Interest Rate Barrier Options Pricing

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Abstract

Cheuk and Vorst's method [1996a] can be applied to price barrier options using one-factor interest rate models when recombining trees are available. For the Hull-White model, barriers on bonds or swap rates are transformed to time-dependent barriers on the short rate and we use a time-dependent shift to position the tree optimally with respect to the barrier. Comparison with barrier options on bonds or swaps when the observation frequency is discrete confirms that the method is faster than the Monte Carlo method. Unlike other methods which are only applicable in the continuously observed case, the lattice methods can be used in both the continuously and discretely observed cases. We illustrate the methodology by applying it to value single-barrier swaption and single-barrier bond options. Moreover, we extend Cheuk and Vorst's idea [1996b] to double-barrier swaption pricing.

Chapter 1

Introduction

1.1 Setting the Ground

The financial world has witnessed an explosive growth in the trading of derivative securities since the opening of the first options exchange in Chicago in 1973. The growth in derivatives markets has not only been that of volume but also of complexity. Many of these more complex derivative contracts are not exchange traded, but are traded "over-the-counter." Over-the-counter contracts provide tailor made products to reduce financial risks for clients.

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. In the 1980s and 1990s, the volume of trading in interest rate derivatives in both the over-the-counter and exchange-traded markets increased very quickly. Interest rate derivatives have become most popular products in all derivatives markets. Exhibits 1 and 3 show the amount of interest rate derivatives in the OTC and exchange-traded markets (also plotted in Exhibits 2 and 4). Moreover, many new products have been developed to meet particular needs of end-users. A key challenge for derivatives traders is to find better and more robust procedures for pricing and hedging these contracts.

EXHIBIT 1

OTC DERIVATIVES: AMOUNT OUTSTANDING

Global market, by instruments, in billion of US dollars						
	1998	1999	2000	2001	2002	2003
Interest rate derivatives	50015	60091	64668	77568	101658	141991
Forward rate agreements	5756	6775	6423	7737	8792	10769
Interest rate swaps	36262	43936	48768	58897	79120	111209
Interest options	7997	9380	9476	10933	13746	20012
Other derivatives	30303	28111	30531	33610	40021	55186

Source: Bank of International Settlements.

EXHIBIT 2

OTC DERIVATIVES GRAPH
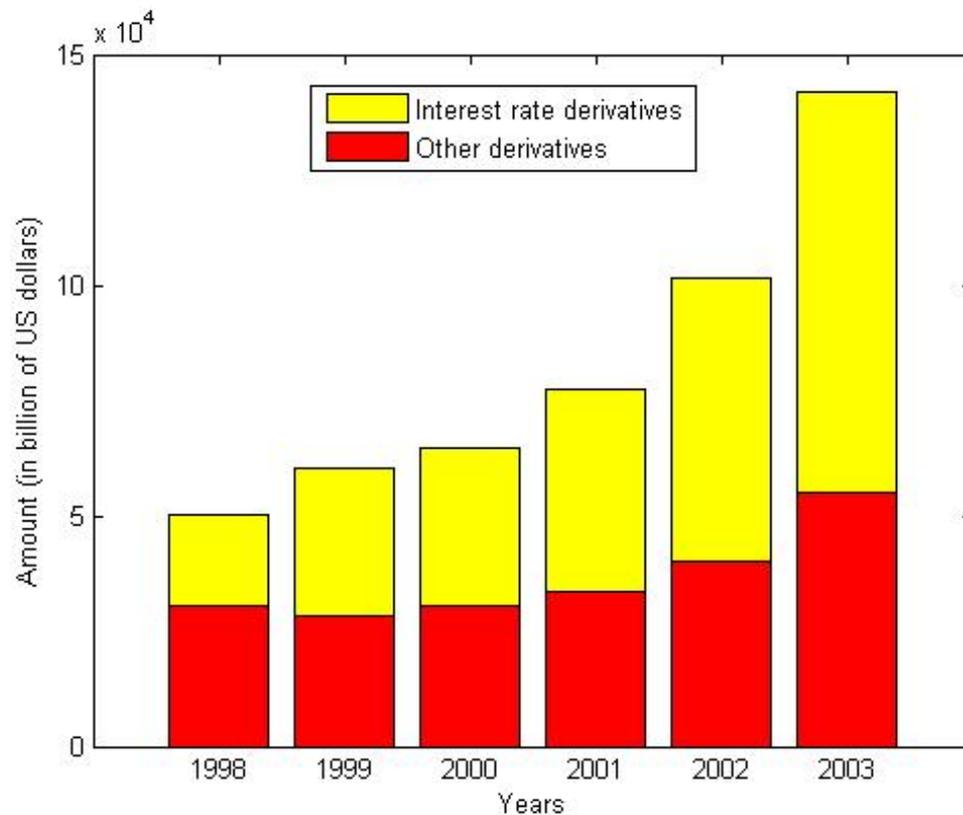
EXHIBIT 3

EXCHANGE-TRADED DERIVATIVES: AMOUNTS OUTSTANDING

Notional principal in billions of US dollars											
	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
Interest rate derivatives	7304	8431	8618	9257	11227	12655	11681	12642	21758	21711	33933
Interest rate futures	4943	5808	5876	5979	7587	8031	7925	7908	9265	9951	13132
Interest rate options	2361	2623	2742	3278	3640	4624	3756	4734	12493	11760	20801
Other derivatives	453	467	664	761	1180	1280	1909	1616	2002	2099	2818

Source: Bank of International Settlements.

EXHIBIT 4

EXCHANGE-TRADED DERIVATIVES GRAPH
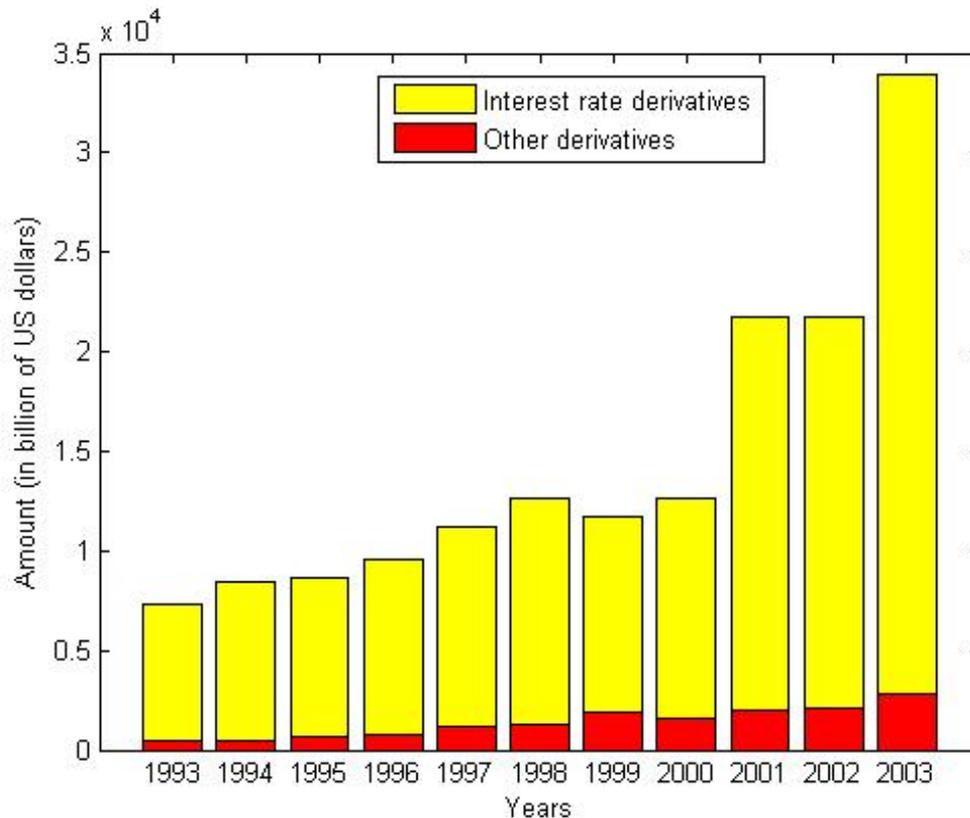
Interest rate derivatives are more difficult to value than equity and foreign

exchange derivatives. Reasons include

1. The behavior of interest rate is harder to capture than that of a stock price or an exchange rate, so many models are introduced to approach it.

2. For the valuation of many products, it is necessary to develop a model describing the behavior of the entire zero-coupon yield curve, but it is not easy to depict the curve exactly.

3. The volatilities of different points on the yield curve are different.

4. Interest rates are considered as variables for discounting as well as for defining the payoff form the derivative.

In this thesis we focus on interest rate options with barriers. Barrier options are options where the payoff depends on whether the underlying asset's price/level reaches a certain threshold during a certain period of time. A number of different types of barrier options are regularly traded in the over-the counter market. They are attractive to some market participants because they are less expensive than the standard options. These barrier options can be classified as either knock-out or knock-in types. A knock-out option ceases to exist when the underlying asset's price hits a certain barrier; a knock-in option comes into existence only when the underlying asset's price or level hits a barrier.

Interest rate barrier options usually involve one or two time-dependent boundaries affecting the option prices, and exact closed-form solutions are not available for most interest rate models.

It is well-known that for barrier options, the positions of nodes in the tree with respect to the barrier value are critical. Cheuk and Vorst [1996b] proposed a trinomial tree model for barrier options. It uses a time-dependent shift to position the tree optimally with respect to the barrier. The model they constructed is flexible and can be used to price options with time-varying barrier structures such as interest rate

derivatives.

For now, we will focus our attention on down-and-out swaptions and down-and
-out bond options. The single-barrier down-and-out swaption and the single-barrier
down-and-out bond options are priced using the methodology of Cheuk and Vorst
[1996a]. The double-barrier down-and-out swaption is priced by extending the idea of
Cheuk and Vorst [1996a]. Other interest rate barrier options can be priced using
similar ideas.

1.2 Survey of Literature

Valuation approaches have largely focused on equity barrier options, where in
certain instances analytical expressions may be available. Works on interest rate
barrier options pricing are relatively rare. Accurate and efficient valuation techniques
are required since barrier options have become very popular in recent years as useful
hedging instruments for risk management strategies.

Several researches address the pricing of interest rate barrier options. Cheuk
and Vorst [1996a] extended Ritchken's method by introducing a time-dependent shift
in trinomial lattice. The Hull-White model is selected and single-barrier swaptions are
priced in both the continuously and discretely observed cases. Kuan and Webber
[2003] use one-factor interest rate models including the Hull-White model and the
swap market model to value barrier knock-in bond options and barrier knock-in
swaptions. A numerical integration method is used to price interest barrier options
when the transition distribution function of underlying rate is known but explicit
pricing formulas are not available. Although the convergence is fast, the drawback is
that the valuation only can be applied to the continuously observed case. Monte Carlo

simulation is known for its high flexibility. However, in the case of barrier option it produces biased results for options, which depend on the continuously monitored sample path of some stochastic variable. Barone-Adesi and Sorwar [2003] price continuously observed barrier bond options in the corrected Monte-Carlo simulation of the CKLS diffusion process:

$$dr = (\theta - ar)dt + \sigma r^\gamma dz \tag{1.1}$$

Using the results of Baldi et al [1999], they set up a Monte-Carlo scheme to value interest rate barrier options which takes into account the possibility of breaching the barrier between successive intervals of time. It has enough flexibility to price all kinds of continuously observed interest rate barrier options. This procedure then provides an almost unbiased Monte Carlo estimator. However, the speed of convergence is still very slow and need many paths of the underlying interest rate process to obtain accurate results.

1.3 Thesis Structure

The remainder of this thesis will be organized as follows. Chapter 2 reviews the Black's model pricing technologies and explains how to calibrate Hull-White model with the Black's model consistently. Chapter 3 covers the pricing of single-barrier swaptions and bond options. The pricing of double-barrier swaptions is presented in Chapter 4. Chapter 5 concludes.

Chapter 2

Preliminaries

First, define the following variables:

T_0 : Current time,

T : Time to maturity of the option,

$P(t,T)$: Price at time t of a zero-coupon bond paying \$1 at time T ,

$y(t,T)$: the $(T-t)$-period interest rate (annualized) at time t .

2.1 General Framework

Vanilla European contingent claims such as caps, floors, bond options, and swaptions are priced correctly using the simple model developed by Black [1973]. This model makes several simplifying assumptions which allow closed-form valuation formulae to be possible. Svoboda [2004] referred the class of vanilla contingent claims as first-generation products.

Second- and third-generation derivatives, such as path-dependent and barrier options, provide exposure to the relative levels and correlated movements of various portions of the yield curve, and the market prices of these first-generation instruments are taken as given. This does not necessarily imply a belief in the intrinsic correctness of the Black model. Distributional assumptions which are not included in the Black model, such as mean reverting and skewness, are incorporated by adjusting the implied volatility input.

More sophisticated models allow the pricing of instruments dependent on the changing level and slope of the yield curve. A crucial factor is that these models must price the exotic derivatives in a manner that is consistent with the pricing of vanilla instruments. When assessing the correctness of any more sophisticated model, its

ability to reproduce the Black prices of vanilla instruments is vital. Svoboda [2004] remarked that it is not a model's a priori assumptions but rather the correctness of its hedging performance that plays a pivotal role in its market acceptance.

The calibration of a model is an integral part of its specification. So the usefulness of a model cannot be assessed without considering the reliability and robustness of any parameter estimation scheme.

2.2 The Standard Market Models

2.2.1 Black's Model

We will show how the Black's model for valuing vanilla European interest rate options is derived. The market price will be given, and the parameters of more complex models can be valued by minimizing deviations from market prices.

Black's model is used to value options on foreign exchanges, options on indices, and options on future contracts. Traders have become very comfortable with both the lognormal assumption that underlies the model and the volatility measure that describes uncertainty. It is no surprise that there have been attempts to extend the model so that it covers interest rate derivatives.

In the following we will discuss three of the most popular interest rate derivatives (bond options, interest rate caps, and swap options) and describe how the lognormal assumption underlying the Black-Scholes model can be used to value these instruments. Now we use Black's model to price European options.

Consider a European call option g on a variable whose value is V. Define

T : Time to maturity of the option,

F : Forward price of V for a contract with maturity T,

F_0 : Value of F at time T_0,

K : Strike price of the option,

V_T: Value of V at time T,

σ: Volatility of F.

Black's model calculates the expected payoff from the option assuming:

1. V_T has a lognormal distribution with the standard deviation of $\ln V_T$ equal to $\sigma \sqrt{T}$.

2. The expected value of V_T is F_0.

3. The world is forward risk neutral with respect to $P(t,T)$; i.e., $g_t / P(t,T)$ is a martingale in the world.

So the value of the option at time T is $\max(V_T - K, 0)$ and the lognormal assumption implies that the expected payoff is

$$E(V_T) N(d_1) - KN(d_2), \tag{2.1}$$

where $E(V_T)$ is the expected value of V_T and

$$d_1 = \frac{\ln\left(E(V_T)/K\right) + \sigma^2 T/2}{\sigma \sqrt{T}}.$$
$$d_2 = d_1 - \sigma \sqrt{T}$$

Because $g_t / P(t,T)$ is a martingale with respect to $P(t,T)$, it follows that

$$\frac{g_0}{P(T_0,T)} = E_T\left(\frac{g_T}{P(T,T)}\right) = E_T(g_T). \tag{2.2}$$

Besides, we are assuming that $E(V_T) = F_0$ and the validity of discounting at the risk-free rate, so the value of the option is given by (2.1) and (2.2)

$$g_0 = P(T_0,T)\left[F_0 N(d_1) - KN(d_2)\right], \tag{2.3}$$

where
$$d_1 = \frac{\ln\left(F_0/K\right) + \sigma^2 T/2}{\sigma \sqrt{T}}.$$
$$d_2 = d_1 - \sigma \sqrt{T}$$

2.2.2 Bond Options

A bond option is an option to long or short a particular bond by a certain date for a particular price. Consider a zero-coupon bond $P(T_0, T^*)$ and we assume the world is forward risk-neutral with respect to a zero-coupon bond maturing at time T ($T \le T^*$). So the price of a call option with strike price K and maturity T (years) on a bond $P(T_0, T^*)$ is

$$c = P(T_0, T) E_T \left[\max \left(P(T, T^*) - K, 0 \right) \right],\tag{2.4}$$

where E_T denotes expected value in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T. It implies that

$$E_T \left(P(T, T^*) \right) = F_0,$$

where F_0 is the forward price of $P(T, T^*)$ at time T_0.

Assuming the bond price is lognormal with the standard deviation of the logarithm of the bond price equal to $\sigma\sqrt{T}$, the equation (2.3) becomes:

$$c = P(T_0, T) \left[F_0 N(d_1) - K N(d_2) \right],\tag{2.5}$$

where

$$d_1 = \frac{\ln\left[F_0 / K \right] + \sigma^2 T / 2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

This reduces to Black's model. We have shown that we can use today's T-year maturity interest rate for discounting provided that we also set the expected bond price equal to the forward bond price.

2.2.3 Interest Rate Caps

A popular interest rate option offered by financial institutions in the over-the-counter market is the interest rate cap. Consider a cap with a total life of T,

a principal of C, and a cap rate of K. Suppose that the reset dates are $T_1, T_2, ..., T_n$ and define $T_{n+1} = T$. Define $y(T_k, T_{k+1})$ as the simply compounded interest rate for the period between times T_k and T_{k+1} observed at time T_k $(1 \le k \le n)$:

$$y(T_{k-1}, T_k) = \frac{1 - P(T_{k-1}, T_k)}{\delta P(T_{k-1}, T_k)}.$$ (2.6)

The caplet corresponding to the rate observed at time T_k provides a payoff at time T_{k+1} of

$$C\delta_k \max\left(y(T_k, T_{k+1}) - K, 0 \right),$$

where $\delta_k = T_{k+1} - T_k$. If the rate $y(T_k, T_{k+1})$ is assumed to be lognormal with volatility σ_k, the value of the caplet is

$$caplet = C\delta_k P(T_0, T_{k+1})\left[F_k N(d_1) - KN(d_2) \right],$$ (2.7)

where

$$d_1 = \frac{\ln\left[F_k / R_K \right] + \sigma_k^2 T_k / 2}{\sigma_k \sqrt{T_k}}$$

$$d_2 = d_1 - \sigma_k \sqrt{T_k}$$

and F_k is the forward rate for the period between time T_k and T_{k+1}. Note that $y(T_k, T_{k+1})$ and F_k are expressed with a compounding frequency equal to frequency of resets in these equations.

The cap is a portfolio of n such options and the formula also reduces to a summation of Black-like formulas.

2.2.4 European Swap Options

Swap options, or swaptions, are options on swap rates and are a very popular type of interest rate option. They give the holder the right to enter a certain interest

rate swap that pays a fixed rate, the strike rate, and receives a floating interest rate at a certain time in the future.

Consider a swaption which gives the right to pay a rate K and receive $y(T_{k-1}, T_k)$ on s swap settled in arrears at time $T_k = T + k\delta, k = 1,...,n$ with a notional principal C. Suppose that the swap rate for an n-payments swap at the maturity of the swap option is s_T. Assume the relevant swap rate at the maturity of the option is lognormal. By comparing the cash flows on a swap where the fixed rate is K. The total value of the swaption is

$$\sum_{i=1}^{n} C\delta P(T_0, T_i)\left[s_0 N(d_1) - KN(d_2)\right], \tag{2.8}$$

where

$$d_1 = \frac{\ln\left[s_0 / K\right] + \sigma^2 T / 2}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and s_0 is the forward swap rate starting at time T and will be introduced below.

Forward swap rate

The forward swap rate $\kappa_T(T_0, n)$ at time T can also be determined using the formula (Brace and Musiela 1997) that makes the value of the forward swap zero, i.e.,

$$\kappa_T(T_0, n) = \frac{P(T_0, T) - P(T_0, T + n\delta)}{\delta\sum_{j=1}^{n} P(T_0, T + n\delta)}. \tag{2.9}$$

2.2.5 Generalizations

We have presented three versions of Black's model: one for bond options, one for caps, and one for swaptions. Each of the models is internally consistent with each other, but they are not consistent with each other. For example, when future bond prices are lognormal, future zero rates and swap rates can not be, and when future zero rates are lognormal, future bond prices and swap rates can not be.

With the market price by Black's formula we now proceed to introduce how to calibrate more complex models.

2.3 Hull-White Model

2.3.1 Model Formulation

We will use the short rate model to price barrier options in the thesis. Hull and White proposed an extension to the Vasicek model of the one-factor form:

$$dr = \left(\theta(t) - ar\right)dt + \sigma dz, \tag{2.10}$$

where $\theta(t)$ is a time-dependent reversion level chosen so that the spot yield rate curve implied by the model matches the yield curve observed initially, a is the speed of mean reversion, and σ is a known constant.

It provides enough degrees of freedom to fit the current interest rate term structure. The process describing the evolution of the short rate can be deduced from the observed term structure of interest rates and interest rate volatilities.

We will value the European call option on a zero coupon bond and describe how to use the information from observed term structure of interest rate and volatilities. We then go on to make sure that the model is consistent with market prices by calibrating the Hull-White model for constant mean reversion and volatility parameter. After finishing all work above the more complicated interest rate derivatives, barrier options, will be investigated in the next chapter.

2.3.2 Pricing Bond Options within the Hull-White Framework

Let $P(t,T)$ be the time t price of a discount bond maturing at time T. The bond price formula is shown in Hull and White [1990, 1994a] to be

$$P(t,T) = A(t,T)e^{-B(t,T)r}, \tag{2.11}$$

where $P(t,T)$ is the price at time t of a zero-coupon bond maturing at time T.

Furthermore, $A(t,T)$ and $B(t,T)$ are functions only of t and T, and r is the short

rate at time t. The function $A(t,T)$ is determined from the initial values of discount

bonds $P(0,T)$ as

$$A(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[B(t,T)F(0,t) - \sigma^2 B(t,T)^2 \left(1-e^{-2at}\right)\Big/(4a) \right] \qquad (2.12)$$

$$B(t,T) = \left(1-e^{-a(T-t)}\right)\Big/a, \qquad (2.13)$$

$F(0,t)$ is the instantaneous forward rate that applies to time t as observed at time

zero. It can be computed from the initial price of discount bond as

$$F(0,t) = -\frac{\partial \log\left[P(0,t)\right]}{\partial t}. \qquad (2.14)$$

By Ito's Lemma we have:

$$\begin{aligned}
dP &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}drdr \\
&= \frac{\partial P}{\partial t}dt - ABe^{-Br}\left[\left(\theta(t)-ar\right)dt + \sigma dz\right] + \frac{1}{2}AB^2 e^{-Br}\sigma^2 dt \qquad (2.15) \\
&= P_t dt - BP\left[\theta(t)-ar\right]dt - BP\sigma dz + \frac{1}{2}B^2 P\sigma^2 dt
\end{aligned}$$

Hence the price process of the discount bond is described by the stochastic equation:

$$dP = \left[P_t - BP\left(\theta(t)-ar\right) + \frac{1}{2}B^2 P\sigma^2\right]dt - PB\sigma dz. \qquad (2.16)$$

The relative volatility of $P(r,t,T)$ is $B(t,T)\sigma$. Since it is independent of r, the

distribution of the bond price at any time t^*, conditional on its value at an earlier

time t, must be lognormally distributed.

Consider a European option on this discount bond. This option has the

following characteristics:

K : exercise price,

T : option expiry time,

T^* : bond maturity time,

T_0 : current (valuation) time, where $T_0 \le T \le T^*$.

Since $P(T, T^*)$ is lognormally distributed and represents the forward bond price, the

Black's formula can be used to price the discount bond option:

$$C = P(T_0, T^*) N(h) - KP(T_0, T^*) N(h - \sigma_p),\qquad(2.17)$$

where $h = \dfrac{\ln\left(\dfrac{P(T_0, T^*)}{\left(KP(T_0, T^*)\right)}\right)}{\sigma_p} + \dfrac{1}{2}\sigma_p$

and $\sigma_p^2 = \dfrac{\sigma^2}{2a^3}\left(1 - e^{-a(T^* - T)}\right)^2 \left(1 - e^{-2a(T - T_0)}\right)$.

2.3.3 Calibrating the Hull-White Model

Having considered the model formulation that allows us to incorporate observed term structure data into the pricing formula, we explain how actual data are used in the calibration exercise.

Cubic spline interpolation

An interpolation technique must be applied to term structure so that values for any maturity term maybe extracted. Cubic spline interpolation was favored for the smoothness of curve it produces. Cubic spline interpolation is a type of piecewise polynomial approximation that uses a cubic polynomial between successive pairs of nodes. At each of nodes across which the cubic spline is fitted, the following hold:

- The values of the fitted splines equal the values of the original function at the node points.
- The first and the second derivatives of the fitted splines are continuous.

The algorithm is presented in Appendix A.

Using the market data

For each day, the continuously compounded rate of interest and historical volatility are available for a discrete set of node points corresponding to terms to maturity, t_i, $i = 1,...,N$ where $t_1 = \frac{1}{365}$ year and $t_N = T$.

- The interest rate with term to maturity 1 day and its corresponding historical volatility are taken as proxies for the instantaneous short-term interest rate $r(T_0)$ and its volatility $\sigma_r(r,T_0)$.

- $B(T_0,T)$ is calculated using $B(T_0,T) = \frac{(1-e^{-aT})}{a}$. Applying the initial term structure of interest rates $y(T_0,t_i)$, we determine the time T_0 discount bond prices as $P(T_0,T) = e^{-y(T_0,T)T}$.

- Apply $B(T_0,t_i)$ and $P(T_0,t_i)$ to find $A(T_0,t_i)$.

Calibration methodology

Calibration of the model to the observed market prices involves retrieving values of σ and a such that these market prices may be recovered from the model. The reversion speed a and the associated volatility parameter σ should give rise to the smallest pricing error. That is, we want the a and σ such that

$$\sum_{i=1}^{n} \left(p_{model}(a,\sigma) - p_{market} \right)^2 ,$$

is minimized for several different maturities t_i, $i = 1,...,n$ and (2.17) is used to calculate $p_{model}(a,\sigma)$.

Time-dependent mean reversion level $\theta(t)$

As shown by Hull and White (1993), the time-dependent mean reversion level $\theta(t)$ is determined at the initial time T_0 as

$$\theta(t) = \frac{\partial F(T_0, t)}{\partial t} + aF(T_0, t) + \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-T_0)}\right).$$ (2.18)

Thereafter, our model can be consistent with market prices, and exotic options pricing will be followed under the calibration parameters a and σ.

Chapter 3

Cheuk and Vorst's Method

First, define the following variables:

T_0 : current time,

T : option maturity,

$w(t,n)$: the spot swap rate at time t which makes the value of the swap

(n payments) zero,

β : the barrier swap rate fixed at the spot swap rate $w(T_0,n)$ minus the same fixed

rate throughout (for example, 25 basis points),

h_i : the barrier short rate at time t_i is found by setting the value of the swap at the

fixed rate β zero,

ϕ_i : the barrier Δt -period rate (annualized) at time t_i that corresponds to h_i,

$\kappa_T(t,n)$: the at-the-money forward swap rate at time T which makes the value of

forward swap zero.

In our tree, we define three variables at node (i, j):

$R(i, j)$: the Δt -period interest rate at time t_i that is associated with node (i, j),

$r(i, j)$: the short rate that is associated with node (i, j),

$s(i, j)$: the swap rate at time t_i that is associated with node (i, j).

3.1 Single-Barrier Swaptions

3.1.1 Definitions

With a down-and-out swaption the holder can choose to enter into a swap starting in the swaption's maturity or choose not to exercise the swaption if the swap rate at the swaption's maturity is less than the strike rate. However, if the corresponding swap rate falls below a certain barrier value before the swaption's maturity, the swaption expires worthless.

3.1.2 Time-Dependent Barrier

For the Hull-White model, barriers are transformed to smooth time-dependent barriers on the short rate. We position the nodes optimally on the time-varying barrier that enables us to price barrier options efficiently.

Since the short rate is the only state variable, the time-dependent barrier can be transformed to the equivalent time-dependent barrier on short rate, h_i. The short rate h_i can be found through Newton-Raphson iteration or any other iterative scheme to be detailed later. The determination of h_i can be done *before* constructing the tree.

Interest rate swap

Consider a payer swap on principal C settled in arrears at times $T_j = T_0 + j\delta, j = 1,...,n$. The floating rate $y(T_{j-1}, T_j)$ received at time T_j is set at time T_{j-1}.

The swap cash flows at times T_j, $j = 1,...,n$, are $C \cdot y(T_{j-1}, T_j)\delta$ (the floating leg) and $-Ck\delta$ (the fixed leg), where k is the fixed rate determined at T_0. Hence the value of the swap is (Brace and Musiela 1997):

$$E\left(C\sum_{j=1}^{n} P\left(T_0,T_j\right)\left(y\left(T_{j-1},T_j\right)-k\right)\delta \right)$$

$$= CP\left(T_0,T_0\right)-C\sum_{j=1}^{n} d_j P\left(T_0,T_j\right)$$

$$= C\left(1-\sum_{j=1}^{n} d_j P\left(T_0,T_j\right)\right)$$

(3.1)

where $d_j = k\delta$ for $j=1,...,n-1$ and $d_n = 1+k\delta$.

The identity of (3.1) is explained as follows. The swap can be viewed as a portfolio of a zero coupon bond and a coupon bearing bond. It receives interest at a floating rate on a notional principal C, and the value of the notional principal at time T_0 is C. It pays interest at a fixed rate on the notional principal C. Its value thus equals n zero coupon bonds with notional principals Cd_j for $j=1,...,n$ maturing at time T_j. Therefore, the value is

$$C\sum_{j=1}^{n} d_j P\left(T_0,T_j\right).$$

Swap rate

The spot swap rate $w\left(T_0,n\right)$ at time T_0 is that value of the fixed rate $w\left(T_0,n\right)$ which makes the value of the swap zero, i.e.,

$$C\sum_{j=1}^{n} P\left(T_0,T_j\right) w\left(T_0,n\right)\delta = C*P\left(T_0,T_0\right)-C*P\left(T_0,T_n\right).$$

(3.2)

The left expression is fixed payment value at T_0 and the right expression is floating reception value at T_0. Using equation (3.2), we find that

$$w\left(T_0,n\right)=\frac{1-P\left(T_0,T_n\right)}{\delta\sum_{j=1}^{n} P\left(T_0,T_j\right)}.$$

(3.3)

Barrier swap rate and barrier short rate

Suppose the contractual barrier swap rate β is fixed at the spot swap rate

$w\left(T_0, n\right)$ minus the same fixed rate λ throughout (for example, 25 basis points) and

the swaption knocks out if the swap rate for $T_0 \le t_i \le T$ (option maturity) is less than

or equal to β. So the barrier short rate h_i can be found by setting the value of the

swap at the fixed rate β zero for $T_0 \le t_i \le T$, i.e., we want to make:

$$\frac{1 - P\left(t_i, t_i + n\delta\right)}{\delta \sum_{j=1}^{n} P\left(t_i, t_i + j\delta\right)} - \beta = 0. \tag{3.4}$$

We can approximate the value h_i through Newton-Raphson iteration given the bond

price formula $P\left(t_i, T\right) = A\left(t_i, T\right) e^{-B\left(t_i, T\right) h_i}$.

3.1.3 Initial Hull-White Tree

Barrier Δt -period rate

Let ϕ_i be the Δt -period barrier interest rate (annualized) at time t_i that

corresponds to the short rate h_i. We can find ϕ_i by using the bond price formula

with short rate h_i:

$$e^{-\phi_i \Delta t} = A\left(t_i, t_i + \Delta t\right) e^{-B\left(t_i, t_i + \Delta t\right) h_i}. \tag{3.5}$$

So

$$\phi_i = \frac{B\left(t_i, t_i + \Delta t\right) h_i - \log\left(A\left(t_i, t_i + \Delta t\right)\right)}{\Delta t}. \tag{3.6}$$

A swaption is at-the-money when the strike rate equals the forward swap rate

at swaption's maturity. We illustrate the results using the example of a continuously

observed at-the-money knockout swaption on a five-year swap with a principal of 100

and the fixed leg paying *annually*. The option maturity is 2 months from now, and the

barrier swap rate is fixed at the spot swap rate at time T_0 minus 70 basis points. The

zero yield curve is given by $y\left(T_0, t\right) = 0.08 - 0.05 e^{-0.18t}$, the parameter a in the

Hull-White model equals 0.1, and σ equals 0.015. We build the Hull-White model with four time steps, and the results are shown in Exhibit 5.

EXHIBIT 5

A TRINOMIAL TREE FOR A BARRIER SWAPTION

Forward swap rate :				6.20%	
Spot swap rate:				6.03%	
Barrier swap rate:				5.33% (=6.03% − 0.70%)	
Time t_i	1	2	3	4	
h_i	2.17%	2.19%	2.21%	2.24%	
ϕ_i	2.21%	2.23%	2.25%	2.27%	

By the Hull-White model, we first construct a tree that has the form shown in Exhibit 6. We just reveal the first three out of a total of four steps for brevity. The starting nodes $R(0,0)$ and the step size Δr can be determined according to Hull and White (1993):

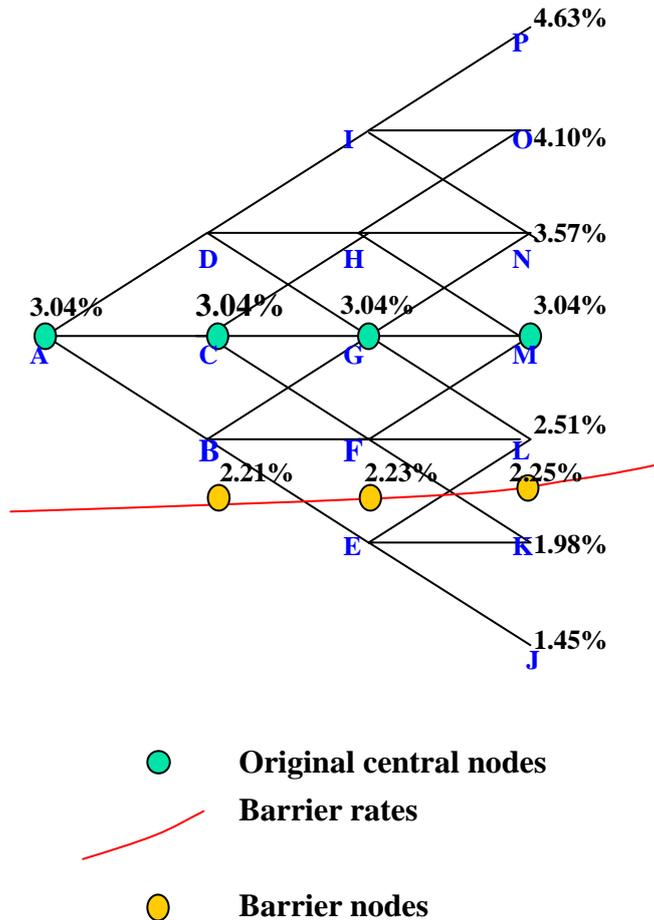$$R(0,0) = y(T_0, T_0 + \Delta t)$$

$$\Delta r = \sqrt{3 \frac{\sigma^2 \left(1 - e^{-2a\Delta t}\right)}{2a}} .$$

The three nodes that can be reached by the branches emanating from any given node (i, j) are $(i+1, j+1), (i+1, j)$, and $(i+1, j-1)$. The nodes $R(i, j)$ in the tree are constructed by:

$$R(i, j) = R(0,0) + j\Delta r .$$

EXHIBIT 6

THE INITIAL TREE

4.63% P

4.10% O

3.57% N

D   H   N

3.04%   3.04%   3.04%   3.04%
A   C   G   M

2.51% L

B   F   L
2.21%   2.23%   2.25%

E   K 1.98%

1.45% J

● Original central nodes
   Barrier rates

● Barrier nodes

3.1.4 Adjusted Hull-White Tree

The second stage in the construction of the tree involves central node adjustments. As remarked above, it is important that the barrier Δt-period rate lies on one of the nodes. The best positioning depends on whether observation of the barrier is continuous or discrete. For continuously observed barriers, it is preferred that the barrier lies exactly on one of the nodes. For discretely observed barriers, it is best to let the barrier fall exactly half way between two nodes. To unify the treatment for

continuous and discrete barrier observations, we introduce a new variable $\hat{\phi}_i$. If the barrier is observed continuously:

$$\hat{\phi}_i = \phi_i. \tag{3.7}$$

If the barrier Δt rate is below the spot Δt rate at time T_0 for discrete observations:

$$\hat{\phi}_i = \phi_i - \frac{\Delta r}{2}. \tag{3.8}$$

If the barrier Δt rate is above the spot Δt rate at time T_0 for discrete observations:

$$\hat{\phi}_i = \phi_i + \frac{\Delta r}{2}. \tag{3.9}$$

We want to construct a tree that aligns $\hat{\phi}_i$ on a tree node for all $1 \le i \le n$ and then adjust the tree according to the following procedure: The node (i, j) with a value $R(i, j)$ closest to $\hat{\phi}_i$ sets the value of $R(i, j)$ equal to $\hat{\phi}_i$.

Let α_i be the Δt-period interest rate at time t_i that is associated with the adjusted central node at time t_i. The first one equals the current spot yield rate maturing at $t_0 + \Delta t$:

$$\alpha_0 = y(T_0, T_0 + \Delta t). \tag{3.10}$$

Integers e_i for $1 \le i \le n$ can be calculated to ensure that barrier rates are an integer number of steps (with step size Δr) away from central rates α_i:

$$e_i = \left[\frac{\hat{\phi}_i - \alpha_{i-1}}{\Delta r} + \frac{1}{2} \right], \tag{3.11}$$

where $[x]$ is the largest integer less than or equal to x. The adjusted central rates α_i can be found with e_i thus:

$$\alpha_i = \hat{\phi}_i - e_i \Delta r .$$
(3.12)
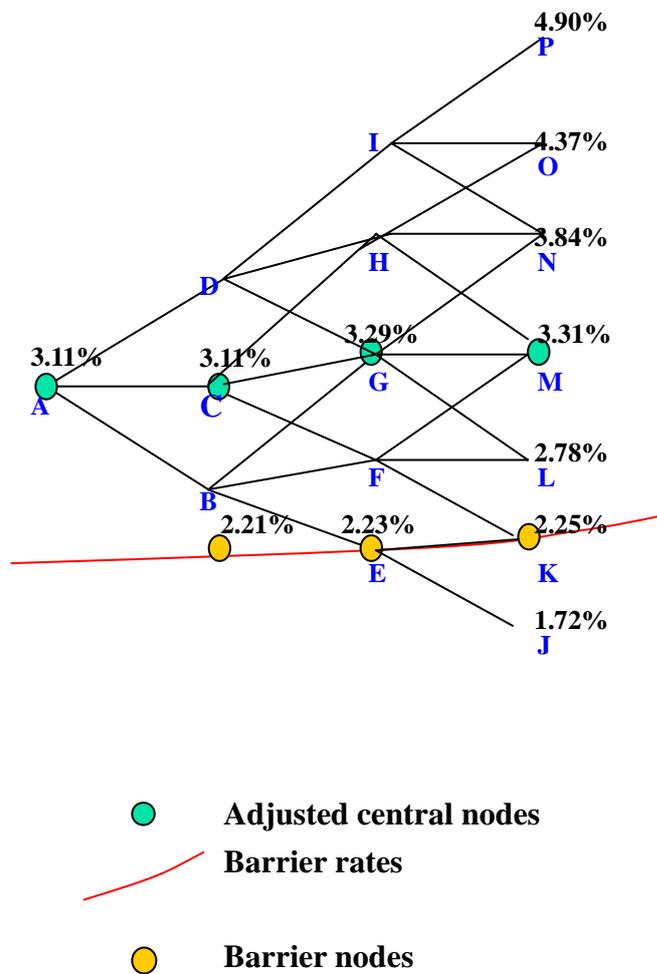
The adjusted interest rate at node (i, j) is then given by

$$R(i, j) = \alpha_i + j\Delta r .$$
(3.13)

The final result is shown in Exhibit 7.

EXHIBIT 7

THE ADJUSTED TREE

4.90% P

4.37% O

3.84% N

3.31% M

3.29% G

3.31% M

3.11% A

3.11% C

2.78% L

2.21% 2.23% 2.25% K

1.72% J

●   **Adjusted central nodes**

     Barrier rates

●   **Barrier nodes**

3.1.5 Calculating Prices

Suppose that the tree has already been constructed up to time T so that it can match the barrier. A value of $\theta(i\Delta t)$ for $0 \le i \le n$ must be chosen so that the tree is consistent with $y(T_0, t_{i+2})$. The procedure is explained by Hull and White [1993], and we give the details in Appendix C.

Let $r(i, j)$ be the short rate that is associated with node (i, j). It can be calculated by bond price formula (2.11) from $R(i, j)$. Once $\theta(i\Delta t)$ has been determined, the corresponding probabilities to nodes (i, j) can be determined by matching the mean $r(i, j) + (\theta(i\Delta t) - ar(i, j))\Delta t$ and variance $\dfrac{\sigma^2(1 - e^{-2a\Delta t})}{2a}$ of the short rate to the continuous-time interest rate model and the condition that the sum of the probabilities equals 1. By the Lindeberg-Feller theorem (see, for example, Billingsley (1986)), a tree constructed in the way will converge to the underlying continuous-time Hull-White model. The probabilities are:

$$
\begin{aligned}
P_u(i, j) &= \frac{V}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} + \frac{\eta}{2\Delta r} \\
P_m(i, j) &= 1 - \frac{V}{\Delta r^2} - \frac{\eta^2}{\Delta r^2} \\
P_d(i, j) &= \frac{V}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} - \frac{\eta}{2\Delta r}
\end{aligned}
\tag{3.14}
$$

with

$$
\eta = (\theta(i\Delta t) - ar(i, j))\Delta t + \alpha_i - \alpha_{i+1}
$$

and

$$
V = \frac{\sigma^2(1 - e^{-2a\Delta t})}{2a}.
$$

Exhibit 8 illustrates the results calculated form Exhibit 7.

EXHIBIT 8

TRINOMIAL TREE FOR EXHIBIT 7

$\theta(i\Delta t)(\%)$	0	1	2	3	4			
	2.08	2.07	2.06	2.05	2.04			
Nodes	A	B	C	D	E	F	G	H
R (%)	3.11	2.58	3.11	3.64	2.23	2.76	3.29	3.82
$P_u(i,j)$	0.247	0.055	0.055	0.054	0.225	0.223	0.220	0.218
$P_m(i,j)$	0.647	0.554	0.551	0.548	0.656	0.656	0.657	0.658
$P_d(i,j)$	0.106	0.391	0.394	0.398	0.119	0.121	0.123	0.124
Nodes	I	J	K	L	M	N	O	P
R (%)	4.35	1.72	2.25	2.78	3.31	3.84	4.37	4.90
$P_u(i,j)$	0.215	0.227	0.225	0.222	0.220	0.217	0.215	0.212
$P_m(i,j)$	0.659	0.655	0.656	0.657	0.657	0.658	0.659	0.660
$P_d(i,j)$	0.126	0.118	0.120	0.121	0.123	0.125	0.126	0.128

First, the short rates $r(i,j)$ at nodes (i,j), where $i=4$, is computed using the bond price formula (2.11) with rates $R(i,j)$. With the short rates $r(i,j)$ given, the swap rates $s(i,j)$ at nodes (i,j), where $i=4$, are found using the swap rate formula (3.3) and therefore the payoff $V(i,j)$ at nodes (i,j), where $i=4$, at maturity T is determined:

$$V(i,j) = C\sum_{i=1}^{5} P(T,T+i\delta)\max\left(s(i,j)-\kappa_T(T_0,n),0\right)\delta. \tag{3.15}$$

Exhibit 9 shows the calculations required to compute the payoff at the option maturity, two months from now.

EXHIBIT 9

OPTION PAYOFF AT TERMINAL NODES (*i*=4)

The barrier swap rate(%) :	5.33			
j	R (%)	short Rate (%)	Swap Rate (%)	Option Payoff
4	5.45	5.42	7.99	7.24073
3	4.92	4.89	7.54	5.49238
2	4.39	4.36	7.09	3.70903
1	3.86	3.83	6.65	1.88994
0	3.33	3.30	6.21	0.03439
-1	2.80	2.77	5.77	0.00000
-2	2.27	2.24	5.33	0.00000
-3	1.74	1.70	4.9	0.00000
-4	1.22	1.17	4.47	0.00000

Finally, Exhibit 10 shows the discounting of the option value back through the tree. If the node (i, j) is touched, the value $V(i, j)$ is set to zero—if not, $V(i, j)$ is calculated using

$$V(i,j) = \left(\begin{array}{l} P_u(i,j)V(i+1,j+1) + P_m(i,j)V(i+1,j) \\ + P_d(i,j)V(i+1,j) \end{array} \right) \exp\left(-R(i,j)\Delta t\right). \quad (3.16)$$

EXHIBIT 10

DISCOUNTING THE OPTION PRICE BACK THROUGH THE TREE

		$Time\ (i\Delta t)$
		0	1	2	3	4
	4					7.24073
	3				5.62349	5.49238
	2			4.00051	3.85494	3.70903
	1		1.73436	2.23866	2.05030	1.88994
j	0	0.80491	0.56870	0.73839	0.43712	0.03439
	-1		0.09728	0.10218	0.00763	0.00000
	-2			0.00000	0.00000	0.00000
	-3				0.00000	0.00000
	-4				0.00000	0.00000

☐ : The barrier node

3.1.6 Numerical Results

We first consider a six-month at-the-money payer's swaption with a notional value of 100 and a barrier fixed at the spot swap rate minus 25 basis points. The underlying is a five-year swap with fixed payments made annually. The zero yield curve is given by $y(T_0,t) = 0.08 - 0.05e^{-0.18t}$, the parameter a in the Hull-White model equals 0.1, and σ equals 0.015.

Exhibits 11 and 12 give the price for such a single-barrier option, when continuously observed, for different numbers of steps in the trinomial tree. It can be seen that convergence is fast due to our specially constructed tree. Exhibit 11 shows that the method gives very accurate answers with as little as 30 steps, and Exhibit 12 shows how fast and smoothly the method converges.

EXHIBIT 11

CONTINUOUSLY OBSERVED SINGLE-BARRIER SWAPTION PRICES

Product	At-the-money single-barrier knockout swaption on five-year swap with the principal 100 and fixed payments made annually	
Barrier	Spot swap rate minus 25 basis points	
Option maturity	0.5 year	
Steps	Prices	Time (sec)
2	1.270025	0.015
5	1.021106	0.031
10	0.986277	0.062
30	0.965275	0.172
100	0.963571	0.547
200	0.963272	1.157
500	0.963417	3.125
1000	0.963320	5.813

Note: Calculations were made in Dev C++ on Window XP system with Inter(R) Pentium(R) 4 CPU 2.40GHz.

EXHIBIT 12

CONVERGENCE OF CONTINUOUSLY OBSERVED SINGLE-BARRIER SWAPTION PRICES
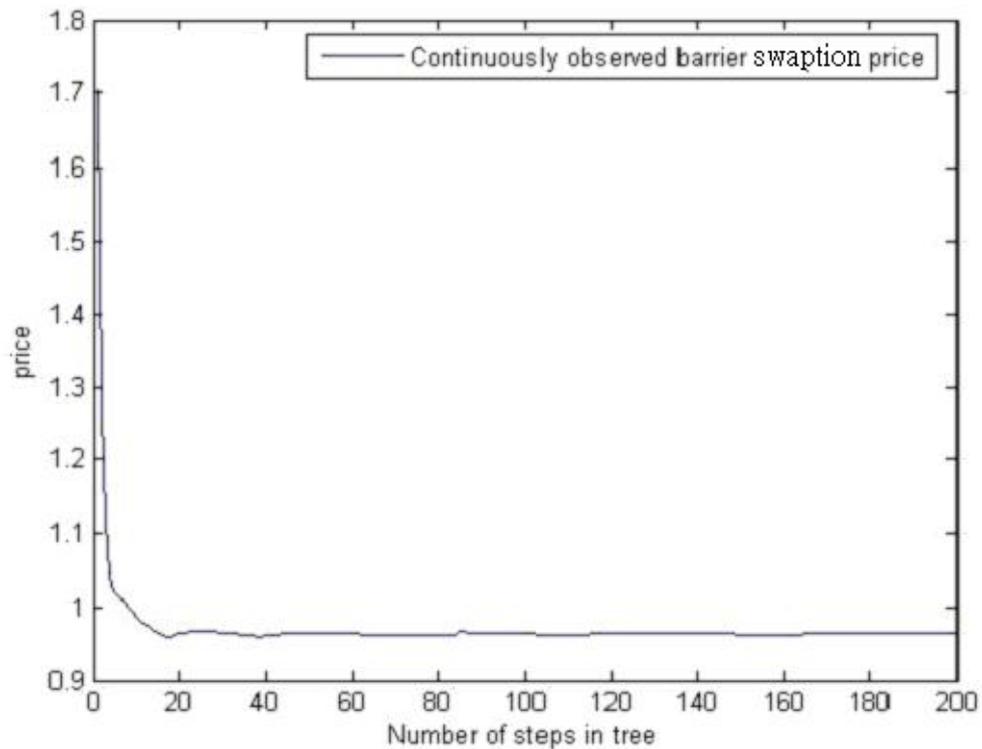
More time steps are needed for a swaption whose barrier is checked at discrete intervals. For example, if the barrier is checked daily and two periods between subsequent observations are allocated, 250 steps are needed for a maturity of six-months (125 days). The speed of convergence is related to the number of periods between observations. So the higher the frequency we observe, the more steps we need to calculate for the same accuracy. While $1250 \ (125 \times 10)$ steps are needed daily monitoring, only $60 \ (6 \times 10)$ steps are needed for the same accuracy (10 steps between observations) for monthly monitoring. As can be seen from Exhibits 13 and 14, the speed of convergence is very good. The prices generally converge with only 10 periods between observations and are very close to Monte Carlo results when our

method is applied with only 50 periods between observations.

EXHIBIT 13

DISCRETELY OBSERVED SINGLE-BARRIER SWAPTION PRICES

|	Number of periods between observations						Monte Carlo
Observation Frequency	2	5	10	20	50	100	
Semi-annually (Time (sec))	1.415429 (0.016)	1.471626 (0.031)	1.396712 (0.047)	1.436232 (0.078)	1.424825 (0.219)	1.428639 (0.438)	1.42821 (1141.423)
Quarterly (Time (sec))	1.465117 (0.016)	1.376347 (0.063)	1.408221 (0.094)	1.404073 (0.204)	1.399401 (0.547)	1.398472 (1.125)	1.39813 (1015.674)
Monthly (Time (sec))	1.318001 (0.063)	1.297323 (0.125)	1.292777 (0.266)	1.289523 (0.531)	1.287248 (1.392)	1.286702 (3.047)	1.28654 (798.811)
Weekly (Time (sec))	1.178672 (0.218)	1.160131 (0.562)	1.154721 (1.188)	1.151438 (2.562)	1.149227 (7.953)	1.148500 (18.341)	1.14856 (891.615)
Daily (Time (sec))	0.704829 (2.226)	0.690894 (3.156)	1.061978 (7.484)	1.059989 (15.383)	1.058798 (48.312)	1.058396 (108.120)	1.0586 (899.254)

Note 1. For the Monte-Carlo method (5000,000 paths and about 200 time steps), the

process is $dr = (\theta(t) - ar)dt + \sigma dZ$ and the discount rate $\exp\left(-\Delta t \sum r_i\right)$ is

calculated using Simpson's method.

Note 2. For example, 250 (125 days $\times$ 2) steps are needed for daily observed swaptions and the number of periods between observations is 2.

EXHIBIT 14

CONVERGENCE OF DISCRETELY OBSERVED SINGLE-BARRIER SWAPTION PRICES
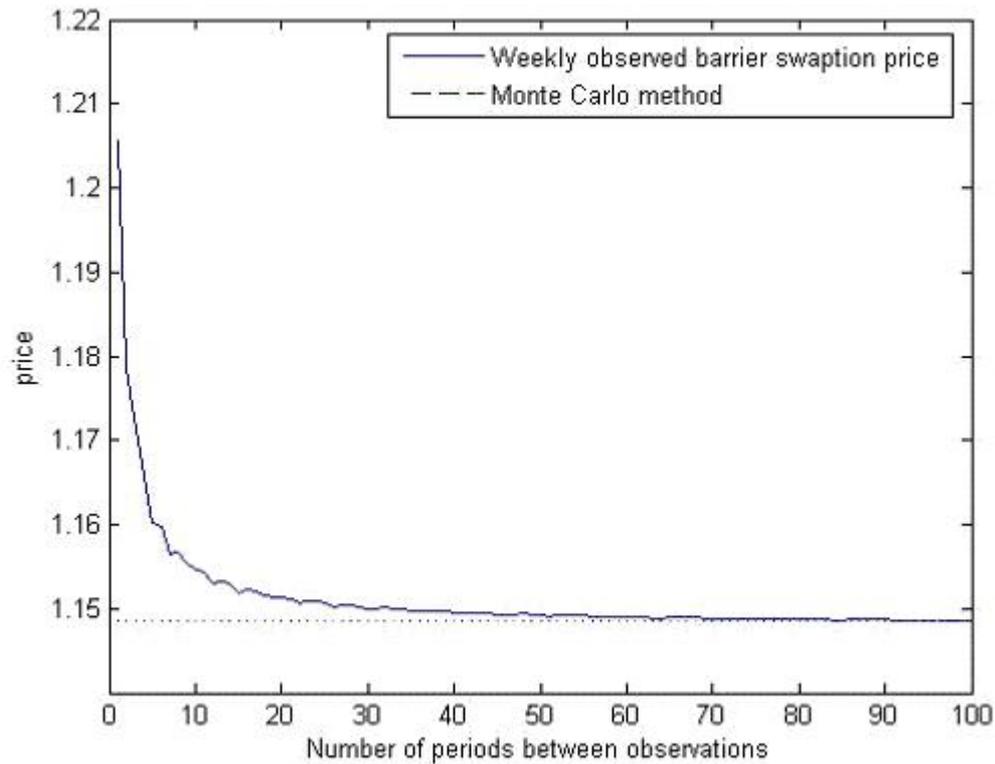
3.2 Single-Barrier Bond Options

3.2.1 Definition

An up-and-out bond option is one type of knock-out bond option. It is a standard option but it ceases to exist if the bond price reaches a certain barrier level, H.

3.2.2 Time-Dependent Barrier

The lattice method can also be applied to price single-barrier bond options. Since the short rate is the only state variable, the time-dependent barrier can be transformed to an equivalent time-dependent barrier on short rate h_i and the

determination of h_i can be done before constructing the tree.

As shown in Hull and White [1990, 1994a]:

$$P(t_i, T) = A(t_i, T) e^{-B(t,T)h_i}. \tag{3.17}$$

Let the barrier bond price H be fixed. The barrier short rate h_i can be calculated for all $0 \le t_i \le T$ (option maturity) by

$$H = A(t_i, T) e^{-B(t_i, T)h_i} \tag{3.18}$$

so that

$$h_i = \frac{\log\left(A(t_i, T)/H\right)}{B(t_i, T)}. \tag{3.19}$$

After finding the barrier short rates, all the remaining procedure follows that for single-barrier swaptions earlier.

3.2.3 Numerical Results

A six-month barrier bond option with a notional value of 100 and strike price 0.85 is considered. We fix the barrier price at 0.91. The zero yield curve is given by $y(T_0, t) = 0.08 - 0.05 e^{-0.18t}$, the parameter a in the Hull-White model equals 0.1, and σ equals 0.015.

Exhibits 15 and 16 give the price for such a barrier option, when continuously observed, for different numbers of steps in the trinomial tree. It can be seen that convergence is faster than barrier swaptions because barrier swaption wasted some time in short rates calculation with Newton-Raphson iteration. Exhibit 15 shows the method gives very accurate answers with as little as 100 steps and Exhibit 16 shows how fast and smoothly the method converges.
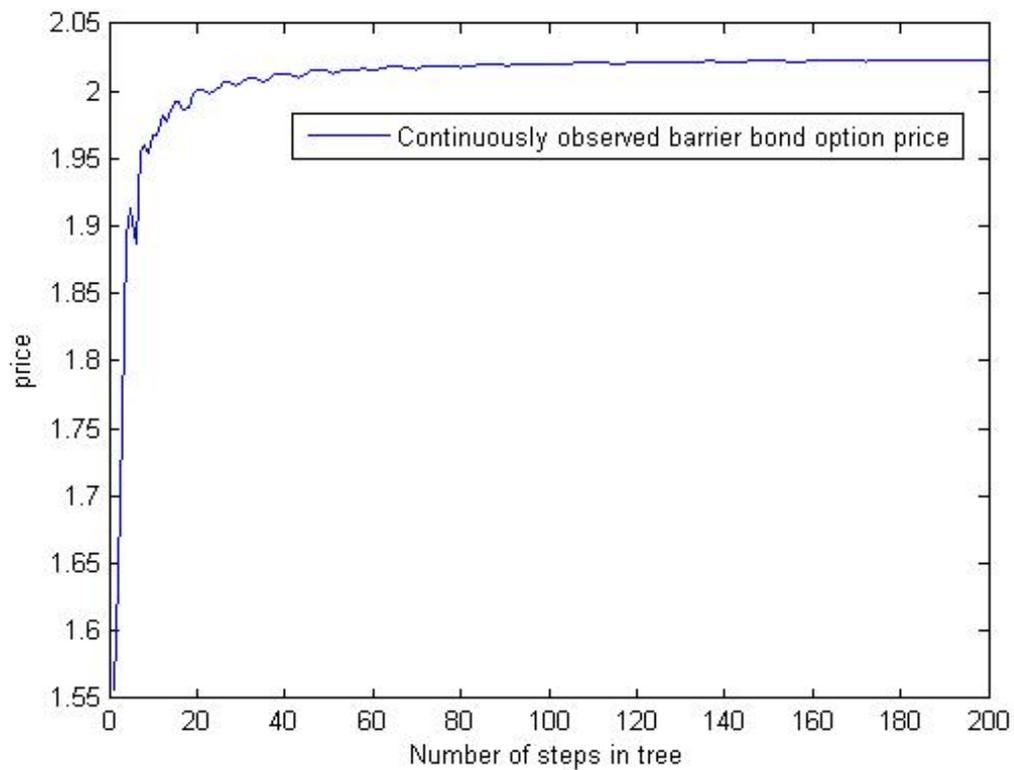
EXHIBIT 15

CONTINUOUSLY OBSERVED SINGLE-BARRIER BOND OPTION

PRICES

Product		Single knock-out call options on three-year discount bond with principal 100
Barrier price		0.91
Strike price		0.85
Option maturity		0.5 year
Steps	Prices	Time (sec)
2	1.641741	0.000
5	1.912776	0.000
10	1.967791	0.000
30	2.006969	0.000
100	2.020153	0.015
200	2.023302	0.063
500	2.024933	0.36
1000	2.025322	1.531
1500	2.025501	3.359

EXHIBIT 16

CONVERGENCE OF CONTINUOUSLY OBSERVED SINGLE-BARRIER BOND OPTION PRICES

More time steps are needed for a bond option whose barrier is checked at discrete intervals. As can be seen from Exhibits 17 and 18, the speed of convergence is also very good. The prices generally converge with only 10 periods between observations and are very close to Monte-Carlo results, with only 50 periods between observations.

EXHIBIT 17

DISCRETELY OBSERVED SINGLE-BARRIER BOND OPTION PRICES

Observation Frequency	Number of periods between observations								Monte Carlo
	2	5	10	20	50	100	150	200	
Semiannually	2.04100	2.15258	2.17812	2.16727	2.17103	2.17033	2.16944	2.16981	2.1698
(Time (sec))	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.016)	(0.047)	(0.062)	(892.344)
Quarterly	2.14913	2.17597	2.16427	2.16761	2.16689	2.16629	2.16628	2.16589	2.1661
(Time (sec))	(0.000)	(0.000)	(0.000)	(0.000)	(0.016)	(0.063)	(0.140)	(0.250)	(1093.132)
Monthly	2.15467	2.14355	2.14117	2.13852	2.13804	2.13763	2.13742	2.13743	2.13777
(Time (sec))	(0.000)	(0.000)	(0.000)	(0.015)	(0.140)	(0.578)	(1.312)	(2.359)	(879.516)
Weekly	2.10666	2.09808	2.09542	2.09413	2.09334	2.09315	2.09310	2.09307	2.09271
(Time (sec))	(0.016)	(0.016)	(0.109)	(0.422)	(2.718)	(11.063)	(24.39)	(43.359)	(798.812)
Daily	2.06718	2.06298	2.06146	2.06080	2.06037	2.06023	2.06019	2.06016	2.06016
(Time (sec))	(0.110)	(0.609)	(2.500)	(10.172)	(62.437)	(250.859)	(562.062)	(1008.250)	(977.311)

EXHIBIT 18

CONVERGENCE OF DISCRETELY OBSERVED SINGLE-BARRIER

BOND OPTION PRICES
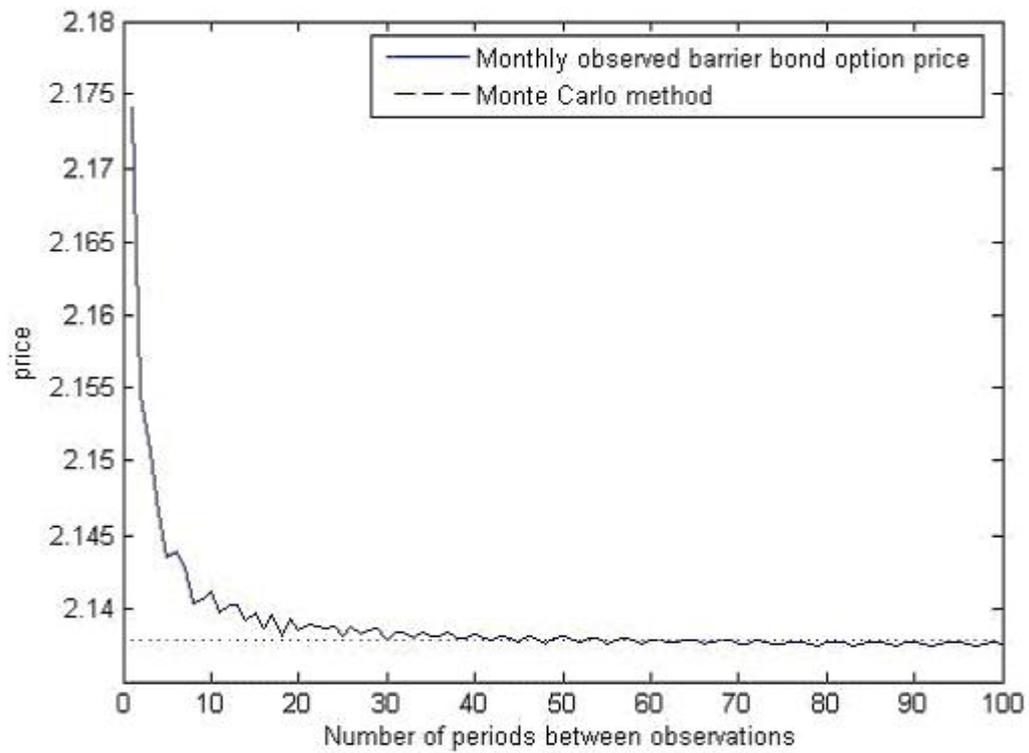
Chapter 4

Extending Cheuk and Vorst's Method

4.1 Double-Barrier Swaptions

4.1.1 Moving Barriers

It has shown how a time-dependent barrier can be matched, and here we explain how a second barrier can be matched by extending Cheuk and Vorst's method [1996b].

To distinguish the two barriers, we will refer to an upper and a lower barrier. The adjectives "upper" and "lower" describe the locations of the barriers relative to each other. Double-barrier knock-out swaptions pricing is covered, and the same method is applicable to other types of interest options including barrier bond options and interest barrier caps, etc.

Trinomial models can be constructed so that nodes are always situated on the two barriers. Our method is to change dr_i to match two barriers for $i = 1,...,n$.

We first unify the treatment of continuous and discrete observation barriers and variables $\hat{h}_i^u$ and $\hat{h}_i^d$ are introduced for $i = 1,...,n$:

$$\hat{h}_i^u = h_i^u$$

$$\hat{h}_i^d = h_i^d$$

for continuously observed barriers, and:

$$\hat{h}_i^u = h_i^u + \frac{dr_i}{2}$$

$$\hat{h}_i^d = h_i^d - \frac{dr_i}{2}$$

for discretely observed barriers, where h_i^u is the upper barrier and h_i^d is the lower barrier.

Then we define the distance as

$$M_i = \hat{h}_i^{\,u} - \hat{h}_i^{\,d}$$

To begin with, we choose a dr_i close to $\sqrt{3V}$ that satisfies

$$x_i dr_i = M_i$$

for continuously observed barriers, and:

$$(x_i - 1)dr_i = \left(h_i^{\,u} - h_i^{\,d}\right)$$

for discretely observed barriers, where x_i is an integer. That is, x_i is chosen as

$$x_i = \left[\frac{M_i}{dr_i} + \frac{1}{2}\right].$$

When x_i is known, dr_i follows from the equation above.

After determining the dr_i, we match the lower barrier by shifting the central nodes α_i we have explained earlier. As the distance between the two barriers is fixed for $i = 1,...,n$, matching the lower implies that the upper barrier is matched also, given dr_i.

All formulas earlier can be used by treating dr as dr_i. The resulting probabilities also match the mean and variance of the short rate in the continuous-time interest rate model.

4.1.2 Numerical Results

We consider a six-month at-the-money payer's swaption with a notional value of 100. A lower-barrier is fixed at the spot swap rate minus 25 basis points and a upper-barrier is fixed at the spot rate plus 200 basis points. The underlying is a five-year swap with fixed payments made annually. The zero yield curve is given by $y(T_0,t) = 0.08 - 0.05e^{-0.18t}$, the parameter a in the Hull-White model equals 0.1, and σ equals 0.015.

Double-barrier swaptions require more time than single-barrier option in pricing because $\overset{\wedge}{h_i}{}^u$ and $\overset{\wedge}{h_i}{}^d$ are calculated simultaneously. Besides, the speed of convergence is also slower than single-barrier option because dr_i changes with the time t_i. But we can still get very accurate answer with as little as 200 steps.

Exhibits 19 and 20 give the price for such a barrier option, when continuously observed, for different numbers of steps in the trinomial tree. It can be seen that convergence is fast due to our specially constructed tree. Exhibit 19 shows the method gets an accurate answer in a short time, and Exhibit 20 shows how fast and smoothly the method converges.
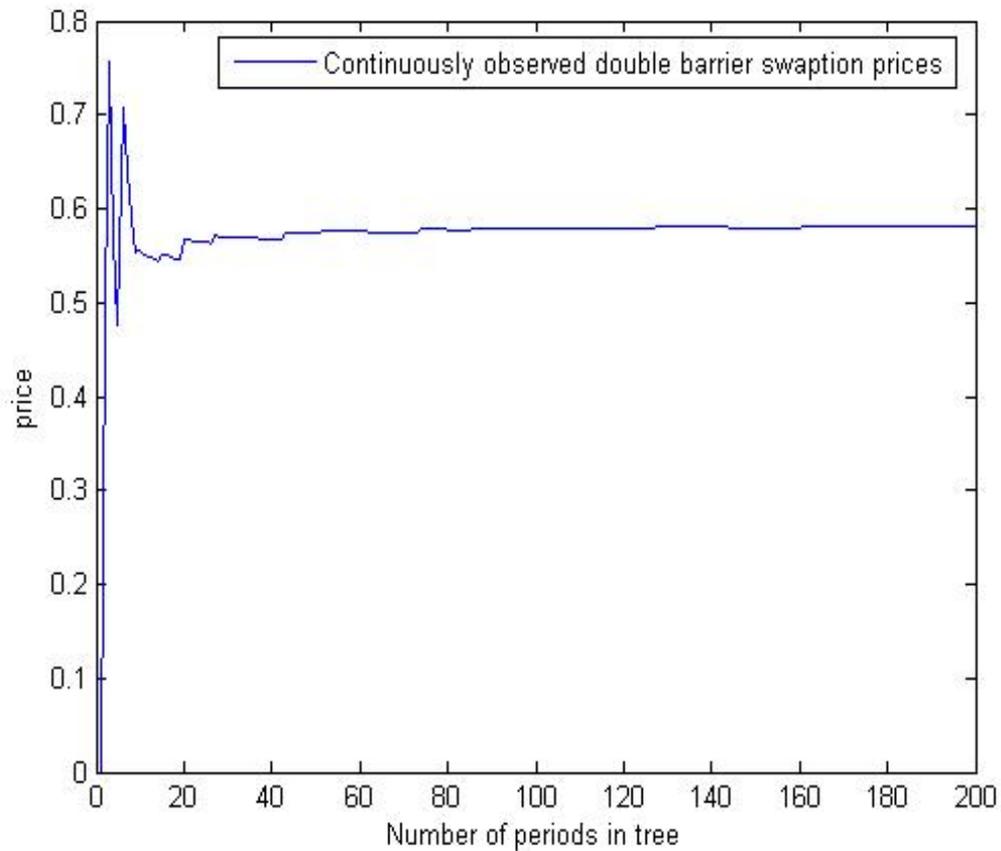
EXHIBIT 19

CONTINUOUSLY OBSERVED DOUBLE-BARRIER SWAPTIONS

PRICES

Product	At-the-money double-barrier knockout swaptions on five-year swap with the principal 100 and fixed payments made annually	
Lower barrier	Spot swap rate minus 25 basis points	
Upper barrier	Spot swap rate plus 200 basis points	
Option maturity	0.5 year	
Steps	Prices	Time (sec)
2	0.495313	0.015
5	0.475678	0.031
10	0.554802	0.062
30	0.569651	0.266
100	0.578415	0.847
200	0.580903	1.782
500	0.582516	4.734
1000	0.582964	10.281

EXHIBIT 20

CONVERGENCE OF CONTINUOUSLY OBSERVED DOUBLE-BARRIER SWAPTION PRICES

More time steps are still needed for a swaption whose barrier is checked at discrete intervals. As can be seen from Exhibits 21 and 22, the speed of convergence is very good. The prices generally converge with only 20 periods between observations and are very close to Monte-Carlo results when our method is used with only 50 periods between observations.
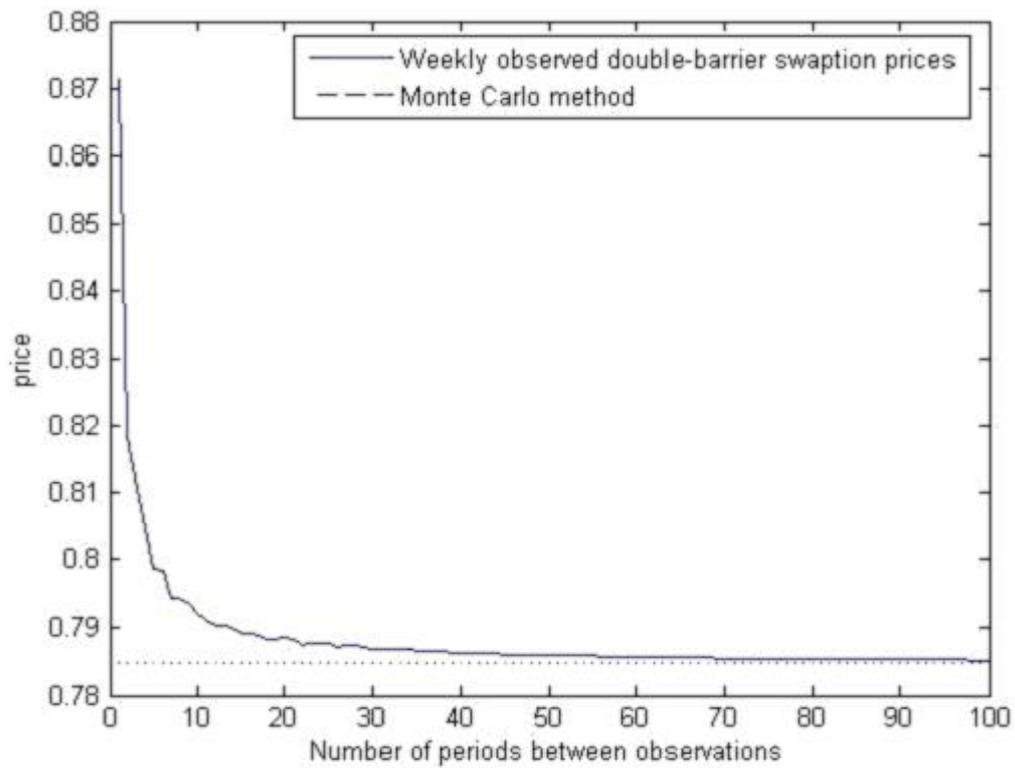
EXHIBIT 21

DISCRETELY OBSERVED DOUBLE-BARRIER SWAPTION PRICES

Observation Frequency	Number of periods between observations						Monte Carlo
	2	5	10	20	50	100	
Semi-annually	1.090210	1.155344	1.122846	1.121167	1.113194	1.115203	1.112772
(Time (sec))	(0.016)	(0.047)	(0.075)	(0.188)	(0.419)	(0.875)	(1134.31)
Quarterly	1.166872	1.098719	1.091995	1.087980	1.083037	1.081246	1.080515
(Time (sec))	(0.016)	(0.094)	(0.172)	(0.344)	(0.875)	(1.766)	(1124.674)
Monthly	0.987741	0.963259	0.953694	0.951150	0.948474	0.947544	0.946642
(Time (sec))	(0.110)	(0.266)	(0.532)	(1.062)	(2.345)	(5.127)	(937.118)
Weekly	0.818323	0.798535	0.792114	0.788520	0.785881	0.785205	0.784679
(Time (sec))	(0.432)	(1.125)	(2.328)	(4.732)	(13.672)	(38.86)	(1291.615)
Daily	0.704829	0.690894	0.686753	0.684701	0.683409	0.682989	0.682474
(Time (sec))	(2.159)	(7.015)	(15.487)	(36.979)	(48.312)	(432.563)	(1189.254)

EXHIBIT 22

CONVERGENCE OF DISCRETELY OBSERVED DOUBLE-BARRIER SWAPTION PRICES

Chapter 5

Conclusions

We have described computing procedures that implement the barrier methodology in Cheuk and Vorst [1996a] to value single-barrier swaptions under the Hull-White interest rate model. We have also applied the same idea to price single-barrier bond options. The prices of discretely observed single-barrier swaptions and bond options are very close to those computed by the Monte-Carlo method, and the rate of convergence of our method is excellent. A second time-dependent barrier can be accommodated using the parameter dr_i and two barriers are completely matched. The price of discretely observed double-barrier swaptions computed by our method is close to that derived by the Monte-Carlo method, and the rate of convergence of our method is also excellent.

Both continuously and discretely observed barriers have been considered. It is no surprise accuracy is highly sensitive to the number of periods in tree in the continuously observed case but not in the discretely observed case. Our results show that the observation frequency is a very important determinant of barrier options prices.

We have presented a novel idea that can deal with all kinds of interest rate options with barriers. The extensive applicability of our methodology makes it extremely useful in practice.

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APPENDIX A Calculating Time-Dependent Level $\theta(t)$

In this appendix, it is assumed that the tree has been constructed up to time $i\Delta t$ and it is shown how $\theta(i\Delta t)$ is obtained. Define $Q(i,j)$ as the value of a security that pays off $1 if node (i,j) is reached and zero otherwise. It is assumed that $Q(i,j)$ is calculated as the tree is being constructed using the relationship:

$$Q(i,j) = \sum_{j^*} Q(i-1,j^*)q(j^*,j)e^{-r(i-1,j^*)\Delta t},$$

where $q(j^*,j)$ is the probability of moving from node $(i-1,j^*)$ to node (i,j).

The value as seen at node (i,j) of a bond maturing at time $(i+2)\Delta t$ is

$$e^{-r(i,j)\Delta t}E\left[e^{-r(i+1)\Delta t} \mid r(i) = r(i,j)\right]$$

where E is the risk-neutral expectations operator and $r(i)$ is the value of instantaneous rate at time $i\Delta t$. The value at time zero of a discount bond maturing at time $(i+2)\Delta t$ is therefore given by:

$$e^{-(i+2)y(i+2)\Delta t} = \sum_j Q(i,j)e^{-r(i,j)\Delta t}E\left[e^{-r(i+1)\Delta t} \mid r(i) = r(i,j)\right].$$

The value of $E\left[e^{-r(i+1)\Delta t} \mid r(i) = r(i,j)\right]$ is known analytically:

$$E\left[e^{-r(i+1)\Delta t} \mid r(i) = r(i,j)\right] = e^{-r(i,j)\Delta t}e^{(-\theta(i\Delta t)+ar(i,j)+V/2)\Delta t^2}$$

so that this leads to

$$\theta(i\Delta t) = \frac{1}{\Delta t}(i+2)\,yield\,(i+2) + \frac{V}{2} + \frac{1}{\Delta t^2}\log\sum_j Q(i,j)e^{-2r(i,j)\Delta t+ar(i,j)\Delta t^2}$$

and $P_u(i,j), P_m(i,j), P_d(i,j)$ are calculated with $\theta(i\Delta t)$. Therefore $\theta(i\Delta t)$ for $0 \le i \le n$ can be determined recursively with $Q(i,j)$ and

$$P_u\left(i-1, j\right), P_m\left(i-1, j\right), P_d\left(i-1, j\right).$$