Option Pricing with Stochastic Volatility

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Contents

Chapter 1

Introduction

1.1 Introduction

The volatility smile is frequently observed in options prices. But in the pure Black-Scholes world, there should not be any smile as the volatility should be constant across the strike price and time. Although the Black-Scholes formula has been successfully used in pricing stock option prices, it does have known biases. This is not surprising since the Black-Scholes model makes the strong assumption that stock returns are normally distributed with known variance, but the constant variance assumption is somewhat simplisitc.

Pricing models with stochastic volatility have been addressed in the literature by many authors (see Scott [1987,1991], Hull and White [1987], and Wiggins [1987)); they generalize the Black-Scholes model to allow stochastic volatility. However, these models all assume zero correlation between volatility and price. Heston [1993] provides a closed-form solution for European options when the stochastic volatility is correlated with the spot asset.

The bivariate binomial framework presented by Hilliard and Schwartz [1996] not only allows non-zero correlation between the volatility and the underlying process but can also be used to value American options. It uses a simple recombining binomial tree with a set of four joint, but possibly non-independent, probabilities. The node probabilities, P_{ij} , can be determined by simple calculations.

Although the underlying process has stochastic volatility, it is still driven by the usual Brownian motion. The stochastic volatility process is driven by Brownian motion except that the drift term can be quite general; for example, it can be mean-reverting.

Hilliard and Schwartz [1996] present an efficient method to calculate option prices on the bivariate binomial model. But, strangely, they do not use the tree method to derive the values in their paper; instead, they only use simulation to obtain the numbers. In other words, the tree does not seem to have been implemented. This thesis fills that gap by implementing the bivariate binomial tree method to price options. It then uses Monte-Carlo simulation to compare the accuracy of the tree method.

1.2 Structures of the Thesis

There are four chapters in this thesis. The first chapter introduces stochastic volatility model. The second chapter covers the bivariate binomial model which is developed by Hilliard and Schwartz [1996]. The third chapter presents the numerical results using the bivariate binomial model. The fourth chapter concludes this thesis.

Chapter 2

Mathematical Models

2.1 Stochastic-Volatility Model

We consider continues-time risk-neutral diffusion process of the form

$$\begin{aligned} dS &= m_S dt + f(S)h(V)dZ_S \\ dV &= m_V dt + bVdZ_V \end{aligned} \tag{2.1}$$

where S is the spot asset, V is the stochastic volatility, dZ_S and dZ_V are Wiener processes with correlation $\mathrm{Corr}(dZ_S,\, dZ_V) = \rho_{SV}$, and $f(S)h(V)$ is typically of the form $S^{\theta}V^{\alpha}$.

2.2 Constructing the Lattice

We follow the tree method of Hilliard and Schwartz [1996]. Consider first the volatility transformation. The transformation is

$$Y = \frac{\ln(V)}{b},$$

which yields a process with unit volatility (see Appendix A for the proof of Eq. (2.2)):

$$\begin{aligned} dY &= \left(\frac{m_V}{bV} - \frac{b}{2}\right)dt + dZ_V \\ &= m_y dt + dZ_V \end{aligned} \tag{2.2}$$

where m_y is the drift term of Y.

Since the coefficient of dZ_V is a constant, the lattice in Y recombines as required. But the transformation of S to constant volatility is not straightforward because the volatility of S includes both random variables S and V.

We use a two-step transformation. First, we consider a transformation H of the form

$$H = H(S,V) = h^{-1}(V)\int^{S} \frac{dS}{f(S)},$$

and the diffusion process is

$$dH = H_S dS + H_V dV + \frac{1}{2}[H_{SS} dS^2 + 2H_{SV} dSdV + H_{VV} dV^2]$$
$$= H_S f(S)h(V)dZ_S + H_V bVdZ_V + m_h dt$$

where m_h is the drift term of H and depends on m_S, m_V, and second-order partials.

The second transformation of H to Q is

$$Q = (\alpha b)^{-1} \ln(\alpha bH - \rho_{SV} + \sigma_h)$$

where

$$\sigma_h = \sqrt{1 - 2\alpha bH\rho_{SV} + \alpha^2 b^2 H^2},$$

and the diffusion is of the form

$$dQ = m_q dt + dZ_h.$$

The diffusion for Q has unit volatility now, as required.

By Ito's theorem, we can get the drift terms of H and Q as follows (see Appendix B and C for the proofs of Eqs. (2.3) and (2.4)):

$$m_h = \frac{m_S}{f(S)V^\alpha} - \frac{m_V \alpha H}{V} - \frac{1}{2} f_S V^\alpha + \frac{1}{2}\alpha H(1 + \alpha)b^2 - \alpha b\rho_{SV} \quad (2.3)$$

and

$$m_q = \frac{m_h}{\sigma_h} + \frac{1}{2}\frac{(\alpha b\rho_{SV} - \alpha^2 b^2 H)}{\sigma_h} \quad (2.4)$$

Since both Y and Q have unit volatility, the bivariate binomial grid can be easily constructed on the $Y \times Q$ space.

The values of V and S variables are given by the inverse transformation (see Appendix D for the proof of Eq. (2.5)):

$$V = \exp(bY)$$
$$H = \frac{2r_{SV} - (1 - r_{SV}{}^2) \times \exp(-abQ) + \exp(abQ)}{2ab} \quad (2.5)$$
$$S = \begin{cases} [V^a(1-q)H]^{\frac{1}{1-q}}, & q \neq 1 \\ \exp(h(V)H), & q = 1 \end{cases}$$

Under these transformations, the increments dZ_V and dZ_h have correlation

$$\text{Corr}(dZ_V,\, dZ_h) \;=\; \frac{(\rho_{SV} \;-\; \alpha bH)}{\sigma_h}$$

and $\text{Corr}(dY,\, dQ) \;=\; \text{Corr}(dZ_V,\, dZ_h)$.

2.3 Binomial Jumps and Probabilities

As in the standard univariate model with unit volatility, the binomial jumps for the transformation process Y and Q are given by:

$$Y_1^{\pm} \;=\; Y_0 \pm \sqrt{\Delta t}$$

and

$$Q_1^{\pm} \;=\; Q_0 \pm \sqrt{\Delta t}$$

where Δt is the size of time step. The associated probabilities for upward jumps for Y and Q are, respectively:

$$p \;=\; 0.5(1 \;+\; m_y\sqrt{\Delta t})$$

and

$$q \;=\; 0.5(1 \;+\; m_q\,\sqrt{\Delta t})$$

Joint probabilities are defined by:

$$\begin{aligned}
P_{11} &= \text{prob}(Q_1^{+}, Y_1^{-}) \\
P_{12} &= \text{prob}(Q_1^{+}, Y_1^{+}) \\
P_{21} &= \text{prob}(Q_1^{-}, Y_1^{-}) \\
P_{22} &= \text{prob}(Q_1^{-}, Y_1^{+})
\end{aligned}$$

When dZ_V and dZ_h are independent, joint probabilities are easily derived by multiplication. For example,

$$P_{11} \;=\; q(1-p),\; P_{12} \;=\; pq,\; P_{21} \;=\; (1-q)(1-p),\; P_{22} \;=\; p(1-q).$$

When they are dependent, the joint probabilities are given by:

$$
\begin{aligned}
P_{11} &= q(1-p) - \mathrm{Corr}(dY, dQ)\kappa \\
P_{12} &= pq + \mathrm{Corr}(dY, dQ)\kappa \\
P_{21} &= (1-q)(1-p) + \mathrm{Corr}(dY, dQ)\kappa \\
P_{22} &= p(1-q) - \mathrm{Corr}(dY, dQ)\kappa
\end{aligned}
$$

where $\kappa = \sqrt{p(1-p)q(1-q)}$.

Chapter 3

Numerical Results

3.1 Bivariate Binomial Option Pricing

We consider a special diffusion processes examined by Hull and White [1987]:

$$\begin{aligned} dS &= rSdt + S\sqrt{V}dZ_S \\ dV &= bVdZ_V \end{aligned}$$

that is $m_s = rS$, $f(S)h(V) = S\sqrt{V}$, and $m_v = 0$ in Eq. (2.1). The resulting transformations from Q and Y back to the original variables V and S are given by equations illustrated in section 2.2 with $\theta = 1$ and $\alpha = 0.5$.

We first calculate the values of Q and Y at each node by a bivariate binomial tree. We can also calculate the values of V and S at the expiration day. Then we can get the value of option price by backward induction.

We notice that the probabilities of upward jumps p and q are determined by m_y of Eq. (B) and m_q of Eq. (D), respectively. These two values should be calculated by the initial values of each variable involved in the equations in section 2.2.

3.2 Evaluating European Put Options

The following is an example of the effect of stochastic volatility on the prices of European put options when the volatility is uncorrelated with the underlying asset price.

Exhibit 1 shows the value of a European put with stochastic volatility parameter b = 25% and zero correlation between volatility and price. The bivariate binomial values with 270 time steps are compared with the values generated by Hilliard and Schwartz [1996], the Hull-White stochastic volatility model [1987], and the standard Black-Sholes model [1973] which the volatility is fixed and equal to the initial volatility.

The simulation values are from Hilliard and Schwartz [1996]. Observe that the simulation values are almost indistinguishable from the Black-Scholes and Hull-White values. Recall that Hilliard and Schwartz do not use the bivariate binomial tree method to obtain the values of European puts. This thesis uses the bivariate binomial tree method to get the values of European options, in contrast. Observe that our values are close to the Black-Scholes, Hull-White, and Hilliard and Schwartz's values. In fact, the difference of bivariate binomial tree method from the other three methods is less than 0.1. We also use Monte Carlo simulation to get the values of European puts with 100,000, 1,000,000, and 10,000,000 sample paths. The values by Monte Carlo simulation also match the above values. In fact, the difference of Monte Carlo simulation from Black-Scholes, Hull-White, and Hilliard and Schwartz's simulation values is less than 0.01.

Exhibit 2 shows the convergence of put prices as the number of time steps varies. The exercise prices are 80, 100, and 120. The parameters are identical to those in Exhibit 1 except the numbers of time steps. We first set the number of time steps to be 6 and calculate the put values. After this, we set the number of time steps to be 7 and recalculate again. And we keep increase the number of time steps and calculate the put values until the number of time steps reaches 358.

In Exhibit 2, we can see that the prices change a lot when the number of time steps is less than 25 and they converge to a level when the number of time steps exceeds 200.

Exhibit 1

Effect of Stochastic Volatility on European Put Prices

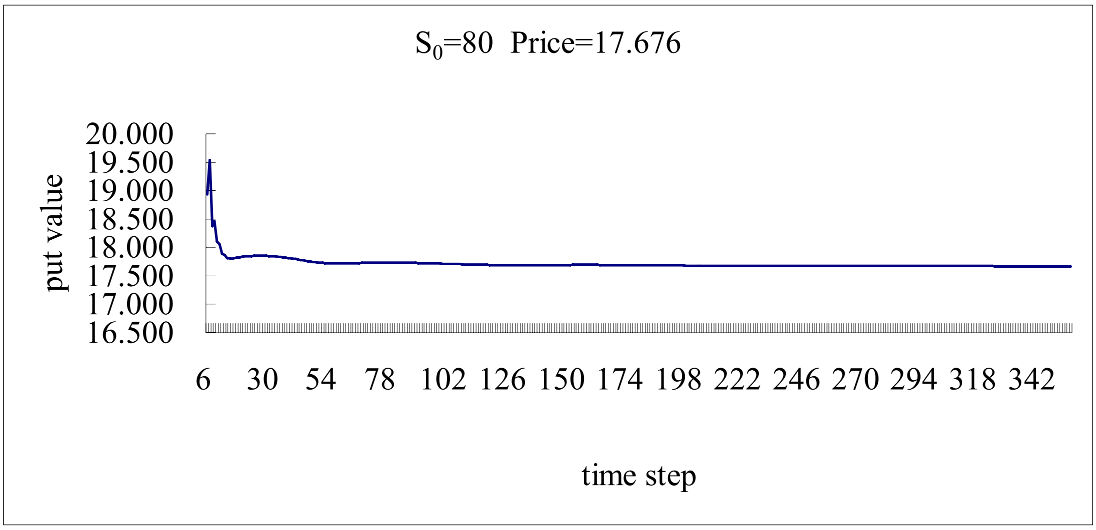
S/X X=100	Black- Scholes	Hull-White	Simulation values (H&S)	Bivariate Binomial	Monte Carlo *N*=100,000
0.80	17.643	17.645	17.646	17.6781	17.6432
0.84	13.876	13.878	13.878	13.9317	13.8900
0.88	10.398	10.397	10.397	10.4597	10.3868
0.92	7.365	7.362	7.361	7.4094	7.3654
0.96	4.903	4.898	4.898	4.9042	4.8937
1.00	3.058	3.053	3.054	3.0155	3.0529
1.04	1.784	1.782	1.782	1.7408	1.7698
1.08	0.975	0.975	0.975	0.9501	0.9753
1.12	0.500	0.501	0.501	0.4772	0.5008
1.16	0.241	0.244	0.243	0.2291	0.2485
1.20	0.110	0.112	0.112	0.1091	0.1182

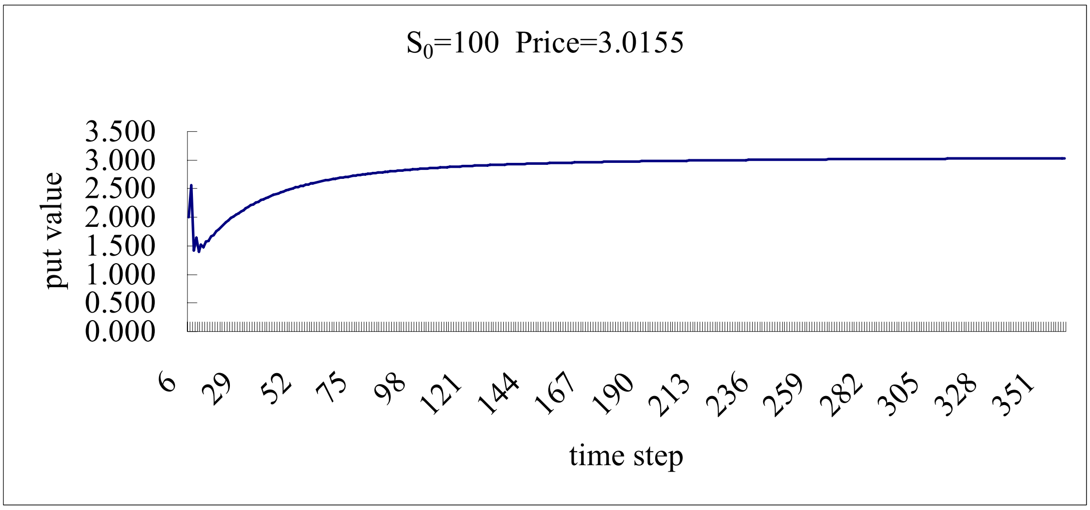
S/X X=100	Monte Carlo *N*=1,000,000	Monte Carlo *N*=10,000,000
0.80	17.6476	17.6479
0.84	13.8799	13.8786
0.88	10.4004	10.3993
0.92	7.3524	7.3570
0.96	4.8864	4.8898
1.00	3.0409	3.0401
1.04	1.7754	1.7746
1.08	0.9726	0.9743
1.12	0.5044	0.5038
1.16	0.2484	0.2488
1.20	0.1172	0.1168

Bivariate binomial is a stochastic volatility model with 270 time steps. The volatility parameter of the volatility diffusion (dV) is b = 25%. There is zero correlation between price and volatility. European puts are priced with the parameters: risk-free rate = 5%, time to maturity = 0.5 year, stock volatility = 15%, and exercise price = \$100. For stochastic volatility models, the initial volatility, $\sqrt{V_0}$, is equal to 15%, the stock volatility. There are no dividends. *N* is the number of sample paths.

Exhibit 2

The Convergence of Put Prices





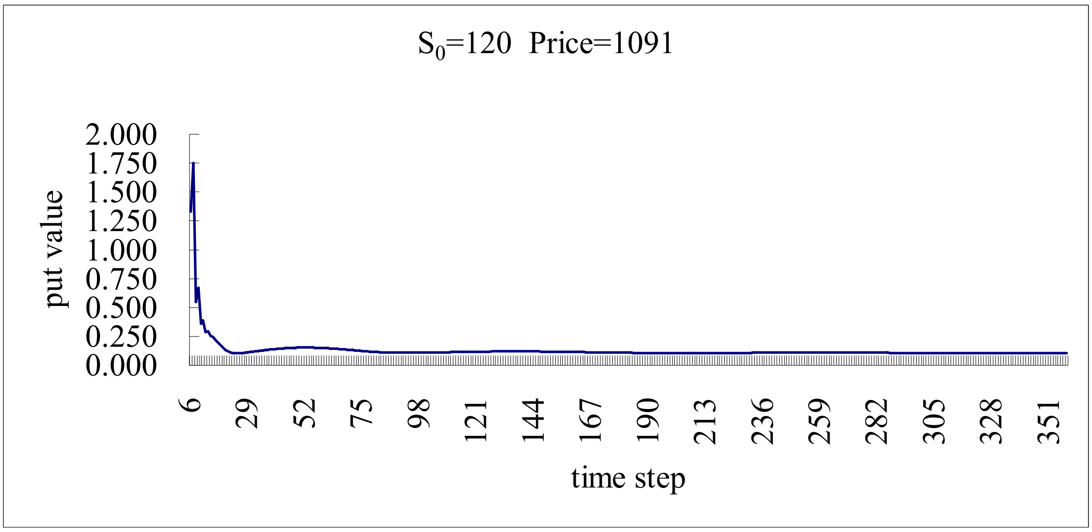
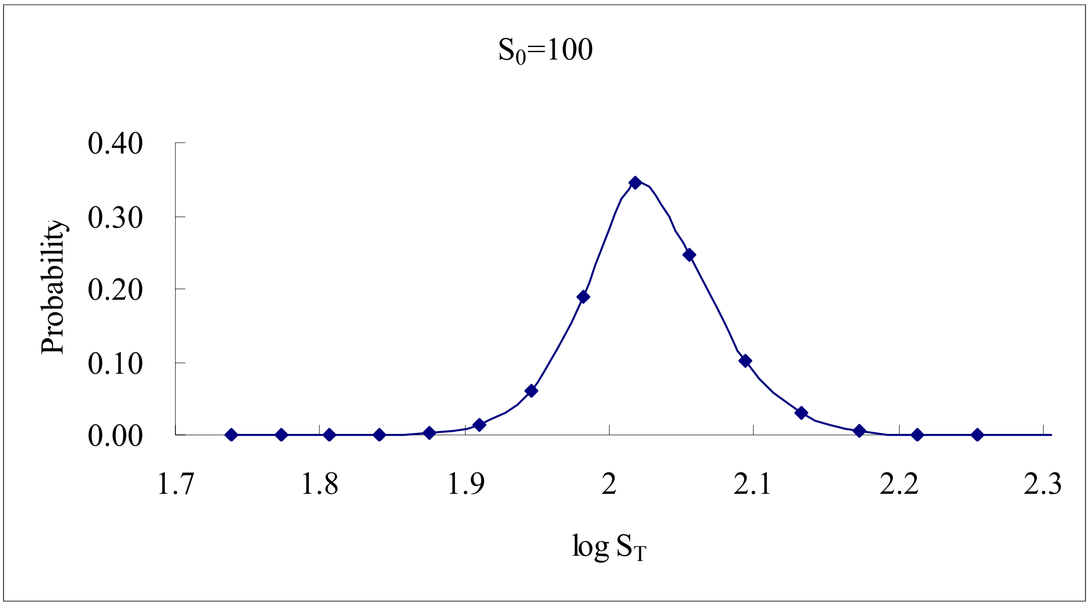


Exhibit 3 is the distribution of stock at maturity and the initial stock price is 100 with all parameters equal to Exhibit 1 except time steps are 100. There would be 101*101 stock prices after we use bivariate binomial tree method. We sort these values from small to large and then add their probabilities every 100 stock prices, so we can get 102 stock prices with their probabilities. The horizontal axis is the logarithm stock price at maturity. The vertical axis is the probability of the stock price. Some logarithm stock prices less than 1.7 and more than 2.3 with almost zero probability are cut.

In Exhibit 3, we can find that the distribution of the logarithm stock prices at maturity is like a normal distribution.

Exhibit 3

The Distribution of the Stock Price at Maturity



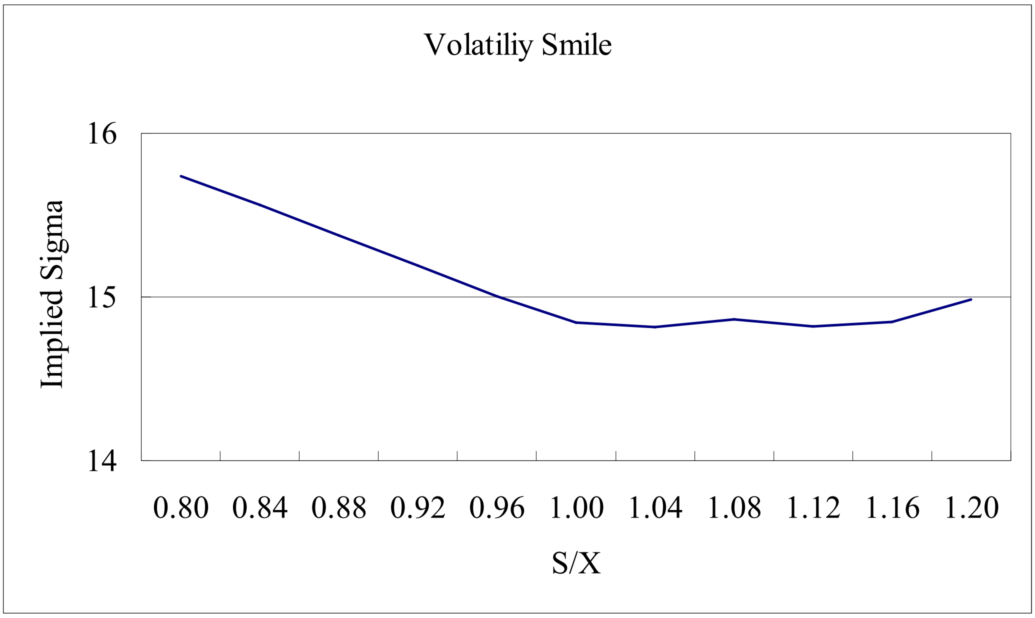
The volatility smile is a widely accepted phenomenon. The smile describes the convex shape of the implied volatility with respect to moneyness (*S/X*) computed via the Black-Scholes formula. Hull and White [1998] use Monte Carlo simulation to evaluate the effects of stochastic volatility and correlations on option prices. Their results are consistent with the smile when implied volatilities are computed by matching Black-Scholes prices to theoretically correct values.

Similarly, the bivariate binomial model also reveals significant convexities. Exhibit 4 graphs the implied volatility when the initial stock volatility is 15%, the risk-free rate is 5%, the time to maturity is 0.5 year, and the volatility drift rate is zero. There is indeed a smile across the initial stock prices. All points below the implied volatility of 15% represent overpricing by Black-Scholes.

Exhibit 4

Black-Scholes Implied Volatility Smile

S/X	Implied Volatility
0.80	15.736432
0.84	15.563522
0.88	15.376780
0.92	15.192946
0.96	15.002478
1.00	14.842511
1.04	14.814724
1.08	14.863096
1.12	14.818848
1.16	14.845893
1.20	14.984344



European puts are priced by the bivariate binomial model with 270 time steps. The exercise price is $100, the risk-free rate is 5%, the time to maturity is 0.5 year, the volatility drift is zero, and the stock volatility is 15%. The volatility parameter of the diffusion process (dV) is b = 25%. The initial volatility, $\sqrt{V_0}$, is equal to15%, the stock volatility.

Exhibit 5 is another illustration of the effects of stochastic volatility on European put options with a long maturity. We let the volatility be correlated with the underlying asset price.

Exhibit 5 shows the value of a European put with stochastic volatility parameter $b = 1.00$ and the correlation between volatility and price being -0.5, -0.25, 0, 0.25, and 0.5, respectively. The bivariate binomial model with stochastic volatility uses 1000 time steps.

Some values calculated by the tree method are close to the simulation values, but others are different from the simulation values. We can see the convergence of the put prices when the number of time steps varies in Exhibit 6. The convergence pattern is not clear even when the number of time steps exceeds 300. So the tree method for pricing long-maturity options is not without problems, and we should let the time steps be large to achieve convergence.

Compared with the values calculated by Black-Scholes which the volatility is constant and equal to the initial volatility, we find that the Black-Scholes model overprices in-the-money puts when there is negative correlation as well as out-of-money when there is positive correlation.

Exhibit 5

Effect of Stochastic Volatility on European Put Prices (Long Maturities)

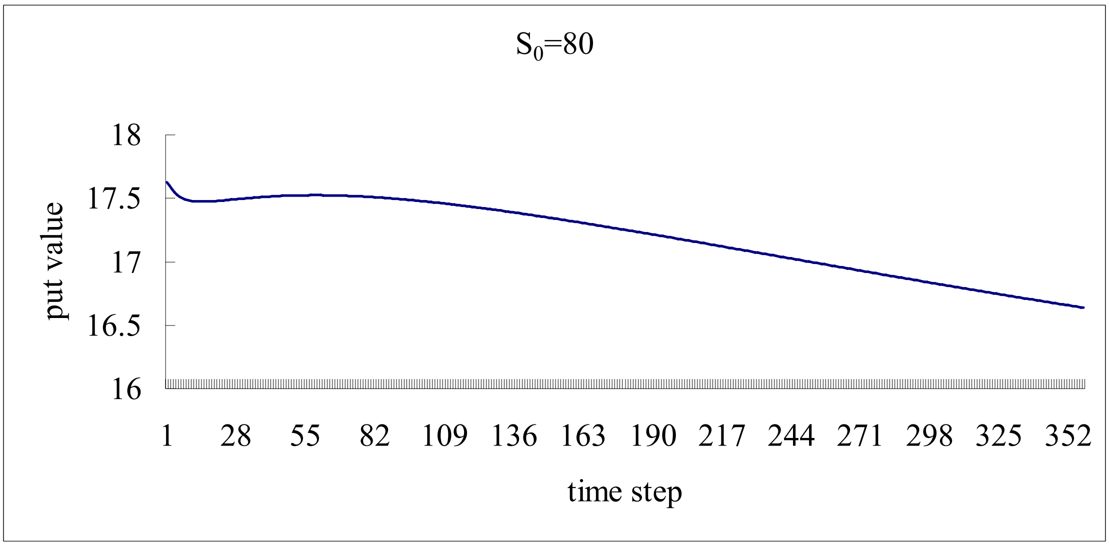
S/X X=100	Black-Scholes	ρ_{SV} = -0.50		ρ_{SV} = -0.25	
		Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate
0.80	15.716	14.267	15.7476	14.735	15.9531
0.85	12.848	11.465	12.5868	11.816	13.1262
0.90	10.393	9.217	9.9550	9.424	10.4335
0.95	8.325	7.451	7.9368	7.512	8.0610
1.00	6.611	6.073	6.3011	6.014	6.4325
1.05	5.208	4.999	4.9053	4.849	5.1444
1.10	4.074	4.159	3.8004	3.945	3.9978
1.15	3.167	3.495	2.9870	3.240	2.9660
1.20	2.449	2.965	2.3760	2.689	2.2606

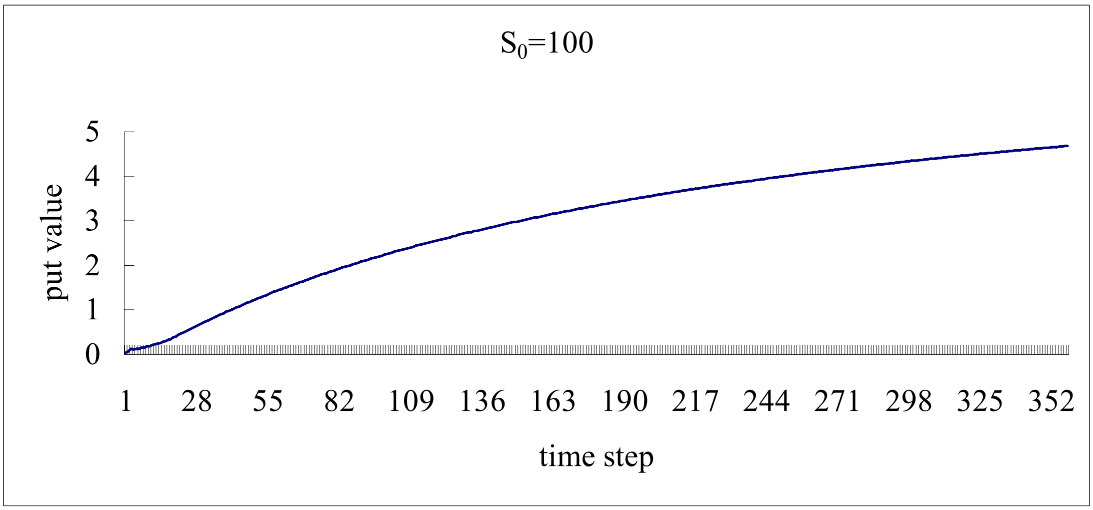
S/X X=100	ρ_{SV} = 0		ρ_{SV} = 0.25		ρ_{SV} = 0.50	
	Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate
0.80	15.145	15.8232	15.499	15.8975	15.796	15.7668
0.85	12.110	13.0572	12.351	12.7745	12.538	12.7522
0.90	9.581	10.6938	9.683	10.4086	9.721	10.0662
0.95	7.518	8.3190	7.464	8.2786	7.338	7.8827
1.00	5.890	6.0560	5.694	6.2737	5.409	6.1161
1.05	4.626	5.0036	4.319	4.6571	3.905	4.5511
1.10	3.655	4.0464	3.274	3.7122	2.775	3.2651
1.15	2.910	3.1141	2.492	2.9329	1.961	2.3614
1.20	2.340	2.1989	1.912	2.2015	1.385	1.8019

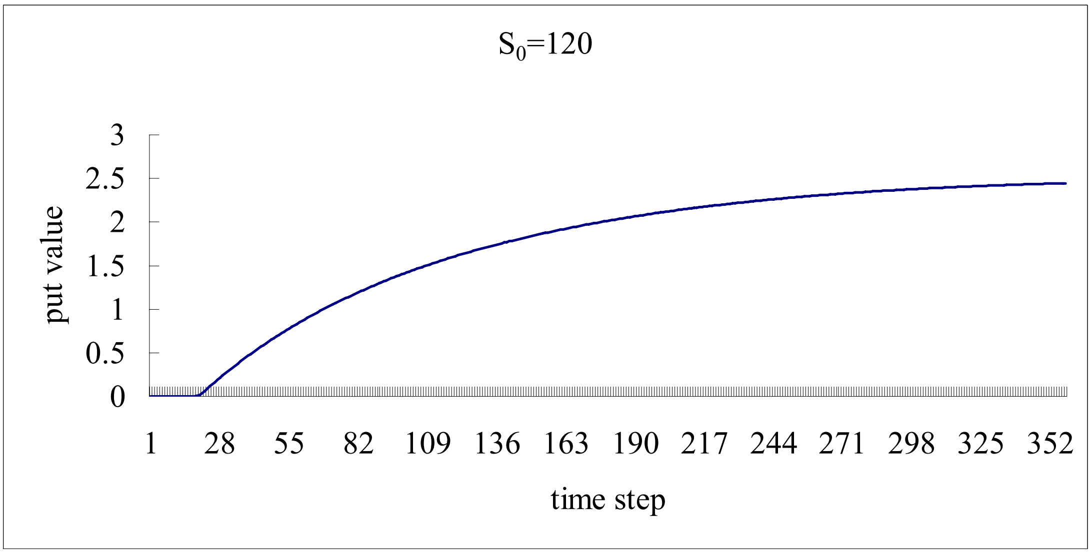
Bivariate binomial model with stochastic volatility uses 1000 time steps. The volatility parameter of the volatility diffusion (dV) is b = 1.00. European puts are priced with the parameters: risk-free rate = 5%, time to maturity = 2.0 years, stock volatility = 20%, and exercise price = \$100. For stochastic volatility models, the initial volatility, $\sqrt{V_0}$, is equal to 20%, the stock volatility. There are no dividends.

Exhibit 6

The Convergence of Put Prices







3.3 Evaluating American Put Options

The bivariate binomial model is more general than the other methods because it can evaluate American options with stochastic volatility. The following is an example for pricing American put options when the volatility is correlated with the underlying asset price.

Because the Black-Scholes formula can not evaluate American options, and Monte Carlo simulation is not appropriate for the evaluation of the early exercise premium found in American options, we use the CRR binomial tree to serve as a benchmark for the bivariate binomial model. At each node, a comparison of the early exercise value and the continuation value of the option is made. The maximum of these two values is then retained.

Exhibit 7 shows the value of an American put with stochastic volatility parameter b = 1 and correlation between volatility and price being -0.5, –0.25, 0, 0.25, and 0.5. The bivariate binomial values are estimated with 500 time steps.

In Exhibit 7, we find that the values calculated by the bivaviate binomial tree method are close to the values given by Hilliard and Schwartz [1996]. Comparing with the values calculated by CRR binomial tree, we find that the CRR binomial tree (fixed volatility) overprices in-the-money puts when there is negative correlation as well as out-of-money when there is positive correlation.

Exhibit 7

Effect of Correlation and Moneyness on Univariate and Bivariate American Put Prices

S/X X=100	CRR Binomial Tree	$\rho_{SV} = -0.50$		$\rho_{SV} = -0.25$	
		Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate
0.80	20.000	20.000	20.0000	20.000	20.0000
0.85	15.020	15.000	15.0000	15.000	15.0000
0.90	10.668	10.429	10.3857	10.550	10.4973
0.95	7.226	6.992	6.8253	7.081	6.8686
1.00	4.649	4.566	4.3683	4.570	4.2006
1.05	2.864	2.935	2.7325	2.865	2.6736
1.10	1.675	1.872	1.6576	1.763	1.6107
1.15	0.927	1.193	0.9961	1.076	0.8882
1.20	0.500	0.765	0.6010	0.656	0.5251
S/X X=100	$\rho_{SV} = 0$		$\rho_{SV} = 0.25$		$\rho_{SV} = 0.50$	
	Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate	Simulation (H&S)	Bivariate
0.80	20.000	20.0000	20.000	20.0000	20.000	20.0000
0.85	15.037	15.0000	15.092	15.0000	15.158	15.0241
0.90	10.663	10.4441	10.767	10.4747	10.863	10.6131
0.95	7.160	7.1005	7.228	7.0672	7.285	7.0577
1.00	4.565	3.8554	4.549	4.1369	4.523	4.3217
1.05	2.785	2.6226	2.689	2.4743	2.581	2.4127
1.10	1.642	1.6898	1.504	1.5935	1.352	1.3547
1.15	0.946	0.7821	0.806	0.8500	0.651	0.7843
1.20	0.539	0.4703	0.419	0.4233	0.293	0.4088

American put prices computed by the univariate and bivariate binomial models. Both models use 500 time steps. The volatility parameter of the volatility diffusion (*dV*) is *b* = 1.00. European puts are priced with the parameters: risk-free rate = 5%, time to maturity = 0.5 year, stock volatility = 20%, and exercise price = \$100. For stochastic volatility models, the initial volatility, $\sqrt{V_0}$, is equal to 20%, the stock volatility. There are no dividends.

Chapter 4

Conclusions

In the paper by Hilliard and Schwartz [1996], they develop a stochastic volatility model that is simple and accurate. The basis of the model is a lattice formed from a possibly correlated volatility process and an underlying price process. These processes are then transformed to form a recombining bivariate binomial tree with attractive convergence properties.

However, they do not seem to implement the method they develop. Instead, they only use simulation to get the prices. In this thesis, we use the model they develop and get the numbers by bivariate binomial tree method. The numbers show that the values given by the bivariate binomial tree method are almost identical to those in Hull and White [1987] and Hilliard and Schwartz [1996]. Unlike the Hull-White model, the methods in this thesis are also appropriate for non-zero correlations. In addition, they are effective for value American options.

Appendix A The Derivation of the Process of *Y*

The transformation of *Y* is defined by:

$$Y = \frac{\ln(V)}{b}$$

$$\therefore dY = Y_V dV + \frac{1}{2} Y_{VV} dV^2$$

From Eq. (2.3):

$$dV = m_V dt + bVdZ_V \ ,$$

so we can get

$$dV^2 = b^2 V^2 dt$$

$$\therefore dY = \frac{1}{bV}(m_V dt + bVdZ_V) + \frac{1}{2}(-\frac{1}{bV^2})(b^2 V^2 dt)$$

$$= (\frac{m_V}{bV} - \frac{b}{2})dt + dZ_V$$

Appendix B The Derivation of m_h

The transformation $H = H(S,V)$ is defined by:

$$H = H(S,V) = h^{-1}(V)\int^{S} \frac{dS}{f(S)}$$

With $h(V) = V^{\alpha}$, the derivatives are given by:

$$H_S = \frac{1}{f(S)V^{\alpha}}, \; H_{SS} = -\frac{f_S}{f^2 V^{\alpha}}$$

$$H_V = -\frac{\alpha H}{V}, \; H_{VV} = \frac{-\alpha(-\frac{\alpha H}{V})V - (-\alpha H)}{V^2} = \frac{\alpha H(1+\alpha)}{V^2}$$

$$H_{SV} = -\frac{\alpha}{f(S)V^{\alpha}V}$$

$$\begin{aligned}
\therefore dH &= H_S dS + H_V dV + \frac{1}{2}[H_{SS} dS^2 + 2H_{SV} dSdV + H_{VV} dV^2] \\
&= H_S(m_S dt + f(S)h(V)dZ_S) + H_V(m_V dt + bVdZ_v) \\
&\quad + \frac{1}{2}[H_{SS} f^2(S)V^{2\alpha} dt + 2H_{SV} bf(S)V^{\alpha}V\rho_{SV} dt + H_{VV} b^2 V^2 dt] \\
&= H_S f(S)V^{\alpha} dZ_S + H_V bVdZ_v + m_h dt
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\therefore m_h &= H_S m_S + H_V m_V + \frac{1}{2} H_{SS} f^2(S)V^{2\alpha} + H_{SV} bf(S)V^{\alpha}V\rho_{SV} + \frac{1}{2} H_{VV} b^2 V^2 \\
&= \frac{m_S}{f(S)V^{\alpha}} - \frac{m_V \alpha H}{V} + \frac{1}{2}[-\frac{f_S}{f^2 V^{\alpha}} f^2(S)V^{2\alpha}] - \frac{\alpha bf(S)V^{\alpha}V\rho_{SV}}{f(S)V^{\alpha}V} + \frac{1}{2}\frac{\alpha H(1+\alpha)}{V^2} b^2 V^2 \\
&= \frac{m_S}{f(S)V^{\alpha}} - \frac{m_V \alpha H}{V} - \frac{1}{2} f_S V^{\alpha} + \frac{1}{2}\alpha H(1+\alpha)b^2 - \alpha b\rho_{SV}
\end{aligned}$$

Appendix C The Derivation of m_q

The transformation $Q = Q(H)$ is defined by:

$$Q = Q(H) = (\alpha b)^{-1} \ln(\alpha bH - \rho_{SV} + \sigma_h),$$

where $\sigma_h = \sqrt{1 - 2\alpha bH\rho_{SV} + \alpha^2 b^2 H^2}$. The derivatives are given by:

$$(\sigma_h)_h = \frac{-2\alpha b\rho_{SV} + 2\alpha^2 b^2 H}{2\sqrt{1 - 2\alpha bH\rho_{SV} + \alpha^2 b^2 H^2}} = \frac{-\alpha b\rho_{SV} + \alpha^2 b^2 H}{\sigma_h}$$

$$Q_h = (\alpha b)^{-1} \frac{\alpha b + (\sigma_h)_h}{\alpha bH - \rho_{SV} + \sigma_h} = (\alpha b)^{-1} \frac{\alpha b + \dfrac{-\alpha b\rho_{SV} + \alpha^2 b^2 H}{\sigma_h}}{\alpha bH - \rho_{SV} + \sigma_h} = \frac{1 + \dfrac{-\rho_{SV} + \alpha bH}{\sigma_h}}{\alpha bH - \rho_{SV} + \sigma_h}$$

$$= \frac{\dfrac{\sigma_h - \rho_{SV} + \alpha bH}{\sigma_h}}{\alpha bH - \rho_{SV} + \sigma_h} = \frac{1}{\sigma_h}$$

$$Q_{hh} = (Q_h)_h = (\frac{1}{\sigma_h})_h = \frac{-(\sigma_h)_h}{\sigma_h^{\ 2}} = \frac{-\dfrac{-\alpha b\rho_{SV} + \alpha^2 b^2 H}{\sigma_h}}{\sigma_h^{\ 2}} = \frac{\alpha b\rho_{SV} - \alpha^2 b^2 H}{\sigma_h^{\ 3}}$$

From Eq. (B.1):

$$dH = m_h dt + H_S f(S) V^\alpha dZ_S + H_V bV dZ_v .$$

So we can get

$$dH^2 = [H_S f(S) V^\alpha dZ_S + H_V bV dZ_v]^2 = [dZ_S - \alpha bH dZ_V]^2 = [1 - 2\alpha bH\rho_{SV} + \alpha^2 b^2 H^2]dt = \sigma_h^{\ 2} dt$$

$$\therefore dQ = Q_h dH + \frac{1}{2} Q_{hh} dH^2 = \frac{1}{\sigma_h}(m_h dt + H_S f(S) V^\alpha dZ_S + H_V bV dZ_v) + \frac{1}{2} \frac{\alpha b\rho_{SV} - \alpha^2 b^2 H}{\sigma_h^{\ 3}} \sigma_h^{\ 2} dt$$

$$= \frac{1}{\sigma_h} H_S f(S) V^\alpha dZ_S + \frac{1}{\sigma_h} H_V bV dZ_v + [\frac{m_h}{\sigma_h} + \frac{1}{2} \frac{\alpha b\rho_{SV} - \alpha^2 b^2 H}{\sigma_h^{\ 3}} \sigma_h^{\ 2}]dt$$

$$= \frac{1}{\sigma_h} H_S f(S) V^\alpha dZ_S + \frac{1}{\sigma_h} H_V bV dZ_v + m_q dt$$

$$\therefore m_q = \frac{m_h}{\sigma_h} + \frac{1}{2} \frac{\alpha b\rho_{SV} - \alpha^2 b^2 H}{\sigma_h^{\ 3}} \sigma_h^{\ 2} = \frac{m_h}{\sigma_h} + \frac{1}{2} \frac{\alpha b\rho_{SV} - \alpha^2 b^2 H}{\sigma_h}$$

Appendix D The Derivation of *H* from *Q* (Proof of Eq. (2.5))

The transformation of $Q = Q(H)$ is defined by:

$$Q = (\alpha b)^{-1} \ln(\alpha bH - \rho_{SV} + \sigma_h) = (\alpha b)^{-1} \ln(\alpha bH - \rho_{SV} + \sqrt{1 - 2\alpha bH \rho_{SV} + \alpha^2 b^2 H^2})$$

$$\therefore \alpha bQ = \ln(\alpha bH - \rho_{SV} + \sqrt{1 - 2\alpha bH \rho_{SV} + \alpha^2 b^2 H^2})$$

$$\therefore \exp(\alpha bQ) = \alpha bH - \rho_{SV} + \sqrt{1 - 2\alpha bH \rho_{SV} + \alpha^2 b^2 H^2}$$

$$\therefore \exp(\alpha bQ) + \rho_{SV} - \alpha bH = \sqrt{1 - 2\alpha bH \rho_{SV} + \alpha^2 b^2 H^2}$$

$$\therefore [\exp(\alpha bQ) + \rho_{SV} - \alpha bH]^2 = 1 - 2\alpha bH \rho_{SV} + \alpha^2 b^2 H^2$$

$$\therefore [\exp(\alpha bQ) + \rho_{SV}]^2 - 2[\exp(\alpha bQ) + \rho_{SV}]\alpha bH = 1 - 2\alpha bH \rho_{SV}$$

$$\therefore [\exp(\alpha bQ) + \rho_{SV}]^2 - 1 = 2\alpha bH \exp(\alpha bQ)$$

$$\therefore H = \frac{2\rho - (1 - \rho_{SV}{}^2)\exp(-\alpha bQ) + \exp(\alpha bQ)}{2\alpha b}$$

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