0.0004	5.8021	0.0664	0.0004	6.9490	6.9351	0.1388	0.0020	6.9351	0.1354	0.0020
	38	4.6715	4.6698	0.0482	0.0004	4.6697	0.0474	0.0004	5.9766	5.9741	0.1251	0.0004	5.9741	0.1291	0.0004
	40	3.6939	3.6861	0.0443	0.0021	3.6861	0.0452	0.0021	5.0539	5.0400	0.0948	0.0028	5.0400	0.0983	0.0028
	42	2.9327	2.9294	0.0425	0.0011	2.9294	0.0433	0.0011	4.3345	4.3321	0.0789	0.0006	4.3321	0.0837	0.0006
	44	2.2849	2.2836	0.0417	0.0006	2.2836	0.0429	0.0006	3.6371	3.6266	0.0735	0.0029	3.6266	0.0747	0.0029

		T=1							T=2						
σ	S	binomial model	CV	S.E (%)	MAPE	CV+ EMS	S.E (%)	MAPE	binomial model	CV	S.E (%)	MAPE	CV+ EMS	S.E (%)	MAPE
0.05	36	4.0000	4.0045	0.0170	0.0011	4.0045	0.0170	0.0011	4.0000	4.0014	0.0376	0.0003	4.0014	0.0356	0.0003
	38	2.0000	2.0082	0.0374	0.0041	2.0081	0.0354	0.0041	2.0000	2.0142	0.0583	0.0071	2.0142	0.0587	0.0071
	40	0.2825	0.2728	0.1600	0.0342	0.2729	0.1654	0.0341	0.2936	0.2786	0.2646	0.0509	0.2786	0.2645	0.0509
	42	0.0176	0.0177	0.0369	0.0066	0.0177	0.0378	0.0070	0.0243	0.0243	0.1118	0.0015	0.0242	0.1106	0.0035
	44	0.0007	0.0006	0.0080	0.0906	0.0006	0.0079	0.0912	0.0019	0.0019	0.0247	0.0185	0.0018	0.0245	0.0267
0.1	36	4.0000	4.0076	0.1311	0.0019	4.0076	0.1286	0.0019	4.0000	4.0234	0.7864	0.0058	4.0234	0.7863	0.0058
	38	2.0554	2.0424	0.2309	0.0063	2.0424	0.2310	0.0063	2.1077	2.0950	0.4239	0.0060	2.0950	0.4237	0.0060
	40	0.8894	0.8775	0.1849	0.0134	0.8775	0.1894	0.0134	1.0188	1.0012	0.3712	0.0173	1.0012	0.3814	0.0173
	42	0.3558	0.3549	0.1522	0.0026	0.3549	0.1555	0.0026	0.4874	0.4847	0.3222	0.0055	0.4847	0.3363	0.0056
	44	0.1276	0.1279	0.0766	0.0025	0.1279	0.0780	0.0026	0.2267	0.2273	0.2072	0.0027	0.2273	0.2087	0.0026
0.2	36	4.4845	4.4722	0.3610	0.0027	4.4722	0.3475	0.0027	4.8512	4.8475	0.5288	0.0008	4.8476	0.5208	0.0008
	38	3.2529	3.2421	0.2553	0.0033	3.2420	0.2533	0.0033	3.7546	3.7562	0.4683	0.0004	3.7562	0.4766	0.0004
	40	2.3130	2.2967	0.2157	0.0071	2.2967	0.2235	0.0071	2.8800	2.8535	0.4498	0.0092	2.8536	0.4612	0.0092
	42	1.6239	1.6163	0.1686	0.0047	1.6163	0.1750	0.0047	2.2249	2.2272	0.3702	0.0010	2.2271	0.3828	0.0010
	44	1.1212	1.1192	0.1396	0.0018	1.1192	0.1450	0.0018	1.6907	1.6887	0.3168	0.0012	1.6887	0.3287	0.0012
0.3	36	5.7476	5.7399	0.2565	0.0013	5.7399	0.2506	0.0013	6.6160	6.5993	0.5140	0.0025	6.5993	0.5262	0.0025
	38	4.7055	4.7062	0.2296	0.0001	4.7062	0.2361	0.0001	5.1707	5.7116	0.4534	0.1046	5.7116	0.4815	0.1046
	40	3.8016	3.7820	0.2002	0.0052	3.7820	0.2024	0.0052	4.8797	4.8461	0.4179	0.0069	4.8461	0.4395	0.0069
	42	3.1002	3.0987	0.1858	0.0005	3.0987	0.1879	0.0005	4.2214	4.2240	0.4182	0.0006	4.2241	0.4457	0.0006
	44	2.4738	2.4689	0.1682	0.0020	2.4688	0.1736	0.0020	3.6232	3.6126	0.3780	0.0029	3.6127	0.4028	0.0029
0.4	36	7.0997	7.0854	0.2724	0.0020	7.0855	0.2725	0.0020	8.5310	8.5338	0.5882	0.0003	8.5339	0.6032	0.0003
	38	6.1805	6.1784	0.2400	0.0003	6.1784	0.2427	0.0003	7.6990	7.6943	0.5167	0.0006	7.6944	0.5507	0.0006
	40	6.3028	5.2804	0.2161	0.1622	5.2805	0.2223	0.1622	6.9036	6.8665	0.4681	0.0054	6.8665	0.5161	0.0054
	42	4.6137	4.6129	0.2047	0.0002	4.6129	0.2139	0.0002	6.2670	6.2682	0.4657	0.0002	6.2683	0.5115	0.0002
	44	3.9616	3.9503	0.1908	0.0028	3.9503	0.1998	0.0028	5.6781	5.6767	0.4536	0.0002	5.6768	0.5056	0.0002
0.5	36	8.5205	8.5181	0.2791	0.0003	8.5182	0.2760	0.0003	10.4470	10.4480	0.6471	0.0001	10.4475	0.7124	0.0000
	38	7.6491	7.6438	0.2661	0.0007	7.6438	0.2748	0.0007	9.6697	9.6552	0.5931	0.0015	9.6547	0.6680	0.0015
	40	6.7993	6.7754	0.2316	0.0035	6.7755	0.2476	0.0035	8.9128	8.8707	0.5587	0.0047	8.8703	0.6297	0.0048
	42	6.1322	6.1311	0.2179	0.0002	6.1312	0.2321	0.0002	8.3091	8.3051	0.5202	0.0005	8.3047	0.5774	0.0005
	44	5.4964	5.4900	0.1974	0.0012	5.4901	0.2185	0.0012	7.7441	7.7442	0.4541	0.0000	7.7438	0.5161	0.0000

*When T=0.25 and σ =0.05, the option price of binomial model is nearly zero. In this case, MAPE is infinite and has no meaning..

Table 3. This is Monte Carlo valuation of the Bermudan options. We select S.E. and MAPE of CV-at-exercise and CV+EMS from Table 2 and define new ratios called S.E-ratio and MAPE-ratio. S.E.-ratios are derived by S.E of EMS+CV divided by S.E. of CV-at-exercise and MAPE-ratios are derived by MAPE of EMS+CV divided by MAPE of CV-at-exercise.

S.E.-ratio (EMS+CV/CV)							**MAPE-ratio (EMS+CV/CV)**						
σ	t	S=36	S=38	S=40	S=42	S=44	σ	t	S=36	S=38	S=40	S=42	S=44
0.05	0.25	0.9045	1.0000	0.9941	0.9608	*	0.05	0.25	0.9557	0.9874	1.0003	1.0780	*
	0.5	1.0001	1.0000	1.0151	0.9985	1.0889		0.5	1.0009	0.9998	0.9999	0.9878	1.0000
	1	1.0002	0.9448	1.0341	1.0247	0.9776		1	1.0002	0.9978	0.9989	1.0562	1.0069
	2	0.9456	1.0060	0.9997	0.9892	0.9940		2	1.0075	0.9997	0.9999	2.3503	1.4443
0.1	0.25	1.0000	0.9965	1.0011	1.1127	0.9728	0.1	0.25	1.0005	1.1457	1.0003	0.6734	0.9488
	0.5	0.9998	0.9848	1.0066	1.0122	0.9609		0.5	1.0010	0.9844	1.0007	0.9635	1.0295
	1	0.9812	1.0003	1.0243	1.0216	1.0191		1	1.0045	1.0009	1.0000	0.9927	1.0362
	2	0.9999	0.9993	1.0275	1.0437	1.0070		2	1.0008	1.0008	1.0002	1.0075	0.9884
0.2	0.25	0.9503	0.9638	1.0125	0.9894	1.0577	0.2	0.25	0.9909	1.0031	0.9999	0.9977	1.0138
	0.5	0.9610	0.9890	0.9942	1.0061	1.1122		0.5	1.0026	1.0013	0.9992	0.9900	1.2660
	1	0.9624	0.9922	1.0362	1.0381	1.0390		1	1.0030	1.0046	1.0000	1.0042	0.9995
	2	0.9849	1.0177	1.0254	1.0342	1.0374		2	0.9936	1.0080	0.9987	0.9823	0.9968
0.3	0.25	0.9222	0.9866	1.0191	1.0412	0.9658	0.3	0.25	0.9360	0.9955	0.9989	0.9997	1.0535
	0.5	0.9322	0.9731	0.9925	0.9950	1.0939		0.5	1.0088	0.9957	0.9993	0.9940	1.0211
	1	0.9770	1.0281	1.0109	1.0111	1.0324		1	0.9948	1.0017	0.9994	1.0079	1.0070
	2	1.0238	1.0622	1.0518	1.0658	1.0656		2	0.9966	1.0001	1.0008	1.0115	0.9947
0.4	0.25	0.8982	0.9671	1.0138	0.9906	0.9078	0.4	0.25	1.0011	1.0018	1.0018	1.0031	0.9831
	0.5	0.9436	0.9917	1.0248	1.0361	1.0220		0.5	0.9947	0.9924	0.9977	0.9949	1.0070
	1	1.0003	1.0114	1.0287	1.0454	1.0471		1	0.9949	0.9913	1.0000	0.9779	0.9992
	2	1.0256	1.0657	1.1026	1.0984	1.1147		2	1.0544	0.9782	0.9984	1.0939	0.9175
0.5	0.25	0.9237	0.9830	1.0217	1.0183	1.0287	0.5	0.25	0.9962	1.0021	1.0007	1.0045	0.9800
	0.5	0.9752	1.0314	1.0364	1.0614	1.0172		0.5	0.9991	0.9879	0.9994	1.0004	1.0003
	1	0.9891	1.0326	1.0691	1.0654	1.4106		1	0.9741	0.9884	0.9980	0.9208	0.9844
	2	1.1010	1.1263	1.1270	1.1099	1.1367		2	0.4883	1.0348	1.0100	1.0902	4.1065

*When T=0.25 and σ =0.05, the option price of binomial model is nearly zero. In this case, MAPE is infinite and has no meaning..

5.2. American Rainbow Options

To investigate the application of control variates to a more complex example than the single-asset put option investigated in the previous numerical examples, we now investigate the *n*-asset Bermudan rainbow option for the case of n=2,3, and 5 assets. We use the same parameters as the example in Andersen and Broadie (2004) without any use of variance reduction and compare the results.

Andersen and Broadie use a set of 13 basis functions involving the highest and second highest asset prices, as well as polynomials of these, together with the value of European max-call option on the two largest assets and polynomials of this. Inspired by their choice and the concept of Rasmussen (2005), we use another set of basis functions.

For the case of two assets, the European max-call option price can easily be computed according to Stulz (1982). Although the pricing formula of multi-asset max-option has been derived by Johnson (1987) the computation of multivariate normal cumulated probability requires numerical integration. To circumvent this shortcoming we therefore choose to use combinations of two-asset max-call options as basis functions in our regression and as control variates. We list our basis functions as following. For the case of two assets, the six basis functions are

$$f_0(S_1,S_2,t) = K;$$
$$f_1(S_1,S_2,t) = \max(S_1(t),S_2(t));$$
$$f_2(S_1,S_2,t) = \max(S_1(t),S_2(t))^2;$$
$$f_3(S_1,S_2,t) = C(\max(S_1(t),S_2(t)),K);$$
$$f_4(S_1,S_2,t) = C(\max(S_1(t),S_2(t)),K)^2;$$
$$f_5(S_1,S_2,t) = C_{\max}(S_1(t),S_2(t),K);$$
$$F_6(S_1,S_2,t) = C_{\max}(S_1(t),S_2(t),K)^2;$$

For the case of three assets, the seven basis functions are

$$f_0(S_1,S_2,t) = K;$$
$$f_1(S_1,S_2,t) = \max(S_1(t),S_2(t),S_3(t));$$
$$f_2(S_1,S_2,t) = \max(S_1(t),S_2(t),S_3(t))^2;$$
$$f_3(S_1,S_2,t) = C(\max(S_1(t),S_2(t),S_3(t)),K);$$
$$f_4(S_1,S_2,t) = C(\max(S_1(t),S_2(t),S_3(t)),K)^2;$$
$$f_5(S_1,S_2,t) = C_{\max}(S_1(t),S_2(t),K);$$
$$f_6(S_1,S_2,t) = C_{\max}(S_1(t),S_3(t),K);$$
$$f_7(S_1,S_2,t) = C_{\max}(S_2(t),S_3(t),K);$$

For the case of five assets, the fifteen basis functions are

$$
\begin{aligned}
&f_0(S_1,S_2,t) = K;\\
&f_1(S_1,S_2,t) = \max(S_1(t),S_2(t),S_3(t),S_4(t),S_5(t));\\
&f_2(S_1,S_2,t) = \max(S_1(t),S_2(t),S_3(t),S_4(t),S_5(t))^2;\\
&f_3(S_1,S_2,t) = C(\max(S_1(t),S_2(t),S_3(t),S_4(t),S_5(t)),K);\\
&f_4(S_1,S_2,t) = C(\max(S_1(t),S_2(t),S_3(t),S_4(t),S_5(t)),K)^2;\\
&f_5(S_1,S_2,t) = C_{\max}(S_1(t),S_2(t),K);\\
&f_6(S_1,S_2,t) = C_{\max}(S_1(t),S_3(t),K);\\
&f_7(S_1,S_2,t) = C_{\max}(S_1(t),S_4(t),K);\\
&f_8(S_1,S_2,t) = C_{\max}(S_1(t),S_5(t),K);\\
&f_9(S_1,S_2,t) = C_{\max}(S_2(t),S_3(t),K);\\
&f_{10}(S_1,S_2,t) = C_{\max}(S_2(t),S_4(t),K);\\
&f_{11}(S_1,S_2,t) = C_{\max}(S_2(t),S_5(t),K);\\
&f_{12}(S_1,S_2,t) = C_{\max}(S_3(t),S_4(t),K);\\
&f_{13}(S_1,S_2,t) = C_{\max}(S_3(t),S_5(t),K);\\
&f_{14}(S_1,S_2,t) = C_{\max}(S_4(t),S_4(t),K);\\
&f_{15}(S_1,S_2,t) = C_{\max}(S_4(t),S_5(t),K);
\end{aligned}
$$

In the example of Andersen and Broadie (2004), the risk-neutral dynamics of n assets follow correlated geometric Brownian motion processes, i.e.,

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i$$

where $W_i, i = 1,2,\ldots,$n, are standard Brownian motion processes and the instantaneous correlation of W_i and W_j is ρ_{ij}. For simplicity and consistency with Andersen and Broadie (2004), in our numerical results we also take $q_i = q, \sigma_i = \sigma$ and $\rho_{ij} = \rho$ for all $i, j = 1,2,\ldots,n$ and $i \neq j$. The interest rate is assumed to be constant, so the value of the money market account at time t is $B_t = e^{rt}$. Exercise opportunities are equally spaced at times $t = (T/d)e$, $e = 0, 1, \ldots, d$ with d=9.

In the example of Andersen and Broadie (2004), they use 2,000,000 independent paths for the valuation of the Bermudan max-call option. In our example, we only use 100,000 independent paths combined with the use of control variates as outlined above. The results are listed in Table 4.

From the resulting estimates of Bermudan max-call options in Table 4, we see that the use of control variates reduces standard error and obtains more accuracy estimates by fewer simulation paths.

Table 4. This is Monte Carlo valuation of two-, three- and five-asset Bermudan max-call options using CV-at-exercise. Combinations of discounted European two-asset max-call options sampled at exercise are used as control variates. For all the options, the strike K=100, the interest rate equals r=0.05, the dividend rate δ=0.1, the volatility of each asset σ=0.2, and the correlation ρ=0. For all the max-call options, the time to expiry T=3 years with exercise points at $t=(T/d)e$, for e=0,1,2,…,d with d=9. The binomial model estimates and no variance reduction estimates are taken from Andersen and Broadie (2004). The current stock prices S are given. Numbers of simulation paths N are different. Values in parentheses are the standard errors.

n	S	Binomial Model	no-variance-reduction (N=2,000,000)		CV-at-exercise (N=50,000)	
2	90	8.075	8.065	(0.006)	8.074	(0.0043)
	100	13.902	13.907	(0.008)	13.904	(0.0067)
	110	21.345	21.333	(0.009)	21.339	(0.0080)
3	90	11.29	11.279	(0.007)	11.287	(0.0049)
	100	18.69	18.678	(0.009)	18.683	(0.0061)
	110	27.58	27.531	(0.010)	27.568	(0.0077)
5	90		16.618	(0.008)	16.577	(0.0074)
	100		26.128	(0.010)	26.213	(0.0088)
	110		36.725	(0.011)	36.723	(0.0092)

6. Conclusion

From the results of American put options, we see that improvement coming from EMS or CV+EMS is not very significant compared with CV-at-exercise. Therefore, when pricing American options, we can use CV-at-exercise to raise the simulation efficiency.

For the Bermudan max-call option in the multi-asset Black-Sholes model, the improvement coming from CV-at-exercise is very significant in terms of standard error reduction. The estimates of CV-at-exercise also are more close to binomial model estimates than no-variance-reduction LSM. It is thus evident that CV-at-exercise is a good variance reduction method when we price single- or multi-asset American options under Black-Sholes framework.

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