Pricing Asian Options with Fourier Convolution

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Contents

Abstract

This thesis investigates the fast Fourier transform-based pricing algorithm for discrete Asian options by Benhamou [1]. We compare it with other methods and combine it with extrapolation to increase numerical accuracy. We also apply it to the continuous case by using extrapolation. Running the algorithm with different numbers of grid points, we observe the convergence of option values both in the continuous case and in the discrete case. The disadvantages of the algorithm are also discussed.

Chapter 1

Introduction

Asian options are path-dependent contingent claims whose payoff is based on an average of underlying prices, interest rates, indices, or countless others. They have become popular in hedging periodic cash flows in that they cost far less than standard options on the same underlying assets. As pointed out by Levy [7], the reason Asian options are cheaper than the otherwise identical standard options is explained by the fact that the variance of the Asian option is smaller than that of the underlying asset's price process under the Black-Scholes model. Asian options can mitigate the possibility of spot manipulations or extreme movements of underlying prices at settlement. This feature is especially useful in thinly traded assets markets in which case the price manipulation on or near the expiration date has a significant impact on the payoff of standard options. In summary, the averaging feature of Asian options become attractive for hedging because it can avoid the large volatility of the price change and can also remove extreme sensitivity of standard options' payoff to the underlying price.

There are two main classes of Asian options: floating-strike and fixed-strike. The floating-strike Asian option pays the difference between the average and the spot price of the underlying. The fixed-strike Asian option pays the difference between the average price of the underlying and the pre-specified strike price. Asian options can also be classified as discrete or continuous according to the way the average is calculated. When the average is calculated from underlying asset's prices at discrete times, it is called a discrete Asian option. If all of the underlying's prices on the time line take part in the calculation of average, it is called a continuous Asian option. When the initial underlying asset's price does not take part in the calculation of the average, the option is called a forward-starting Asian option. We will deal exclusively with forward-starting discrete Asian options in this thesis unless stated otherwise.

Under the Black-Scholes option model, the asset price follows a geometric Brownian motion. Thus the asset price at any future time is described by the lognormal density function. If the Asian option is based on geometric average, the average is still lognormally distributed because the product of lognormal random variables remains lognormal. In this case, it is possible to derive an explicit formula for geometric averaging Asian options (see, for example, Kemna and Vorst [6] and Zhang [15]). However, if the Asian option is based on arithmetic average, there is no explicit representation for the distribution of the average of the underlying asset's prices because the sum of lognormal random variables is not lognormally distributed any more. Thus, there is no explicit pricing formula for arithmetic averaging Asian options as of now, and this is the source of difficulty in pricing them. This thesis focuses on arithmetic averaging Asian options. The main goal is to approximate its probability density function by using discrete points on the density function's domain to represent the distribution function of average.

Several approaches have been proposed in the literature to tackle the difficulty of pricing Asian options. We classify them as follows.

1. Monte-Carlo simulations with variance reduction techniques.

   Kemna and Vorst [6] derive a pricing formula for geometric-based discrete Asian options and used it as a control variate to reduce the variance of the discrete Asian option prices.

2. Binomial tree.

   Hull and White [5] augment an additional state variable to each node in the tree to record the possible averages of the underlying asset's price realized between time zero and the time of that node. Cho and Lee [3] improve it by deriving the maximum and minimum averages for each node. Hsu and Lyuu [4] further improve it by using non-uniform allocation scheme of states in each node according to its probability. All above are numerical methods for continuous Asian options.

3. Approximation of the density function of the average.

   Turnbull and Wakeman [13] apply the Edgeworth series expansion up to the fourth term around the lognormal distribution function to approximate the density function of the average for discrete Asian options. Levy [7] derives an approximate pricing formula for discrete Asian options by matching the first two moments of the density of the average with that of the lognormal density. Carverhill and Clewlow [2] use discrete points on the density function's domain to represent the density function and evaluate the convolution of density functions to approximate the density function of the average for discrete Asian options. Benhamou [1] improves it by incorporating a re-centering step into the algorithm.

4. Partial differential equations

   Zhang [14] derives an analytical approximate formula and a correction term governed by a partial differential equation, which requires numerical evaluation, for continuous Asian options. Rogers and Shi [12] compute the price of continuous Asian options by reducing the pricing problem into that of solving a PDE with the finite-difference method.

The methodology adopted in this thesis to price Asian options belongs to category 3.

The rest of the thesis is organized as follows. In chapter 2, we set up the framework and then introduce basic facts for later use. Chapter 3 develops a procedure to calculate the density function of the average represented by discrete points in the density function's domain and describes the pricing algorithm. The considerations for the choice of parameters in the algorithm are also detailed. Chapter 4 presents the numerical results. From them important characteristics of the Fourier convolution method can be drawn. Conclusions are given in Chapter 5.

Chapter 2

Background

Consider an Asian call option with maturity T, strike K, and n fixing dates during its life. The underlying's price at time t is denoted by S_t. Only the prices S_t on fixing dates take part in the calculation of the arithmetic average. We divide the total length of the derivative's life into n time intervals of equal length. There is a fixing date between two consecutive intervals. Assume these n fixing dates are denoted by $t_1, t_2, \ldots, t_n$ with $t_n = T$ and the initial time of the option is denoted by t_0. The average price is then defined by

$$A = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \qquad (1)$$

We also define the rate of return for each of these intervals as

$$R_{t_i} = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \qquad (2)$$

We assume the Black-Scholes model, where the underlying's price follows a geometric Brownian motion,

$$dS = \mu S dt + \sigma S dz$$

where dz is a Brownian motion whose increments are uncorrelated, σ is the volatility of the underlying's price, and μ is its expected rate of return. The assumption implies that the underlying's price at any time S_{t_i} can be expressed in terms of the preceding price $S_{t_{i-1}}$ as

$$S_{t_i} = S_{t_{i-1}} e^{\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma z} \qquad (3)$$

where z is a Brownian motion. It is most intuitively to think of z as being normally distributed with mean 0 and variance $t_i - t_{i-1}$. We can view this expression as the product of the preceding price and the rate of return. In particular,

$$S_{t_i} = S_{t_{i-1}} e^{R_{t_i}} \tag{4}$$

Comparing equations (3) and (4), we see that the rate of return R_{t_i} defined by equation (2) follows a normal distribution with mean $\left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})$ and variance $\sigma(t_i - t_{i-1})$.

That is,

$$\mathrm{E}[R_{t_i}] = \left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})$$
$$\mathrm{Var}[R_{t_i}] = \sigma(t_i - t_{i-1})$$

Note that if each interval is not of equal length, then each rate of return R_{t_i} corresponding to each interval still follows a normal distribution but has different mean and variance. Alternatively, equation (4) can also be expressed in terms of the initial price S_{t_0} as

$$S_{t_i} = S_{t_0} e^{R_{t_1} + R_{t_2} + \cdots + R_{t_i}} \tag{5}$$

Substitute the above equation (5) into expression (1) of the average to get

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{t_0} e^{R_{t_1} + R_{t_2} + \cdots + R_{t_i}} \tag{6}$$

In complete markets with no arbitrage opportunities, there exists a unique risk-neutral probability measure under which the price process of the derivative is a martingale. In this case, the mean of the rate of return R_{t_i} will be $\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})$ and the price of the Asian call C is the expected payoff under the measure discounted by the risk-free interest rate r:

$$C = \mathrm{E}^Q[e^{-rT}(A - K)^+] \tag{7}$$

where X^+ stands for $\max(X, 0)$. As stated above, the distribution of the average is unknown; there is no simple closed-form formula to calculate equation (7). Instead, we will numerically compute the density function backwards in the time line by means of representing the density function at discrete points. The method converges to the real density functions as the number of such points tends to infinity.

The convolution of two functions $f(x)$ and $g(y)$ is defined as $C(z) = \sum_{x=-\infty}^{\infty} f(x) g(z-x)$. The pair of Fourier transform and its inverse transform for function $f(x)$ is defined as

$$F(k) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} f(j) e^{-i(k-1)\frac{2\pi}{N}(j-1)}$$

$$f(j) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} F(k) e^{i(k-1)\frac{2\pi}{N}(j-1)}$$

We state the following two facts for later use.

Fact 1 Suppose that X and Y are independent random variables with the joint distribution function $f(x, y)$, and let $Z = X + Y$. The distribution function of Z is the convolution of the distribution function $f(x)$ for X and the distribution function $g(y)$ for Y.

Fact 2 The Fourier transform of the convolution of two functions $f(x)$ and $g(y)$ equals the product of $F(x)$ and $G(y)$, which are the Fourier transforms of $f(x)$ and $g(y)$, respectively. The convolution of the two functions can be obtained by taking the inverse Fourier transform of the product of $F(x)$ and $G(y)$.

Chapter 3

The Fourier Convolution Method

The Fourier convolution method represents the density function by discrete grid points within a fixed-width window on the density function's domain.

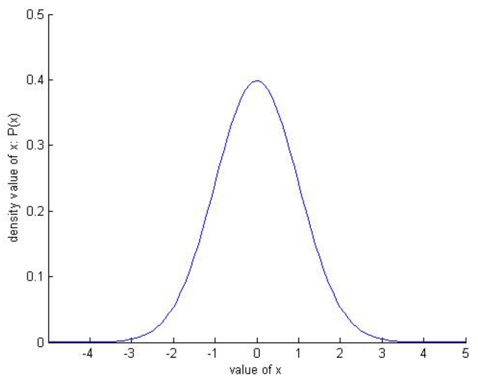

Figure 1

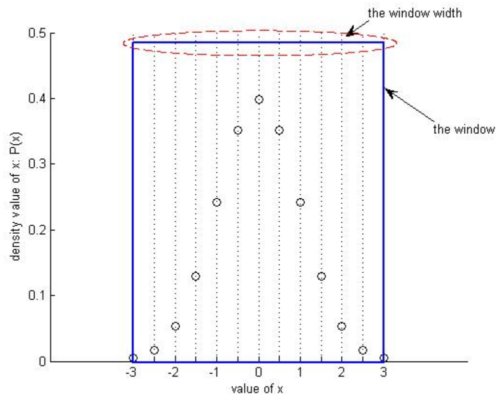

Figure 2

Figure 1 shows the graph of the density function for a standard normal random variable X. The horizontal axis represents the possible values of X, and the vertical axis represents the corresponding density value. Note that the density values for large absolute values of X tend to zero. Figure 2 shows the representation of the density function corresponding to Figure 1 in the Fourier convolution method. The method first needs to determine the two parameters: the number of grid points and the window width. In the case of Figure 2, there are 13 grid points in the window, and the window width is 6 ranging from –3 to +3. Note that the Fourier transform requires that these grid points be equally spaced. Only those density values on grid points are recorded in the representation of the density function. Obviously, the number of grid points has significant impact on the accuracy of representing the density function. Errors will inevitably be produced if we need the density values for non-grid points. The fixed-width window defines the domain of the density function. The density values at grid points out of the window will be assumed to be zero. Thus the window should be wide enough to contain the bulk of the density function, but it should not be too large as to consume too much resource. To sum up, the density function will be limited to these density values on discrete grid points, and computing the density function is equivalent to computing these density values on discrete grid points.

At the beginning, the mean of the initial density function will be located at the center of the window. As the process of computing the density function for the average progresses, the window will be shifted to make sure the location of the mean of the density function remains roughly at the center. The following develops the procedures to compute the density function for the average in detail.

3.1 Steward and Hodges factorization

The average is expressed in equation (6) as a function of the rates of returns. Each rate of return R_{t_i} follows a normal distribution with mean $(r-\sigma^2/2)(t_i - t_{i-1})$ and variance $\sigma(t_i - t_{i-1})$. From the information about the distribution for the rate of return in each interval, we can compute the density function for the underlying asset's price at any particular time, but the density function for the average is still hard to compute. Because the summands in the average are not independent, Fact 1 cannot be applied to compute the density function for the average. In order to apply Fact 1, the following proposition is needed.

Proposition 1 (Steward and Hodges factorization)

The average in equation (6) can be expressed as:

$$A = \frac{S_{t_0}}{n} \exp(R_{t_1} + \ln(1 + \exp(R_{t_2} + \ln(\cdots + \ln(1 + \exp R_{t_n})))))$$

Proof: From equation (6), we have

$$A=\frac{1}{n}\sum_{i=1}^{n}S_{t_0}e^{R_{t_1}+R_{t_2}+\cdots+R_{t_i}}$$

$$=\frac{S_{t_0}}{n}(e^{R_{t_1}}+e^{R_{t_1}}e^{R_{t_2}}+e^{R_{t_1}}e^{R_{t_2}}e^{R_{t_3}}+\cdots+e^{R_{t_1}}\cdots e^{R_{t_n}})$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}}(1+e^{R_{t_2}}(1+e^{R_{t_3}}+(1+\cdots+(1+e^{R_{t_n}}))))$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}}e^{\ln(1+e^{R_{t_2}}(1+e^{R_{t_3}}+(1+\cdots+(1+e^{R_{t_n}}))))}$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}+\ln(1+e^{R_{t_2}}(1+e^{R_{t_3}}+(1+\cdots+(1+e^{R_{t_n}}))))}$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}+\ln(1+e^{R_{t_2}}\exp(\ln(1+e^{R_{t_3}}+(1+\cdots+(1+e^{R_{t_n}})))))}$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}+\ln(1+\exp(R_{t_2}+\ln(1+e^{R_{t_3}}+(1+\cdots+(1+e^{R_{t_n}})))))}$$

$$=\frac{S_{t_0}}{n}e^{R_{t_1}+\ln(1+\exp(R_{t_2}+\ln(1+\exp(R_{t_3}+\ln(1+\cdots+\ln(1+e^{R_{t_n}}))))))}\ \square \qquad (8)$$

After applying Proposition 1, the summands in the exponential term of the equation (8) for the average are independent and now we can apply Fact 1 recursively and backwards in time to compute the density function for the average. Before proceeding, we define the sequence B_i as

$$B_i = R_{t_{n+1-i}} + \ln(1+\exp B_{i-1})\,,\quad i=2,3,\ldots,n \qquad (9)$$

with $B_1 = R_{t_n}$. Then equation (8) can be compactly written as

$$A=\frac{S_{t_0}}{n}e^{B_n} \qquad (10)$$

Our objective is in fact to compute the density function for B_n step by step from B_1, which is known to be normally distributed as discussed before. Equation (9) gives a formula that can be used to compute the density function of B_i from that of B_{i-1}

recursively.

3.2 Re-centering the densities

Notice that the term $\ln(1+\exp B_{i-1})$ in the right hand side of equation (9) would cause the density function computed at the preceding step $i-1$ to shift. More specifically, the probability at any point b would be mapped to the probability at the point $\ln(1+\exp b)$, which results in the shift in location of the density function. In order to prevent the density functions from shifting out of the window, at each step we will move the window so as to fit the density functions as much as possible and make the mean of the density roughly at the center.

We next determine how distant the window would move. If we knew the mean of $\ln(1+\exp B_{i-1})$, we could directly move the center of the window to that location. Suppose that we know the mean m_{i-1} of B_{i-1} whereas the mean of $\ln(1+\exp B_{i-1})$ is not available. We approximate the mean of $\ln(1+\exp B_{i-1})$ by $\ln(1+\exp m_{i-1})$. Then we can define the following sequence to approximate the mean of B_i defined by equation (9):

$$m_i = u_{n+1-i} + \ln(1+\exp m_{i-1}) \tag{11}$$

with $m_1 = u_n$, where $u_n = \mathrm{E}[R_{t_n}]$, which is $(r-\sigma^2/2)(t_i - t_{i-1})$ under the Black-Scholes model. Remember that the total life of option is divided into n intervals of equal length and all R_{t_i} are normally distributed with the same mean and variance. Therefore, all u_{n+1-i} will be the same and equal $(r-\sigma^2/2)(t_i - t_{i-1})$. If these intervals are not of equal length, then these R_{t_i} are still normally distributed but have different means and variances. Equation (11) gives an approximated $\mathrm{E}[B_i]$ from the previously approximated $\mathrm{E}[B_{i-1}]$. The center of the window is then set to the approximated $\mathrm{E}[B_i]$.

Note that the function $\ln(1+\exp x)$ is convex and thus the approximated mean underestimates the true mean, which can be proved by Jensen's inequality: $f(\mathrm{E}[X]) \le \mathrm{E}[f(X)]$.

We define the centered sequence for the equation (9) as $A_i = B_i - m_i$. In effect, this moves the center of the window to the approximated m_i. The expression for the average defined by equation (10) can be expressed in terms of the centered sequence as

$$A = \frac{S_{t_0}}{n} e^{A_n + m_n} \tag{12}$$

where A_n is derived as follows

$$\begin{aligned} A_i &= B_i - m_i \\ &= R_{t_{n+1-i}} + \ln(1+\exp B_{i-1}) - m_i \\ &= R_{t_{n+1-i}} + \ln(1+\exp A_{i-1} \cdot \exp m_{i-1}) - m_i \end{aligned} \tag{13}$$

with initial condition $A_1 = R_{t_n} - m_1$.

With the re-centering step incorporated into the algorithm, we compute the approximate density function for A_n step by step from A_1. Note that the initial random variable A_1 is known to be normally distributed and its mean is centered in the window. Equation (13) gives a formula to compute the density function of A_i from that of A_{i-1} which has been computed at the preceding step.

3.3 The interpolation formula

Suppose we are computing the density function of A_i at step i. From equation (13), we need the convolution of the density functions of $R_{t_{n+1-i}}$ and $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$. The distribution function for $R_{t_{n+1-i}}$ is known and the distribution function for A_{i-1} has been computed in the preceding step. The density function for $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$ is also known and represented by gird points equally spaced in the domain of A_{i-1}, but these grid points are not equally spaced in the domain of $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$ because the function is non-linear. But the Fourier transform requires that the grid points representing the density function be equally spaced. Thus what we need to compute is the density values for $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$ on non-grid points before applying convolution so that these new grid points are also equally spaced. In general, the density value for a function y of a random variable x is $f_x(y^{-1})\left|\frac{d}{dy} y^{-1}\right|$, where f_x represents the density function for x and y^{-1} represents the inverse function for y [11]. This leads to the following interpolation formula:

$$f_{\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i}(a) = \frac{e^{a+m_i}}{e^{a+m_i}-1} f_{A_{i-1}}(\ln(e^{a+m_i}-1) - m_{i-1}) \qquad (14)$$

which gives the density value of $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$ at grid point a in the domain of $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$ from that of A_{i-1} at point $\ln(e^{a+m_i}-1) - m_{i-1}$ in the domain of A_{i-1}. Note that the interpolation formula (14) will introduce errors because of the discretization of the density functions. Specifically, if a is on a grid point in the domain of A_{i-1}, then $\ln(e^{a+m_i}-1) - m_{i-1}$ will not be on a grid point in the domain of A_{i-1}. Thus in applying formula (14), interpolation will be used to get the density value at point $\ln(e^{a+m_i}-1) - m_{i-1}$ between the two nearest grid points in the domain of A_{i-1}. The errors caused by interpolation will accumulate as the number of applying formula (14) increases. Although the formula (14) is exact, the term $f_{A_{i-1}}(\ln(e^{a+m_i}-1) - m_{i-1})$ is not exact and can only be obtained by interpolation [1].

3.4 The pricing algorithm

All the needed procedures to compute the density function for the average have been developed in previous sections. Now we merge them to obtain the pricing algorithm.

The algorithm initially calculates the approximated mean $m_1 = u_n$ for $B_1 = R_{t_n}$ and the centered density function for $A_1 = R_{t_n} - m_1$. Note that the centered density function for A_1 is discretized and represented by discrete grid points in a fixed-width window. The objective is to get the density function of $A_n + m_n$, that is, the density values at grid points. All the operations performed will be on these grid points. Inductively, suppose we know the values of the m_{i-1} and the density function of A_{i-1} computed at step $i-1$. We then recursively execute the following procedures to compute the next approximated mean and centered density function until we get the value of m_n and the density function of A_n:

1. Interpolate the density function for A_{i-1} using formula (14) to get the density function for $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$.
2. Compute the density function for A_i by Fact 2. Note that A_i is the sum of the two independent random variables $R_{t_{n+1-i}}$ and $\ln(1+\exp A_{i-1} \exp m_{i-1}) - m_i$.

Once we have gotten the approximated density function for the average, we compute the expected payoff by numerical integration and then discount it by the risk-free interest rate to obtain the option value. The above completes Benhamou's algorithm [1].

3.5 The choice of parameters

There are two parameters, which are the number of grid points and the window width, in the algorithm. Either of the two parameters can affect the accuracy of the algorithm. How to choose an appropriate number of grid points depends on the required accuracy. If one desires a more accurate value, he can increase the number of grid points at the expense of computation time. The window width should be chosen at the beginning of the algorithm and then fixed so as to be large enough to contain the bulk of the density function. However, the larger the window width is, the less accurate the option value is when the number of grid points is fixed. In Benhamou's original paper, he chooses the window width as $9n\sigma\sqrt{\frac{T}{n}}$. This choice produces less accurate results in our experiments because its width is too large when given the same number of grid points. So we try to reduce the window width so that the accuracy can be improved. From empirical rule, we know that about 99.7% of probability is within the interval between +3 and –3 standard deviations. Because the density function of the initial random variable $A_1 = R_{t_n} - m_1$ is normally distributed, the window width should be at least larger than $6\sigma\sqrt{\frac{T}{n}}$ in order to contain the bulk of the initial density function. When the number of convolution operations increases, more and more probability will be at the tail of the two ends of the distribution and thus the window width should also increase. Through extensive experiments, we find that the increase in the window width should be roughly proportional to $\sqrt{n}$ to achieve good accuracy. So we choose the initial window width as $6\sigma\sqrt{T}$ so as to contain the bulk of the density function. This choice is different from Benhamou's choice, and it produces accurate results.

Chapter 4

Numerical Results

In this chapter, we compare the Fourier convolution method with other methods in pricing discrete Asian options. In order to further speed up the convergence of option values, we incorporate extrapolation into the Fourier convolution. We finally apply extrapolation to the discrete version of Asian options to obtain prices of the continuous version.

4.1 Discrete case

Table 1 shows the results with different methods for pricing discrete Asian options with different strikes. The window width of Fourier convolution (FC) in Table 1 is the same as Benhamou's choice. Its results are far less accurate compared to the other methods because the window width is too wide. Table 2 shows the results corresponding to Table 1 except that the window widths of FC follow our choice. As the results show, its accuracy is improved, although it is still not as accurate as the other methods. But when the number of grid points doubles, the convergence rate also doubles with a concurrent improvement in accuracy. This linear relationship between the number of grid points and convergence rate suggests that extrapolation can be used in order to increase accuracy. Extrapolation is a technique to speed up the convergence rate by using two approximated option values. The formula is

$$f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}$$

where n_1 and n_2 are two different choices of numbers of grid points. When $n_2 = 2n_1$, the extrapolation is called the Richardson extrapolation. Table 3 shows the results when the Richardson extrapolation is incorporated into the FC method. The numbers of grid points under FC in Table 3 are the values of n_2. The FC method without extrapolation is not accurate enough; however, its accuracy outperforms the other methods if it is combined with extrapolation.

4.2 Continuous case

All the underlying asset's prices on the time line take part into the pricing of continuous Asian options. Thus, in theory the life of options should be divided into an infinite number of time intervals. Extrapolation technique is used to approximate the result. As in the discrete case, the same formula is used with n_1 and n_2 being two different choices of number of intervals. Table 4 shows the results when $n_1 = 100$ and uses Richardson extrapolation. Table 5 shows the results when $n_1 = 180$ and $n_2 = 240$.

There are two extrapolation stages in the continuous case. We know from the results in the discrete case that the FC method without extrapolation is not accurate enough. So we use Richardson extrapolation in the first stage to obtain approximated values as the number of grid points tends to infinity. The numbers of grid points under the FC in Table 4 and Table 5 are the values of n_2 for extrapolation in the first stage. Then we apply extrapolation in the second stage by using the results in the previous stage to obtain the approximated values as the number of intervals tends to infinity in the continuous case.

The results in both Table 4 and Table 5 show that the option value converges to some value, although it is different from the exact value. But if higher n_1 and n_2 in the second extrapolation stage are used, the results can be more accurate. Because the results in Table 5 use more time intervals to extrapolate than in Table 4, it is more accurate if the number of grid points increases. However, as the number of time intervals increases, the number of times the interpolation formula (14) is applied also increases, which results in more accumulated errors. That the approximations in Table 5 using 2^{13} are slightly less accurate than those in Table 4 with the same number of grid points verifies this fact. The accumulated errors can be lowered by using more grid points, and this also can be seen from approximations in Table 5 as the number of grid points increases.

Chapter 5

Conclusions

The FC is an efficient pricing algorithm for discrete Asian options. Let m be the number of time intervals and n be the number of grid points used in the algorithm. The complexity of FC is $O(mn\ln n)$. In practice, m is contractual and rather small compared to n in the discrete case. The combined version of FC with extrapolation becomes a fast and accurate pricing algorithm for discrete Asian options.

However, m will be large for the continuous case, which results in more applications of the interpolation formula (14). In order to compensate the accumulated errors caused by interpolation, n must also be increased. Because both m and n increase at the same time, the computation time also increases rapidly. This is the main disadvantage of the FC if it is to be applied to the continuous case.

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	Approximations										
							FC with different numbers of grid points				
Strikes	MC	SD	Levy	TW	Vorst	Hsu-Lyuu	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}
50	50.0506	0.0056	50.0531	50.2409	50.0494	50.0475	54.7577	52.3659	51.1964	50.6205	50.3344
60	41.2315	0.0056	41.2637	41.387	41.2388	41.2301	45.9205	43.5369	42.3732	41.8002	41.5156
70	32.6621	0.0056	32.7573	32.5403	32.6726	32.6617	37.2537	34.9149	33.7774	33.2181	32.9407
80	24.7540	0.0057	24.9110	24.4198	24.7302	24.7492	29.0802	26.8633	25.7933	25.2692	25.0096
90	17.9405	0.0058	18.1138	17.6413	17.8339	17.9357	21.8106	19.8120	18.8583	18.3936	18.1641
100	12.4799	0.0059	12.6129	12.3638	12.2663	12.4816	15.7552	14.0508	13.2488	12.8607	12.6698
110	8.3887	0.0060	8.4489	8.4428	8.0849	8.3925	11.0133	9.6351	8.9965	8.6899	8.5396
120	5.4897	0.0059	5.4761	5.6268	5.1388	5.4900	7.4919	6.4279	5.9429	5.7118	5.5990
130	3.5187	0.0058	3.4510	3.6602	3.1664	3.5164	4.9870	4.1968	3.8423	3.6748	3.5934
140	2.2153	0.0056	2.1221	2.3237	1.8989	2.2177	3.2646	2.6956	2.4444	2.3267	2.2696
150	1.3787	0.0052	1.2777	1.4415	1.1125	1.3839	2.1106	1.7109	1.5371	1.4562	1.4172
160	0.8507	0.0048	0.7560	0.8761	0.6396	0.8578	1.3522	1.0768	0.9588	0.9043	0.8781
170	0.5203	0.0043	0.4415	0.5236	0.3625	0.5293	0.8614	0.6741	0.5950	0.5588	0.5414
180	0.3196	0.0039	0.2555	0.3090	0.2036	0.3255	0.5466	0.4207	0.3683	0.3444	0.3330
190	0.1955	0.0034	0.1471	0.1807	0.1137	0.1998	0.3461	0.2622	0.2277	0.2121	0.2047
200	0.1214	0.0029	0.0844	0.1050	0.0633	0.1223	0.2192	0.1635	0.1409	0.1307	0.1259

Table 1: Comparison of different pricing methods: the case of 3-year discrete Asian options. The options are forward-starting Asian calls with $S = 100$, $T = 3$, $\sigma = 0.25$, $r = 0.04$ and $n = 36$ (monthly averaging). The width of the FC method is $9n\sigma\sqrt{\frac{T}{n}}$, which is 23.3 in this case.

	Approximations										
							FC with different numbers of grid points				
Strikes	MC	SD	Levy	TW	Vorst	Hsu-Lyuu	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}
50	50.0506	0.0056	50.0531	50.2409	50.0494	50.0475	50.5558	50.3017	50.1748	50.1116	50.0800
60	41.2315	0.0056	41.2637	41.387	41.2388	41.2301	41.7357	41.4829	41.3569	41.2940	41.2626
70	32.6621	0.0056	32.7573	32.5403	32.6726	32.6617	33.1553	32.9088	32.7860	32.7247	32.6941
80	24.7540	0.0057	24.9110	24.4198	24.7302	24.7492	25.2103	24.9798	24.8651	24.8078	24.7792
90	17.9405	0.0058	18.1138	17.6413	17.8339	17.9357	18.3414	18.1377	18.0365	17.9860	17.9608
100	12.4799	0.0059	12.6129	12.3638	12.2663	12.4816	12.8171	12.6477	12.5637	12.5218	12.5009
110	8.3887	0.0060	8.4489	8.4428	8.0849	8.3925	8.6553	8.5221	8.4561	8.4232	8.4068
120	5.4897	0.0059	5.4761	5.6268	5.1388	5.4900	5.6856	5.5857	5.5363	5.5117	5.4994
130	3.5187	0.0058	3.4510	3.6602	3.1664	3.5164	3.6556	3.5835	3.5479	3.5302	3.5214
140	2.2153	0.0056	2.1221	2.3237	1.8989	2.2177	2.3130	2.2625	2.2376	2.2253	2.2191
150	1.3787	0.0052	1.2777	1.4415	1.1125	1.3839	1.4466	1.4121	1.3951	1.3867	1.3825
160	0.8507	0.0048	0.7560	0.8761	0.6396	0.8578	0.8976	0.8744	0.8631	0.8574	0.8546
170	0.5203	0.0043	0.4415	0.5236	0.3625	0.5293	0.5541	0.5388	0.5313	0.5275	0.5257
180	0.3196	0.0039	0.2555	0.3090	0.2036	0.3255	0.3411	0.3310	0.3261	0.3237	0.3225
190	0.1955	0.0034	0.1471	0.1807	0.1137	0.1998	0.2097	0.2032	0.2000	0.1984	0.1976
200	0.1214	0.0029	0.0844	0.1050	0.0633	0.1223	0.1290	0.1247	0.1227	0.1217	0.1212

Table 2: Comparison of different pricing methods: the case of 3-year discrete Asian options. The options are forward-starting Asian calls with $S = 100$, $T = 3$, $\sigma = 0.25$, $r = 0.04$ and $n = 36$ (monthly averaging). The width of the FC method is 2.6.

	Approximations									
							FC with different numbers of grid points			
Strikes	MC	SD	Levy	TW	Vorst	Hsu-Lyuu	2^{13}	2^{14}	2^{15}	2^{16}
50	50.0506	0.0056	50.0531	50.2409	50.0494	50.0475	50.0476	50.0479	50.0484	50.0484
60	41.2315	0.0056	41.2637	41.387	41.2388	41.2301	41.2301	41.2309	41.2311	41.2312
70	32.6621	0.0056	32.7573	32.5403	32.6726	32.6617	32.6623	32.6632	32.6634	32.6635
80	24.7540	0.0057	24.9110	24.4198	24.7302	24.7492	24.7493	24.7504	24.7505	24.7506
90	17.9405	0.0058	18.1138	17.6413	17.8339	17.9357	17.9340	17.9353	17.9355	17.9356
100	12.4799	0.0059	12.6129	12.3638	12.2663	12.4816	12.4783	12.4797	12.4799	12.4800
110	8.3887	0.0060	8.4489	8.4428	8.0849	8.3925	8.3889	8.3901	8.3903	8.3904
120	5.4897	0.0059	5.4761	5.6268	5.1388	5.4900	5.4858	5.4869	5.4871	5.4871
130	3.5187	0.0058	3.4510	3.6602	3.1664	3.5164	3.5114	3.5123	3.5125	3.5126
140	2.2153	0.0056	2.1221	2.3237	1.8989	2.2177	2.2120	2.2127	2.2130	2.2129
150	1.3787	0.0052	1.2777	1.4415	1.1125	1.3839	1.3776	1.3781	1.3783	1.3783
160	0.8507	0.0048	0.7560	0.8761	0.6396	0.8578	0.8512	0.8518	0.8517	0.8518
170	0.5203	0.0043	0.4415	0.5236	0.3625	0.5293	0.5235	0.5238	0.5237	0.5239
180	0.3196	0.0039	0.2555	0.3090	0.2036	0.3255	0.3209	0.3212	0.3213	0.3213
190	0.1955	0.0034	0.1471	0.1807	0.1137	0.1998	0.1967	0.1968	0.1968	0.1968
200	0.1214	0.0029	0.0844	0.1050	0.0633	0.1223	0.1204	0.1207	0.1207	0.1207

Table 3: Comparison of different pricing methods: the case of 3-year discrete Asian options. The options are forward-starting Asian calls with S = 100, T = 3, σ = 0.25, r = 0.04 and n = 36 (monthly averaging). The width of the FC method is 2.6. The Richardson extrapolation with different $n_2=2n_1$ values is incorporated into the FC.

Strikes	σ	Approximations				FC with different $n_2=2n_1$			
		Exact	AA2	AA3	Hsu-Lyuu	2^{13}	2^{14}	2^{15}	2^{16}
95	0.05	8.8088392	8.80884	8.80884	8.808717	8.80827	8.80836	8.80838	8.80839
100		4.3082350	4.30823	4.30823	4.309247	4.30739	4.30770	4.30778	4.30780
105		0.9583841	0.95838	0.95838	0.960068	0.95689	0.95780	0.95802	0.95808
95	0.1	8.9118509	8.91171	8.91184	8.912238	8.91052	8.91120	8.91137	8.91142
100		4.9151167	4.91514	4.91512	4.914254	4.91283	4.91427	4.91463	4.91472
105		2.0700634	2.07006	2.07006	2.072473	2.06733	2.06919	2.06965	2.06977
95	0.2	9.9956567	9.99597	9.99569	9.995661	9.99093	9.99419	9.99501	9.99521
100		6.7773481	6.77758	6.77738	6.777748	6.77182	6.77572	6.77670	6.77695
105		4.2965626	4.29643	4.29649	4.297021	4.29073	4.29482	4.29585	4.29610
95	0.3	11.6558858	11.65747	11.65618	11.656062	11.64664	11.65263	11.65494	11.65533
100		8.8287588	8.82942	8,82900	8.829033	8.81950	8.82616	8.82782	8.82824
105		6.5177905	6.51763	6.51802	6.518063	6.50845	6.51518	6.51685	6.51726
95	0.4	13.5107083	13.51426	13.51182	13.510861	13.49733	13.50687	13.50925	13.50985
100		10.9237708	10.92507	10.92474	10.923943	10.91007	10.91988	10.92232	10.92292
105		8.7299362	8.72936	8.73089	8.730102	8.71629	8.72607	8.72849	8.72910

Table 4: Comparison with Zhang (2001, 2003) with a wide range of volatilities: the case of 1-year continuous Asian options. The parameters are from Table 2 of Zhang (2003). The options are calls with S=100, r=0.09, and $T = 1$. The width of the FC method is $6\sigma\sqrt{T}$. The Richardson extrapolation with different $n_2=2n_1$ values is incorporated into the FC. The continuous option values for FC are approximated by dividing T into 100 periods and then using Richardson extrapolation.

		Approximations							
						FC with different $n_2=2n_1$			
Strikes	σ	Exact	AA2	AA3	Hsu-Lyuu	2^{13}	2^{14}	2^{15}	2^{16}
95	0.05	8.8088392	8.80884	8.80884	8.808717	8.80957	8.80868	8.80865	8.80863
100		4.3082350	4.30823	4.30823	4.309247	4.30980	4.30841	4.30814	4.30807
105		0.9583841	0.95838	0.95838	0.960068	0.96295	0.95939	0.95854	0.95833
95	0.1	8.9118509	8.91171	8.91184	8.912238	8.91551	8.91248	8.91185	8.91169
100		4.9151167	4.91514	4.91512	4.914254	4.92241	4.91679	4.91544	4.91509
105		2.0700634	2.07006	2.07006	2.072473	2.07960	2.07229	2.07052	2.07008
95	0.2	9.9956567	9.99597	9.99569	9.995661	10.01228	9.99959	9.99651	9.99570
100		6.7773481	6.77758	6.77738	6.777748	6.79731	6.78213	6.77842	6.77747
105		4.2965626	4.29643	4.29649	4.297021	4.31761	4.30152	4.29761	4.29662
95	0.3	11.6558858	11.65747	11.65618	11.656062	11.68719	11.66350	11.65757	11.65605
100		8.8287588	8.82942	8,82900	8.829033	8.86252	8.83694	8.83059	8.82898
105		6.5177905	6.51763	6.51802	6.518063	6.55222	6.52606	6.51963	6.51801
95	0.4	13.5107083	13.51426	13.51182	13.510861	13.55918	13.52229	13.51311	13.51082
100		10.9237708	10.92507	10.92474	10.923943	10.97388	10.93571	10.92626	10.92391
105		8.7299362	8.72936	8.73089	8.730102	8.78023	8.74184	8.73242	8.73009

Table 5: Comparison with Zhang (2001, 2003) with a wide range of volatilities: the case of 1-year continuous Asian option. The parameters are from Table 2 of Zhang (2003). The options are calls with S=100, r=0.09, and $T = 1$. The width of the FC method is $6\sigma\sqrt{T}$. The Richardson extrapolation with different $n_2=2n_1$ values is incorporated into the FC. The continuous option values for FC are approximated by dividing T into 180 periods and using extrapolation with $n_2=240$ periods.